

# Solutions to the 61st William Lowell Putnam Mathematical Competition

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Manjul Bhargava, Kiran Kedlaya, and Lenny Ng

A-1 The possible values comprise the interval  $(0, A^2)$ .

To see that the values must lie in this interval, note that

$$\left( \sum_{j=0}^m x_j \right)^2 = \sum_{j=0}^m x_j^2 + \sum_{0 \leq j < k \leq m} 2x_j x_k,$$

so  $\sum_{j=0}^m x_j^2 \leq A^2 - 2x_0 x_1$ . Letting  $m \rightarrow \infty$ , we have  $\sum_{j=0}^{\infty} x_j^2 \leq A^2 - 2x_0 x_1 < A^2$ .

To show that all values in  $(0, A^2)$  can be obtained, we use geometric progressions with  $x_1/x_0 = x_2/x_1 = \dots = d$  for variable  $d$ . Then  $\sum_{j=0}^{\infty} x_j = x_0/(1-d)$  and

$$\sum_{j=0}^{\infty} x_j^2 = \frac{x_0^2}{1-d^2} = \frac{1-d}{1+d} \left( \sum_{j=0}^{\infty} x_j \right)^2.$$

As  $d$  increases from 0 to 1,  $(1-d)/(1+d)$  decreases from 1 to 0. Thus if we take geometric progressions with  $\sum_{j=0}^{\infty} x_j = A$ ,  $\sum_{j=0}^{\infty} x_j^2$  ranges from 0 to  $A^2$ . Thus the possible values are indeed those in the interval  $(0, A^2)$ , as claimed.

A-2 First solution: Let  $a$  be an even integer such that  $a^2 + 1$  is not prime. (For example, choose  $a \equiv 2 \pmod{5}$ , so that  $a^2 + 1$  is divisible by 5.) Then we can write  $a^2 + 1$  as a difference of squares  $x^2 - b^2$ , by factoring  $a^2 + 1$  as  $rs$  with  $r \geq s > 1$ , and setting  $x = (r+s)/2$ ,  $b = (r-s)/2$ . Finally, put  $n = x^2 - 1$ , so that  $n = a^2 + b^2$ ,  $n+1 = x^2$ ,  $n+2 = x^2 + 1$ .

Second solution: It is well-known that the equation  $x^2 - 2y^2 = 1$  has infinitely many solutions (the so-called ‘‘Pell’’ equation). Thus setting  $n = 2y^2$  (so that  $n = y^2 + y^2$ ,  $n+1 = x^2 + 0^2$ ,  $n+2 = x^2 + 1^2$ ) yields infinitely many  $n$  with the desired property.

Third solution: As in the first solution, it suffices to exhibit  $x$  such that  $x^2 - 1$  is the sum of two squares. We will take  $x = 3^{2^n}$ , and show that  $x^2 - 1$  is the sum of two squares by induction on  $n$ : if  $3^{2^n} - 1 = a^2 + b^2$ , then

$$\begin{aligned} (3^{2^{n+1}} - 1) &= (3^{2^n} - 1)(3^{2^n} + 1) \\ &= (3^{2^{n-1}} a + b)^2 + (a - 3^{2^{n-1}} b)^2. \end{aligned}$$

A-3 The maximum area is  $3\sqrt{5}$ .

We deduce from the area of  $P_1 P_3 P_5 P_7$  that the radius of the circle is  $\sqrt{5}/2$ . An easy calculation using the Pythagorean Theorem then shows that the rectangle  $P_2 P_4 P_6 P_8$  has sides  $\sqrt{2}$  and  $2\sqrt{2}$ . For notational ease, denote the area of a polygon by putting brackets around the name of the polygon.

By symmetry, the area of the octagon can be expressed as

$$[P_2 P_4 P_6 P_8] + 2[P_2 P_3 P_4] + 2[P_4 P_5 P_6].$$

Note that  $[P_2 P_3 P_4]$  is  $\sqrt{2}$  times the distance from  $P_3$  to  $P_2 P_4$ , which is maximized when  $P_3$  lies on the midpoint of arc  $P_2 P_4$ ; similarly,  $[P_4 P_5 P_6]$  is  $\sqrt{2}/2$  times the distance from  $P_5$  to  $P_4 P_6$ , which is maximized when  $P_5$  lies on the midpoint of arc  $P_4 P_6$ . Thus the area of the octagon is maximized when  $P_3$  is the midpoint of arc  $P_2 P_4$  and  $P_5$  is the midpoint of arc  $P_4 P_6$ . In this case, it is easy to calculate that  $[P_2 P_3 P_4] = \sqrt{5} - 1$  and  $[P_4 P_5 P_6] = \sqrt{5}/2 - 1$ , and so the area of the octagon is  $3\sqrt{5}$ .

A-4 We use integration by parts:

$$\begin{aligned} \int_0^B \sin x \sin x^2 dx &= \int_0^B \frac{\sin x}{2x} \sin x^2 (2x dx) \\ &= -\frac{\sin x}{2x} \cos x^2 \Big|_0^B \\ &\quad + \int_0^B \left( \frac{\cos x}{2x} - \frac{\sin x}{2x^2} \right) \cos x^2 dx. \end{aligned}$$

Now  $\frac{\sin x}{2x} \cos x^2$  tends to 0 as  $B \rightarrow \infty$ , and the integral of  $\frac{\sin x}{2x^2} \cos x^2$  converges absolutely by comparison with  $1/x^2$ . Thus it suffices to note that

$$\begin{aligned} \int_0^B \frac{\cos x}{2x} \cos x^2 dx &= \frac{\cos x}{4x^2} \cos x^2 (2x dx) \\ &= \frac{\cos x}{4x^2} \sin x^2 \Big|_0^B \\ &\quad - \int_0^B \frac{2x \cos x - \sin x}{4x^3} \sin x^2 dx, \end{aligned}$$

and that the final integral converges absolutely by comparison to  $1/x^3$ .

An alternate approach is to first rewrite  $\sin x \sin x^2$  as  $\frac{1}{2}(\cos(x^2 - x) - \cos(x^2 + x))$ . Then

$$\int_0^B \cos(x^2 + x) dx = -\frac{2x+1}{\sin(x^2+x)} \Big|_0^B - \int_0^B \frac{2 \sin(x^2+x)}{(2x+1)^2} dx$$

converges absolutely, and  $\int_0^B \cos(x^2 - x)$  can be treated similarly.

A-5 Let  $a, b, c$  be the distances between the points. Then the area of the triangle with the three points as vertices is  $abc/4r$ . On the other hand, the area of a triangle whose vertices have integer coordinates is at least  $1/2$  (for example, by Pick's Theorem). Thus  $abc/4r \geq 1/2$ , and so

$$\max\{a, b, c\} \geq (abc)^{1/3} \geq (2r)^{1/3} > r^{1/3}.$$

A-6 Recall that if  $f(x)$  is a polynomial with integer coefficients, then  $m-n$  divides  $f(m) - f(n)$  for any integers  $m$  and  $n$ . In particular, if we put  $b_n = a_{n+1} - a_n$ , then  $b_n$  divides  $b_{n+1}$  for all  $n$ . On the other hand, we are given that  $a_0 = a_m = 0$ , which implies that  $a_1 = a_{m+1}$  and so  $b_0 = b_m$ . If  $b_0 = 0$ , then  $a_0 = a_1 = \dots = a_m$  and we are done. Otherwise,  $|b_0| = |b_1| = |b_2| = \dots$ , so  $b_n = \pm b_0$  for all  $n$ .

Now  $b_0 + \dots + b_{m-1} = a_m - a_0 = 0$ , so half of the integers  $b_0, \dots, b_{m-1}$  are positive and half are negative. In particular, there exists an integer  $0 < k < m$  such that  $b_{k-1} = -b_k$ , which is to say,  $a_{k-1} = a_{k+1}$ . From this it follows that  $a_n = a_{n+2}$  for all  $n \geq k-1$ ; in particular, for  $m = n$ , we have

$$a_0 = a_m = a_{m+2} = f(f(a_0)) = a_2.$$

B-1 Consider the seven triples  $(a, b, c)$  with  $a, b, c \in \{0, 1\}$  not all zero. Notice that if  $r_j, s_j, t_j$  are not all even, then four of the sums  $ar_j + bs_j + ct_j$  with  $a, b, c \in \{0, 1\}$  are even and four are odd. Of course the sum with  $a = b = c = 0$  is even, so at least four of the seven triples with  $a, b, c$  not all zero yield an odd sum. In other words, at least  $4N$  of the tuples  $(a, b, c, j)$  yield odd sums. By the pigeonhole principle, there is a triple  $(a, b, c)$  for which at least  $4N/7$  of the sums are odd.

B-2 Since  $\gcd(m, n)$  is an integer linear combination of  $m$  and  $n$ , it follows that

$$\frac{\gcd(m, n)}{n} \binom{n}{m}$$

is an integer linear combination of the integers

$$\frac{m}{n} \binom{n}{m} = \binom{n-1}{m-1} \text{ and } \frac{n}{n} \binom{n}{m} = \binom{n}{m}$$

and hence is itself an integer.

B-3 Put  $f_k(t) = \frac{df^k}{dt^k}$ . Recall Rolle's theorem: if  $f(t)$  is differentiable, then between any two zeroes of  $f(t)$  there exists a zero of  $f'(t)$ . This also applies when the zeroes are not all distinct: if  $f$  has a zero of multiplicity  $m$  at  $t = x$ , then  $f'$  has a zero of multiplicity at least  $m - 1$  there.

Therefore, if  $0 \leq a_0 \leq a_1 \leq \dots \leq a_r < 1$  are the roots of  $f_k$  in  $[0, 1]$ , then  $f_{k+1}$  has a root in each of the intervals  $(a_0, a_1), (a_1, a_2), \dots, (a_{r-1}, a_r)$ , so long as we adopt the convention that the empty interval  $(t, t)$  actually contains the point  $t$  itself. There is also a root in the "wraparound" interval  $(a_r, a_0)$ . Thus  $N_{k+1} \geq N_k$ .

Next, note that if we set  $z = e^{2\pi it}$ , then

$$f_{4k}(t) = \frac{1}{2i} \sum_{j=1}^N j^{4k} a_j (z^j - z^{-j})$$

is equal to  $z^{-N}$  times a polynomial of degree  $2N$ . Hence as a function of  $z$ , it has at most  $2N$  roots; therefore  $f_k(t)$  has at most  $2N$  roots in  $[0, 1]$ . That is,  $N_k \leq 2N$  for all  $N$ .

To establish that  $N_k \rightarrow 2N$ , we make precise the observation that

$$f_k(t) = \sum_{j=1}^N j^{4k} a_j \sin(2\pi jt)$$

is dominated by the term with  $j = N$ . At the points  $t = (2i+1)/(2N)$  for  $i = 0, 1, \dots, N-1$ , we have  $N^{4k} a_N \sin(2\pi Nt) = \pm N^{4k} a_N$ . If  $k$  is chosen large enough so that

$$|a_N| N^{4k} > |a_1| 1^{4k} + \dots + |a_{N-1}| (N-1)^{4k},$$

then  $f_k((2i+1)/2N)$  has the same sign as  $a_N \sin(2\pi Nt)$ , which is to say, the sequence  $f_k(1/2N), f_k(3/2N), \dots$  alternates in sign. Thus between these points (again including the "wraparound" interval) we find  $2N$  sign changes of  $f_k$ . Therefore  $\lim_{k \rightarrow \infty} N_k = 2N$ .

B-4 For  $t$  real and not a multiple of  $\pi$ , write  $g(t) = \frac{f(\cos t)}{\sin t}$ . Then  $g(t + \pi) = g(t)$ ; furthermore, the given equation implies that

$$g(2t) = \frac{f(2 \cos^2 t - 1)}{\sin(2t)} = \frac{2(\cos t)f(\cos t)}{\sin(2t)} = g(t).$$

In particular, for any integer  $n$  and  $k$ , we have

$$g(1 + n\pi/2^k) = g(2^k + n\pi) = g(2^k) = g(1).$$

Since  $f$  is continuous,  $g$  is continuous where it is defined; but the set  $\{1 + n\pi/2^k | n, k \in \mathbb{Z}\}$  is dense in the reals, and so  $g$  must be constant on its domain. Since  $g(-t) = -g(t)$  for all  $t$ , we must have  $g(t) = 0$  when  $t$  is not a multiple of  $\pi$ . Hence  $f(x) = 0$  for  $x \in (-1, 1)$ . Finally, setting  $x = 0$  and  $x = 1$  in the given equation yields  $f(-1) = f(1) = 0$ .

B-5 We claim that all integers  $N$  of the form  $2^k$ , with  $k$  a positive integer and  $N > \max\{S_0\}$ , satisfy the desired conditions.

It follows from the definition of  $S_n$ , and induction on  $n$ , that

$$\begin{aligned}\sum_{j \in S_n} x^j &\equiv (1+x) \sum_{j \in S_{n-1}} x^j \\ &\equiv (1+x)^n \sum_{j \in S_0} x^j \pmod{2}.\end{aligned}$$

From the identity  $(x+y)^2 \equiv x^2 + y^2 \pmod{2}$  and induction on  $n$ , we have  $(x+y)^{2^n} \equiv x^{2^n} + y^{2^n} \pmod{2}$ . Hence if we choose  $N$  to be a power of 2 greater than

$\max\{S_0\}$ , then

$$\sum_{j \in S_n} \equiv (1+x^N) \sum_{j \in S_0} x^j$$

and  $S_N = S_0 \cup \{N+a : a \in S_0\}$ , as desired.

B-6 For each point  $P$  in  $B$ , let  $S_P$  be the set of points with all coordinates equal to  $\pm 1$  which differ from  $P$  in exactly one coordinate. Since there are more than  $2^{n+1}/n$  points in  $B$ , and each  $S_P$  has  $n$  elements, the cardinalities of the sets  $S_P$  add up to more than  $2^{n+1}$ , which is to say, more than twice the total number of points. By the pigeonhole principle, there must be a point in three of the sets, say  $S_P, S_Q, S_R$ . But then any two of  $P, Q, R$  differ in exactly two coordinates, so  $PQR$  is an equilateral triangle, as desired.