The 55th William Lowell Putnam Mathematical Competition
Saturday, December 3, 1994

A–1 Let (a_n) be a sequence of positive reals such that, for all n, $a_n \leq a_{2n} + a_{2n+1}$. Prove that $\sum_{n=1}^{\infty} a_n$ diverges.

A–2 Find the positive value of m such that the area in the £rst quadrant enclosed by the ellipse $\frac{x^2}{9} + y^2 = 1$, the x-axis, and the line $y = 2x/3$ is equal to the area in the £rst quadrant enclosed by the ellipse $\frac{x^2}{9} + y^2 = 1$, the y-axis, and the line $y = mx$.

A–3 Prove that the points of an isosceles triangle of side length 1 annot be colored in four colors such that no two points at distance at least $2 - \sqrt{2}$ from each other receive the same color.

A–4 Let A and B be 2×2 matrices with integer entries such that each of $A, A+B, A+2B, A+3B, A+4B$ has an inverse with integer entries. Prove that the same must be true of $A + 5B$.

A–5 Let (r_n) be a sequence of positive reals with limit 0. Let S be the set of all numbers expressible in the form $r_{i_1} + \cdots + r_{i_{1994}}$ for positive integers $i_1 < i_2 < \cdots < i_{1994}$. Prove that every interval (a, b) contains a subinterval (c, d) whose intersection with S is empty.

A–6 Let $f_1, \ldots, f_{10}$ be bijections of the integers such that for every integer n, there exists a sequence $i_1, \ldots, i_k$ for some k such that $f_{i_1} \circ \cdots \circ f_{i_k}(0) = n$. Prove that if A is any nonempty £nite set, there exist at most 512 sequences $(e_1, \ldots, e_{10})$ of zeroes and ones such that $f_1^{e_1} \circ \cdots \circ f_{10}^{e_{10}}$ maps A to A. (Here $f^1 = f$ and f^0 means the identity function.)

B–1 Find all positive integers n such that $|n - m^2| \leq 250$ for exactly 15 nonnegative integers m.

B–2 Find all c such that the graph of the function $x^4 + 9x^3 + cx^2 + ax + b$ meets some line in four distinct points.

B–3 Let $f(x)$ be a positive-valued function over the reals such that $f'(x) > f(x)$ for all x. For what k must there exist N such that $f(x) > e^{kx}$ for $x > N$?

B–4 Let A be the matrix $((32)(42))$ and for positive integers n, de£ne d_n as the greatest common divisor of the entries of $A^n - I$, where $I = ((10)(01))$. Prove that $d_n \to \infty$ as $n \to \infty$.

B–5 Fix n a positive integer. For α real, de£ne $f_\alpha(i)$ as the greatest integer less than or equal to αi, and write f^k for the k-th iterate of f (i.e. $f^1 = f$ and $f^{k+1} = f \circ f^k$). Prove there exists α such that $f_{\alpha^k}(n^2) = f_\alpha^k(n^2) = n^2 - k$ for $k = 1, \ldots, n$.

B–6 Suppose a, b, c, d are integers with $0 \leq a \leq bleq99$, $0 \leq c \leq d \leq 99$. For any integer i, let $n_i = 101i + 1002^i$. Show that if $n_a + n_b$ is congruent to $n_c + n_d$ mod 10100, then $a = c$ and $b = d$.