The 52nd William Lowell Putnam Mathematical Competition
Saturday, December 7, 1991

A–1 A 2×3 rectangle has vertices as $(0,0)$, $(2,0)$, $(0,3)$, and $(2,3)$. It rotates 90° clockwise about the point $(2,0)$. It then rotates 90° clockwise about the point $(5,0)$, then 90° clockwise about the point $(7,0)$, and £nally, 90° clockwise about the point $(10,0)$. (The side originally on the x-axis is now back on the x-axis.) Find the area of the region above the x-axis and below the curve traced out by the point whose initial position is (1,1).

A–2 Let $\mathbf{A}$ and $\mathbf{B}$ be different $n\times n$ matrices with real entries. If $\mathbf{A}^3=\mathbf{B}^3$ and $\mathbf{A}^2\mathbf{B}=\mathbf{B}^2\mathbf{A}$, can $\mathbf{A}^2+\mathbf{B}^2$ be invertible?

A–3 Find all real polynomials $p(x)$ of degree $n\geq 2$ for which there exist real numbers $r_1<r_2<\cdots<r_n$ such that

1. $p(r_i)=0, \qquad i=1,2,\dots,n$, and
2. $p'\left(\frac{r_i+r_{i+1}}{2}\right)=0 \qquad i=1,2,\dots,n-1,$

where $p'(x)$ denotes the derivative of $p(x)$.

A–4 Does there exist an in£nite sequence of closed discs $D_1,D_2,D_3,\dots$ in the plane, with centers $c_1,c_2,c_3,\dots$, respectively, such that

1. the c_i have no limit point in the £nite plane,
2. the sum of the areas of the D_i is £nite, and
3. every line in the plane intersects at least one of the D_i?

A–5 Find the maximum value of

$$\int_0^y \sqrt{x^4+(y-y^2)^2}\,dx$$

for $0\leq y\leq 1$.

A–6 Let $A(n)$ denote the number of sums of positive integers

$$a_1+a_2+\cdots+a_r$$

which add up to n with

$$\begin{gathered} a_1>a_2+a_3, a_2>a_3+a_4,\dots, \\ a_{r-2}>a_{r-1}+a_r, a_{r-1}>a_r. \end{gathered}$$

Let $B(n)$ denote the number of $b_1+b_2+\cdots+b_s$ which add up to n, with

1. $b_1\geq b_2\geq\cdots\geq b_s$,
2. each b_i is in the sequence $1,2,4,\dots,g_j,\dots$ de£ned by $g_1=1$, $g_2=2$, and $g_j=g_{j-1}+g_{j-2}+1$, and
3. if $b_1=g_k$ then every element in $\{1,2,4,\dots,g_k\}$ appears at least once as a b_i.

Prove that $A(n)=B(n)$ for each $n\geq 1$.

(For example, $A(7)=5$ because the relevant sums are $7, 6+1, 5+2, 4+3, 4+2+1$, and $B(7)=5$ because the relevant sums are $4+2+1, 2+2+2+1, 2+2+1+1+1, 2+1+1+1+1+1, 1+1+1+1+1+1+1$.)

B–1 For each integer $n\geq 0$, let $S(n)=n-m^2$, where m is the greatest integer with $m^2\leq n$. De£ne a sequence $(a_k)_{k=0}^\infty$ by $a_0=A$ and $a_{k+1}=a_k+S(a_k)$ for $k\geq 0$. For what positive integers A is this sequence eventually constant?

B–2 Suppose f and g are non-constant, differentiable, real-valued functions de£ned on $(-\infty,\infty)$. Furthermore, suppose that for each pair of real numbers x and y,

$$\begin{aligned} f(x+y) &= f(x)f(y)-g(x)g(y), \\ g(x+y) &= f(x)g(y)+g(x)f(y). \end{aligned}$$

If $f'(0)=0$, prove that $(f(x))^2+(g(x))^2=1$ for all x.

B–3 Does there exist a real number L such that, if m and n are integers greater than L, then an $m\times n$ rectangle may be expressed as a union of 4×6 and 5×7 rectangles, any two of which intersect at most along their boundaries?

B–4 Suppose p is an odd prime. Prove that

$$\sum_{j=0}^{p}\binom{p}{j}\binom{p+j}{j}\equiv 2^p+1 \pmod{p^2}.$$

B–5 Let p be an odd prime and let $\mathbb{Z}_p$ denote (the £eld of) integers modulo p. How many elements are in the set

$$\{x^2 : x\in\mathbb{Z}_p\}\cap\{y^2+1 : y\in\mathbb{Z}_p\}?$$

B–6 Let a and b be positive numbers. Find the largest number c, in terms of a and b, such that

$$a^x b^{1-x}\leq a\frac{\sinh ux}{\sinh u}+b\frac{\sinh u(1-x)}{\sinh u}$$

for all u with $0<|u|\leq c$ and for all x, $0<x<1$. (Note: $\sinh u=(e^u-e^{-u})/2$.)