The Forty-Sixth Annual William Lowell Putnam Competition
Solutions

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A–1 *Answer:* $2^{10}.3^{10} = 6^{10}$.

For $n \geq 1$ we determine by induction on n the number a_n of ordered triples (A_1, A_2, A_3) of sets which have the property that

(i) $A_1 \cup A_2 \cup A_3 = \{1, ..., n\}$, and

(ii) $A_1 \cap A_2 \cap A_3 = \emptyset$.

For $n = 1$ this can happen iff one set only is empty (three possibilities) or if two sets are empty (three possibilities); that is $a_1 = 3 + 3 = 6 = 6^1$. Moving from n to $n + 1$, note that $n + 1$ either belongs to A_1, or A_2 or A_3 only, or to one and only one of the intersections $A_1 \cap A_2$,$A_2 \cap A_3$,$A_1 \cap A_3$. That is, $a_{n+1} = 6a_n$. Thus $a_n = 6^n$.

Note: If no empty set is allowed, the number is $6^{10} - 3 - 3(3^{10} - 2) = 60289032$. To see this, let us count the con£gurations with empty sets. There are three cases where exactly two sets are empty. In cases where only (say) A_3 is empty, one £rst needs to choose $k \geq 1$ elements among n to determine A_1, then choose from the elements of A_1, those that will be common with A_2 (choose $l \geq 0$ elements among k). Among these possibilities, that one where $k = n$ and $l = 0$ has to be excluded. Finally there are:

$\sum_{k=1}^{n} C_n^k . \sum_{l=0}^{k} C_k^l - 1 = \sum_{k=1}^{n} C_n^k .2^k - 1 = 3^n - 2$

cases where only A_3, also where only A_1 or A_2 is empty. In total there are $3 + 3(3^n - 2)$ con£gurations with empty sets, to be deducted from the total number 6^n of con£gurations.

A–2 *Answer:* $\sup_{R,S} \frac{A(R)+A(S)}{A(T)} = \frac{2}{3}$.

Denote the vertices of T by A_1, A_2, A_3 and set $A_1A_2 = a_1,\ A_2A_3 = a_2,\ A_1A_3 = a_3$. Denote the base length corresponding to side i by H_i.

Suppose for convenience, that the side common to R and T is (A_1, A_2). Set the origin at A_1, the £rst co-ordinate axis as that directed by $\overrightarrow{A_1A_2}$ and the second co-ordinate axis perpendicular to the £rst one. If R has height $h \leq H_1$, then it is easily seen (analytically) that $A(R) = a_1 h(1 - \frac{h}{H_1})$.

This expression is maximum for $h^* = \frac{H_1}{2}$. This yields $A(R^*) = \frac{a_1 H_1}{4} = \frac{A(T)}{2}$. The latter expression does not depend on the choice of the side common to R and T. Therefore, £nding the maximum of $A(R) + A(S)$ is equivalent to £nding the maximum of $a_1 h(1 - \frac{h}{H_1}) + \frac{A(T(h))}{2}$, where $T(h)$ is the triangle of height $H_1 - h$ limited by the side of R opposite (A_1, A_2) and the other two sides of T. That is, we want the maximum of:

$$f(h) = \frac{a_1}{4}(1 - \frac{h}{H_1})(H_1 + 3h).$$

This yields $h^* = \frac{H_1}{3}$ and $f(h^*) = \frac{a_1 H_1}{3} = \frac{2}{3}A(T)$.

A–3 *Answer:* $e^d - 1$.

Since $a_m(j + 1) = (a_m(j) + 1)^2 - 1,\ j \geq 0$, put $b_m(j) = a_m(j) + 1$.

Then $b_m(j + k) = (b_m(j))^{2^k},\ k, j \geq 0$.

Thus $b_m(m) = b_m(0)^{2^m} = (d/2^m + 1)^{2^m} \mapsto e^d$ as m goes to in£nity.

A–4 *Answer:* $27,\ 29,\ 61,\ 67,\ 43$.

We £rst prove that if $a \equiv b\ (100)$ then $3^a \equiv 3^b\ (100)$. To see this, check that $3^{100} \equiv 1\ (100)$ e.g. by noting that $3^{10} \equiv 49$ and expanding $3^{100} = (3^{10})^{10} \equiv (50 - 1)^{10}$, using the binomial formula for $x \rightarrow (x - 1)^{10}$. From there on the following is straightforward:

$$\begin{aligned} a_2 &= 3^{a_1} \equiv 27\ (100) \\ a_3 &= 3^{a_2} \equiv 29\ (100) \\ a_4 &= 3^{a_3} \equiv 61\ (100) \\ a_5 &= 3^{a_4} \equiv 67\ (100) \\ a_6 &= 3^{a_5} \equiv 43\ (100) \\ a_7 &= 3^{a_6} \equiv 27\ (100), \end{aligned}$$

so that $a_n \equiv a_{n-5}\ (100),\ n \geq 7$.

A–5 *Answer:* $m = 4k$ or $m = 4k - 1,\ k \geq 1$.

First put $f_m(x) = \cos(x)\cos(2x)\cdots\cos(mx)$ and $u = \pi - x$. Then $\int_\pi^{2\pi} f_m(x)\,dx = (-1)^{\frac{m(m+1)}{2}} \int_0^\pi f_m(u)\,du$. Thus if $m = 4k - 2$ or $m = 4k - 3$ then $I_m = 0$. Next:

$$f_m(x) = \prod_{k=1}^{m} (\frac{e^{ikx} + e^{-ikx}}{2}) = \frac{1}{2^m} \sum_{k_1,\ldots,k_m} e^{i(\sum_{j=1}^{m}(-1)^{k_j}.j)x}$$

where $k_i \in \{0, 1\},\ i = 1, ..., m$.

Now unless $A = \sum_{j=1}^{m} (-1)^{k_j}.j = 0,\ \int_0^{2\pi} e^{iAx}\,dx = 0$.

For $m = 3$, $0 = 1+2-3$; for $m = 4$, $0 = 1-2-3+4$. By induction on k, it is then easy to see that if $m = 4k - 1$ or $m = 4k$, there is a set of $k_i' s$ such that $\sum_{j=1}^{m} (-1)^{k_j}.j = 0$.

Thus in those cases $I_m \neq 0$.

A–6 *Answer:* $g(x) = 6x^2 + 5x + 1$ is a solution.

If $p(x) = a_0 + a_1 x + \cdots + a_m x^m$ is a polynomial with real coef£cients a_i, establish the convention that for $k < 0$ or $k > m$ then $a_k = 0$. Then set:

$$T_i(p) = \sum_{k=0}^{m} a_k a_{k-i} = \sum_{k=-\infty}^{+\infty} a_k a_{k-i},\ i \in \mathbb{Z}.$$

Then: $T_{-i}(p) = T_i(p), i \in \mathbb{Z}$; $\Gamma(p) = T_0(p)$; $T_i(p) = 0, |i| > m$.
For $p(x) = a_0 + a_1 x + a_2 x^2$ de£ne a sequence $b_k(n)$ as follows:
$b_k(0) = a_k,\ k = 0, 1, 2;$
$p^n(x) = \sum_{k=0}^{2n} b_k(n) x^k$; $b_k(n) = 0$ otherwise. Then since:

$$b_k(n+1) = a_0 b_k(n) + a_1 b_{k-1}(n) + a_2 b_{k-2}(n), k, n \in \mathbb{Z}$$

we have for $i, n \in \mathbb{N}$:

$$T_i(p^{n+1}) = T_0(p) T_i(p^n) + \sum_{k=1}^{2} T_k(p)(T_{i-k}(p^n) + T_{i+k}(p^n)).$$

Thus if p, q are two polynomials of degree 2, we see (by induction on n) that:

$$(T_i(p) = T_i(q), i = 0, 1, 2) \Leftrightarrow (T_i(p^n) = T_i(q^n), i, n \in \mathbb{N})$$

Looking for a polynomial g of degree 2 satisfying $T_i(g) = T_i(f), i = 0, 1, 2$, we come to the proposed answer.

B–1 *Answer:* $k = 3$ which is achieved for example, with $m_1 = 0, m_2 = -2, m_3 = -1, m_4 = 1, m_5 = 2$.

Indeed $k = 2$ is not possible, otherwise either zero would be a multiple root or p would have complex roots.

B–2 *Answer:* 101^{99} since 101 is prime.

By induction on n, one sees that $f_n(x) = x(x+n)^{n-1}$ for $n \geq 1$, thus $f_{100}(1) = 101^{99}$.

B–3 *Answer:*

Denote by $[x]$ the largest integer smaller than x and take $N \geq 9$. Extracting any subset of size N^2 from the array will yield at least $N^2 - 8N$ elements which are strictly greater than N, since in this subset, $1, 2, ..., N$ can each appear 8 times at the most.
Now suppose that $\forall m \leq N, n \leq N, a_{m,n} \leq mn$ to arrive at a contradiction. For this, extract rows $m = 1, ..., N$ and columns $n = 1, ..., N$ from the array. Those elements $a_{m,n}$ for which $a_{m,n} \leq N$ occupy at least the positions where $mn \leq N$. The number of couples (m, n) satisfying the latter is equal to:
$\sum_{l=1}^{N} [\frac{N}{l}] \geq N(\sum_{l=1}^{N} \frac{1}{l}) - N \geq N(\int_{1}^{N+1} \frac{dx}{x}) - N = N(\ln(N+1) - 1)$.
Now choose N such that $N(\ln(N+1) - 1) > 8N$ (i.e. $N > e^9 - 1$) to obtain a contradiction, since then, there are not enough positions left in the $N \times N$ sub-matrix to place all the elements (either smaller than N or larger than N) that should be there.

B–4 *Answer:* $\frac{\sqrt{2}}{\pi^2} \approx 0.14$.

It is assumed that the probability distribution associated to p is the uniform distribution on $[0, 2\pi]$ and that associated to q the uniform distribution on C. Given p, R will be inside C iff q is inside the rectangle inscribed in C, with sides parallels to the axes and having p as a vertex. If $(cos\theta, sin\theta)$ are the coordinates of p, this rectangle has the area $2|sin\theta cos\theta|$. Thus the searched probability is:

$$Prob = \frac{1}{2\pi} \int_0^{2\pi} \frac{2|sin\theta cos\theta|}{\pi}\, d\theta = \frac{2}{\pi^2} \int_0^{\pi/4} sinu\, du = \frac{\sqrt{2}}{\pi^2}.$$

B–5 *Answer:* $\sqrt{\frac{\pi}{1985}}.(e^{-2.1985.\sqrt{1985}})$.

First put $u = t^{1/2}$ then:
$I = \int_0^\infty t^{-1/2} e^{-1985(t+t^{-1})}\, dt = 2\int_0^\infty e^{-1985(u^2+u^{-2})}\, du.$
Let $m = 1985$ and for $\alpha \geq 0$,
$f(\alpha) = \int_0^\infty e^{-m(u^2+\alpha.u^{-2})}\, du.$
Then f is differentiable and
$f'(\alpha) = \int_0^\infty -m.u^{-2} e^{-m(u^2+\alpha.u^{-2})}\, du$
$= -m. \int_0^\infty e^{-m(u^{-2}+\alpha.u^2)}\, du$
$= -m.\alpha^{-1/2} \int_0^\infty e^{-m(u^2+\alpha.u^{-2})}\, du$
$= -m.\alpha^{-1/2} f(\alpha).$
Thus $f(\alpha) = f(0) e^{-2m.\sqrt{\alpha}}$.
Now $f(0) = \int_0^\infty e^{-mx^2}\, dx = m^{-1/2} \int_0^\infty e^{-x^2}\, dx = m^{-1/2}(\sqrt{\pi}/2)$ and $I = 2f(m) = \sqrt{\pi/m}.(e^{-2m.\sqrt{m}})$.

B–6 *Answer:*

Put $A = \sum_{i=1}^r M_i$. For a given j, the set of products $\{(M_i.M_j)\}_{i=1,...r}$ is a translation of G. Thus:

$(\sum_{i=1}^r (M_i)) M_j = \sum_{i=1}^r M_i,\ j = 1, ...r.$

Thus $A^2 = (\sum_{i=1}^r M_i)^2 = r(\sum_{i=1}^r M_i) = rA$. From there, it is easy to see that if r is an eigenvalue of A then in some base of $\mathbb{R}^n$, A must have the form:

$$\begin{pmatrix} rI_p & B \\ 0 & 0 \end{pmatrix}$$

where I_p is the $p \times p$ identity matrix and B some $p \times n - p$ matrix (use a base of eigenvectors associated to eigenvalue r, and complete it with a base of a supplementary vector space in $\mathbb{R}^n$).
Since $\mathrm{tr}(A) = 0$, this is impossible and thus $A = 0$.