

# The Forty-Sixth Annual William Lowell Putnam Competition

## Saturday, December 7, 1985

A-1 Determine, with proof, the number of ordered triples  $(A_1, A_2, A_3)$  of sets which have the property that

- (i)  $A_1 \cup A_2 \cup A_3 = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ , and
- (ii)  $A_1 \cap A_2 \cap A_3 = \emptyset$ .

Express your answer in the form  $2^a 3^b 5^c 7^d$ , where  $a, b, c, d$  are nonnegative integers.

A-2 Let  $T$  be an acute triangle. Inscribe a rectangle  $R$  in  $T$  with one side along a side of  $T$ . Then inscribe a rectangle  $S$  in the triangle formed by the side of  $R$  opposite the side on the boundary of  $T$ , and the other two sides of  $T$ , with one side along the side of  $R$ . For any polygon  $X$ , let  $A(X)$  denote the area of  $X$ . Find the maximum value, or show that no maximum exists, of  $\frac{A(R)+A(S)}{A(T)}$ , where  $T$  ranges over all triangles and  $R, S$  over all rectangles as above.

A-3 Let  $d$  be a real number. For each integer  $m \geq 0$ , define a sequence  $\{a_m(j)\}$ ,  $j = 0, 1, 2, \dots$  by the condition

$$a_m(0) = d/2^m, \\ a_m(j+1) = (a_m(j))^2 + 2a_m(j), \quad j \geq 0.$$

Evaluate  $\lim_{n \rightarrow \infty} a_n(n)$ .

A-4 Define a sequence  $\{a_i\}$  by  $a_1 = 3$  and  $a_{i+1} = 3^{a_i}$  for  $i \geq 1$ . Which integers between 00 and 99 inclusive occur as the last two digits in the decimal expansion of infinitely many  $a_i$ ?

A-5 Let  $I_m = \int_0^{2\pi} \cos(x) \cos(2x) \cdots \cos(mx) dx$ . For which integers  $m$ ,  $1 \leq m \leq 10$  is  $I_m \neq 0$ ?

A-6 If  $p(x) = a_0 + a_1x + \cdots + a_mx^m$  is a polynomial with real coefficients  $a_i$ , then set

$$\Gamma(p(x)) = a_0^2 + a_1^2 + \cdots + a_m^2.$$

Let  $F(x) = 3x^2 + 7x + 2$ . Find, with proof, a polynomial  $g(x)$  with real coefficients such that

- (i)  $g(0) = 1$ , and
- (ii)  $\Gamma(f(x)^n) = \Gamma(g(x)^n)$

for every integer  $n \geq 1$ .

B-1 Let  $k$  be the smallest positive integer for which there exist distinct integers  $m_1, m_2, m_3, m_4, m_5$  such that the polynomial

$$p(x) = (x - m_1)(x - m_2)(x - m_3)(x - m_4)(x - m_5)$$

has exactly  $k$  nonzero coefficients. Find, with proof, a set of integers  $m_1, m_2, m_3, m_4, m_5$  for which this minimum  $k$  is achieved.

B-2 Define polynomials  $f_n(x)$  for  $n \geq 0$  by  $f_0(x) = 1$ ,  $f_n(0) = 0$  for  $n \geq 1$ , and

$$\frac{d}{dx} f_{n+1}(x) = (n+1)f_n(x+1)$$

for  $n \geq 0$ . Find, with proof, the explicit factorization of  $f_{100}(1)$  into powers of distinct primes.

B-3 Let

$$\begin{array}{cccc} a_{1,1} & a_{1,2} & a_{1,3} & \cdots \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{array}$$

be a doubly infinite array of positive integers, and suppose each positive integer appears exactly eight times in the array. Prove that  $a_{m,n} > mn$  for some pair of positive integers  $(m, n)$ .

B-4 Let  $C$  be the unit circle  $x^2 + y^2 = 1$ . A point  $p$  is chosen randomly on the circumference  $C$  and another point  $q$  is chosen randomly from the interior of  $C$  (these points are chosen independently and uniformly over their domains). Let  $R$  be the rectangle with sides parallel to the  $x$  and  $y$ -axes with diagonal  $pq$ . What is the probability that no point of  $R$  lies outside of  $C$ ?

B-5 Evaluate  $\int_0^\infty t^{-1/2} e^{-1985(t+t^{-1})} dt$ . You may assume that  $\int_{-\infty}^\infty e^{-x^2} dx = \sqrt{\pi}$ .

B-6 Let  $G$  be a finite set of real  $n \times n$  matrices  $\{M_i\}$ ,  $1 \leq i \leq r$ , which form a group under matrix multiplication. Suppose that  $\sum_{i=1}^r \text{tr}(M_i) = 0$ , where  $\text{tr}(A)$  denotes the trace of the matrix  $A$ . Prove that  $\sum_{i=1}^r M_i$  is the  $n \times n$  zero matrix.