The Fiftieth Annual William Lowell Putnam Competition
Saturday, December 2, 1989

A–1 How many primes among the positive integers, written as usual in base 10, are alternating 1's and 0's, beginning and ending with 1?

A–2 Evaluate $\int_0^a \int_0^b e^{\max\{b^2x^2, a^2y^2\}}\,dy\,dx$ where a and b are positive.

A–3 Prove that if

$$11z^{10} + 10iz^9 + 10iz - 11 = 0,$$

then $|z| = 1$. (Here z is a complex number and $i^2 = -1$.)

A–4 If α is an irrational number, $0 < \alpha < 1$, is there a £nite game with an honest coin such that the probability of one player winning the game is α? (An honest coin is one for which the probability of heads and the probability of tails are both $\frac{1}{2}$. A game is £nite if with probability 1 it must end in a £nite number of moves.)

A–5 Let m be a positive integer and let $\mathcal{G}$ be a regular $(2m+1)$-gon inscribed in the unit circle. Show that there is a positive constant A, independent of m, with the following property. For any points p inside $\mathcal{G}$ there are two distinct vertices v_1 and v_2 of $\mathcal{G}$ such that

$$|\,|p - v_1| - |p - v_2|\,| < \frac{1}{m} - \frac{A}{m^3}.$$

Here $|s - t|$ denotes the distance between the points s and t.

A–6 Let $\alpha = 1 + a_1x + a_2x^2 + \cdots$ be a formal power series with coef£cients in the £eld of two elements. Let

$$a_n = \begin{cases} 1 & \text{if every block of zeros in the binary expansion of } n \text{ has an even number of zeros in the block} \\ 0 & \text{otherwise.} \end{cases}$$

(For example, $a_{36} = 1$ because $36 = 100100_2$ and $a_{20} = 0$ because $20 = 10100_2$.) Prove that $\alpha^3 + x\alpha + 1 = 0$.

B–1 A dart, thrown at random, hits a square target. Assuming that any two parts of the target of equal area are equally likely to be hit, £nd the probability that the point hit is nearer to the center than to any edge. Express your answer in the form $\frac{a\sqrt{b}+c}{d}$, where a, b, c, d are integers.

B–2 Let S be a non-empty set with an associative operation that is left and right cancellative ($xy = xz$ implies $y = z$, and $yx = zx$ implies $y = z$). Assume that for every a in S the set $\{a^n : n = 1, 2, 3, \ldots\}$ is £nite. Must S be a group?

B–3 Let f be a function on $[0, \infty)$, differentiable and satisfying

$$f'(x) = -3f(x) + 6f(2x)$$

for $x > 0$. Assume that $|f(x)| \leq e^{-\sqrt{x}}$ for $x \geq 0$ (so that $f(x)$ tends rapidly to 0 as x increases). For n a non-negative integer, de£ne

$$\mu_n = \int_0^\infty x^n f(x)\,dx$$

(sometimes called the nth moment of f).

a) Express μ_n in terms of μ_0.

b) Prove that the sequence $\{\mu_n \frac{3^n}{n!}\}$ always converges, and that the limit is 0 only if $\mu_0 = 0$.

B–4 Can a countably in£nite set have an uncountable collection of non-empty subsets such that the intersection of any two of them is £nite?

B–5 Label the vertices of a trapezoid T (quadrilateral with two parallel sides) inscribed in the unit circle as A, B, C, D so that AB is parallel to CD and A, B, C, D are in counterclockwise order. Let s_1, s_2, and d denote the lengths of the line segments AB, CD, and OE, where E is the point of intersection of the diagonals of T, and O is the center of the circle. Determine the least upper bound of $\frac{s_1 - s_2}{d}$ over all such T for which $d \neq 0$, and describe all cases, if any, in which it is attained.

B–6 Let $(x_1, x_2, \ldots x_n)$ be a point chosen at random from the n-dimensional region de£ned by $0 < x_1 < x_2 < \cdots < x_n < 1$. Let f be a continuous function on $[0, 1]$ with $f(1) = 0$. Set $x_0 = 0$ and $x_{n+1} = 1$. Show that the expected value of the Riemann sum

$$\sum_{i=0}^{n} (x_{i+1} - x_i) f(x_{i+1})$$

is $\int_0^1 f(t)P(t)\,dt$, where P is a polynomial of degree n, independent of f, with $0 \leq P(t) \leq 1$ for $0 \leq t \leq 1$.