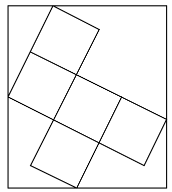
Estonian math competitions 2001/2002

We thank the IMO community for many of these problems which have been taken from various materials distributed at the recent IMO-s.

Autumn Open Contest: October 2001

Juniors (up to 10th grade)

1. A figure consisting of five equal-sized squares is placed as shown in a rectangle of size 7×8 units. Find the side length of the squares.



2. Find the remainder modulo 13 of the sum

$$1^{2001} + 2^{2001} + 3^{2001} + \ldots + 2000^{2001} + 2001^{2001}.$$

3. Find all triples (x, y, z) of real numbers satisfying the system of equations (where $[r]$ and $\{r\}$ denote the integer and fractional part of r, respectively):

$$\begin{cases} x + [y] + \{z\} = 200{,}2 \\ \{x\} + y + [z] = 200{,}1 \\ [x] + \{y\} + z = 200{,}0 \end{cases} .$$

4. Consider a point M inside triangle ABC such that triangles ABM, BCM and CAM have equal areas. Prove that M is the intersection point of the medians of triangle ABC.

5. For any integer $n \geqslant 1$ consider all squares with vertices in points having non-negative integer coordinates not greater than n.
   a) How many such squares are there for $n = 4$?
   b) Find a general formula for the number R_n of such squares for any n.

Seniors (grades 11 and 12)

1. The sum of two distinct positive integers, obtainable from each other by rearrangement of digits, consists of 2001 equal digits. Find all possible values of the digits of the sum.

2. The side lengths of a triangle and the diameter of its incircle, taken in some order, form an arithmetic progression. Prove that the triangle is right-angled.

3. For any positive integer n, denote by $S(n)$ the sum of its positive divisors (including 1 and n).
   a) Prove that $S(6n) \leqslant 12S(n)$ for any n.
   b) For which n does the equality $S(6n) = 12S(n)$ hold?

4. In a triangle ABC we have $\angle B = 2 \cdot \angle C$ and the angle bisector drawn from A intersects BC in a point D such that $|AB| = |CD|$. Find $\angle A$.

5. Let $b_1, b_2, \ldots, b_n$ be a rearrangement of positive real numbers $a_1, a_2, \ldots, a_n$. Prove that
   a) $\left(a_1 + \frac{1}{b_1}\right) \cdot \left(a_2 + \frac{1}{b_2}\right) \cdot \ldots \cdot \left(a_n + \frac{1}{b_n}\right) \geqslant 2^n$;
   b) if equality holds here for an odd n then at least one of the numbers a_i is 1.

Solutions of Autumn Open Contest

J1. *Answer:* $\sqrt{5}$.

Let a be the required side length, then the projections of each side of any square to the sides of the rectangle are x and y where $x^2 + y^2 = a^2$. We have

$$8 = 2x + y + x + y = 3x + 2y$$

and

$$7 = 3x + y,$$

yielding $y = 1$, $x = 2$ and $a = \sqrt{x^2 + y^2} = \sqrt{5}$.

J2. *Answer:* 0.

Arrange all terms of the sum except 1001^{2001} (which is divisible by 13) in pairs $(k^{2001}, (2002-k)^{2001})$. Since 2002 is divisible by 13, $k^{2001} + (2002-k)^{2001}$ is congruent to $k^{2001} + (-k)^{2001} = 0$ modulo 13, i.e. the sum of each pair is divisible by 13 and hence the required remainder is 0.

J3. *Answer:* the only such triple is $x = 100{,}15$; $y = 100{,}95$; $z = 99{,}05$.

Adding all three equations and using $[r] + \{r\} = r$ we have $x + y + z = 300{,}15$. Subtracting from here the first given equation, we obtain $(y - [y]) + (z - \{z\}) = 99{,}95$ or $\{y\} + [z] = 99{,}95$, yielding $\{y\} = 0{,}95$ and $[z] = 99$. Similarly we get $[x] + \{z\} = 100{,}05$ and $\{x\} + [y] = 100{,}15$, i.e. $[x] = 100$, $\{z\} = 0{,}05$, $\{x\} = 0{,}15$ and $[y] = 100$.

J4. It suffices to prove that if triangles ABM and BCM have equal areas then M lies on the median drawn from B. Let K and H be the perpendicular projections of A and C to BM, and let BM intersect AC in a point P (see Figure 1). Then

$$\frac{|BM|\cdot|AK|}{2} = S_{ABM} = S_{BCM} = \frac{|BM|\cdot|CH|}{2}$$

and hence $|AK| = |CH|$. If AC is perpendicular to BM, then $K = H = P$, $|AP| = |PC|$ and BP is a median. If AC is not perpendicular to BM, then clearly one of K and H lies inside triangle ABC and the other one outside of it. Hence $\angle AKP = 90^\circ = \angle CHP$ and $\angle APK = \angle CPH$, i.e. triangles AKP and CHP are congruent, which again yields $|AP| = |PC|$ and BP being a median.

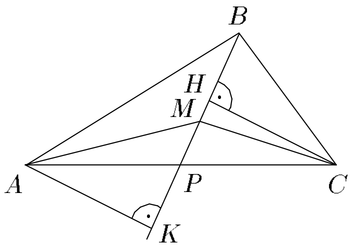


Figure 1

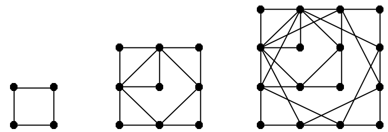

$n = 1$ $n = 2$ $n = 3$

Figure 2

J5. *Answer:* a) 50; b) $1 \cdot n^2 + 2 \cdot (n-1)^2 + \ldots + (n-1) \cdot 2^2 + n \cdot 1^2$.
We first study the possible squares for $n = 1, 2, 3$.
For $n = 1$ we have a single square of side length 1.
For $n = 2$ we have $2 \cdot 2$ possible locations for the square of side length 1 and new possible squares of side lengths 2 and $\sqrt{2}$, one of each.
For $n = 3$, we have $3 \cdot 3$ possible locations for the square of side length 1, $2 \cdot 2$ possible locations for each of the squares of side lengths 2 and $\sqrt{2}$ and three new types of squares, one of each (see Figure 2).

We see that for each n we have $1 = 1^2$ possible location for each of the "new" squares (i.e. squares having all their vertices at the edges of the grid) and for the next values of n we have $2^2, 3^2, 4^2, \ldots$ possible locations for these squares. It remains to notice that the number of the "new" squares is n since we can place one of its vertices either in a corner of the grid or in one of the $n-1$ points on the side of the grid, thereby determining the locations of the other three vertices. Hence for any n we have

$$R_n = 1 \cdot n^2 + 2 \cdot (n-1)^2 + 3 \cdot (n-2)^2 + \ldots + (n-1) \cdot 2^2 + n \cdot 1^2 ,$$

yielding $R_4 = 50$.

Note. Using the identities $R_n - R_{n-1} = n^2 + (n-1)^2 + \ldots + 1^2$ and

$$1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

we can show by induction that

$$R_n = \frac{(n+1)^2 \cdot \left((n+1)^2 - 1\right)}{12} .$$

S1. *Answer:* 1, 2, 3, 4, 5, 6, 7, 8.
Let A be any of the digits 0, 1, 2, 3. Taking

$$n = \overline{A\underbrace{(A{+}5)\ldots(A{+}5)}_{1997\text{ digits}}(A{+}5)A(A{+}6)}, \quad m = \overline{A\underbrace{(A{+}5)\ldots(A{+}5)}_{1997\text{ digits}}(A{+}6)A(A{+}5)},$$

or

$$n = \overline{\underbrace{(A{+}1)\ldots(A{+}1)}_{1999\text{ digits}}A(A{+}2)}, \quad m = \overline{\underbrace{(A{+}1)\ldots(A{+}1)}_{1999\text{ digits}}(A{+}2)A},$$

we have n and m obtainable from each other by rearrangement of digits and

$$n + m = \overline{\underbrace{(2A{+}1)\ldots(2A{+}1)}_{2001\text{ digits}}}, \quad n + m = \overline{\underbrace{(2A{+}2)\ldots(2A{+}2)}_{2001\text{ digits}}},$$

respectively. Hence 1 to 8 are possible digits.
Suppose now that $m + n = \underbrace{99\ldots9}_{2001\text{ digits}}$. Moving from right to left it is easy to see that there can be no carries during the addition. Hence any digit A occurs in n in these and only these positions where the digit $9 - A$ occurs in m. Since n and m are obtainable from each other by rearrangement of digits then any digit A occurs in n the same number of times as $9 - A \neq A$, and the number of digits in n, m and also $n + m$ must be even — a contradiction.

S2. Drawing a line through the incenter of a triangle parallel to each of its sides it is easy to see that the diameter of the incircle is shorter than any of the sides. Let x and $d > 0$ be the diameter of the incircle and the difference of the arithmetic progression, then the side lengths are $x + d$, $x + 2d$ and $x + 3d$. Finding the area S of the triangle in two ways we get

$$p \cdot \frac{x}{2} = S = \sqrt{p \cdot (p - (x+d)) \cdot (p - (x+2d)) \cdot (p - (x+3d))} .$$

Since $p = \dfrac{3(x+2d)}{2}$ we have $\dfrac{3(x+2d)x}{4} = \sqrt{\dfrac{3(x+2d)(x+4d)(x+2d)x}{16}}$, yielding $3x = x + 4d$ and $x = 2d$. Hence the side lengths are $x + d = 3d$, $x + 2d = 4d$

and $x+3d=5d$, i.e. the triangle is right-angled.

S3. *Answer:* b) for n not divisible by 2 or 3.

Considering the representations of n and $6n$ as products of primes we see that any positive divisor of $6n$ is the product of a positive divisor of 6 and a positive divisor of n. Hence the positive divisors of $6n$ are numbers of the form d, $2d$, $3d$ and $6d$ where d is a positive divisor of n, and

$$S(6n) \leqslant S(n)+2S(n)+3S(n)+6S(n)=12S(n) .$$

Here equality holds if and only if the abovementioned four series of divisors do not intersect, i.e. no divisor d of n can be represented as $2d'$ or $3d'$ where d' is another divisor of n, or equivalently n is not divisible by 2 or 3.

S4. *Answer:* 72°.

Let $|AB|=|CD|=a$, $\angle C=\alpha$ and $\angle A=2\beta$, then $\angle CAD=\angle BAD=\beta$, $\angle B=2\alpha$ and $\angle BDA=\alpha+\beta$. Applying the sine rule in triangles ACD and ABD we have

$$\frac{\sin\alpha}{\sin\beta}=\frac{|AD|}{a}=\frac{\sin 2\alpha}{\sin(\alpha+\beta)} ,$$

yielding $2\sin\beta\cos\alpha=\sin(\alpha+\beta)$, or $\tan\alpha=\tan\beta$. Since $0<\alpha,\beta<90^\circ$ we have $\alpha=\beta$. Now from $180^\circ=2\beta+2\alpha+\alpha=5\beta$ we have $\beta=36^\circ$ and $\angle A=2\beta=72^\circ$.

S5. a) Using the AM-GM inequality we have:

$$\left(a_1+\frac{1}{b_1}\right)\cdot\ldots\cdot\left(a_n+\frac{1}{b_n}\right)\geqslant 2\cdot\sqrt{\frac{a_1}{b_1}}\cdot\ldots\cdot 2\cdot\sqrt{\frac{a_n}{b_n}}=2^n\cdot\sqrt{\frac{a_1\cdot\ldots\cdot a_n}{b_1\cdot\ldots\cdot b_n}}=2^n .$$

b) If $n=1$ then $a_1+\frac{1}{a_1}=2$ and hence $a_1=1$. Suppose now that the claim is true for any odd integers less than n. The equality holds if and only if $a_i+\frac{1}{b_i}=2\sqrt{\frac{a_i}{b_i}}$ for each i, i.e. $a_i=\frac{1}{b_i}$. If $a_i=b_i$ for some i then $a_i=b_i=1$ and we are done. If $a_i\neq b_i$ for all i then consider some i_1 such that $a_{i_1}\neq 1$. Then b_{i_1} equals to some a_{i_2} where $i_2\neq i_1$, and $a_{i_2}=b_{i_1}=\frac{1}{a_{i_1}}$. Also, b_{i_2} equals to some a_{i_3} where $i_3\neq i_2$. Hence $a_{i_3}=b_{i_2}=\frac{1}{a_{i_2}}=a_{i_1}$.

If $i_3=i_1$ then

$$\left(a_{i_1}+\frac{1}{b_{i_1}}\right)\cdot\left(a_{i_2}+\frac{1}{b_{i_2}}\right)=\left(a_{i_1}+\frac{1}{a_{i_2}}\right)\cdot\left(a_{i_2}+\frac{1}{a_{i_3}}\right)=\left(a_{i_1}+\frac{1}{a_{i_2}}\right)\cdot\left(a_{i_2}+\frac{1}{a_{i_1}}\right)=$$
$$=(a_{i_1}+a_{i_1})\cdot\left(\frac{1}{a_{i_1}}+\frac{1}{a_{i_1}}\right)=2a_{i_1}\cdot\frac{2}{a_{i_1}}=2^2 .$$

Since $b_{i_1}=a_{i_2}$ and $b_{i_2}=a_{i_1}$, we can omit a_{i_1} and a_{i_2} and use the induction hypothesis.

If $i_3\neq i_1$, we find $a_{i_4}=b_{i_3}=\frac{1}{a_{i_3}}=\frac{1}{a_{i_1}}$, $a_{i_5}=b_{i_4}=\frac{1}{a_{i_4}}=a_{i_1}$ etc. Sooner or later we must have $i_{k+1}=i_1$ for some *even* k (since $a_{i_1}\neq 1$). Similarly to the previous case we can now omit $a_{i_1}, a_{i_2}, \ldots, a_{i_k}$ and use the induction hypothesis.

Spring Open Contest: February 2002

Juniors (up to 10th grade)

1. Is it possible to arrange the integers 1 to 16
   a) on a straight line;
   b) on a circle
   so that the sum of any two adjacent numbers is the square of an integer?

2. Does there exist a rectangle with integer side lengths with the square of its diagonal equal to 2002?

3. In a triangle ABC we have $|AB|=|AC|$ and $\angle BAC=\alpha$. Let $P\neq B$ be a point on AB and Q a point on the altitude drawn from A such that $|PQ|=|QC|$. Find $\angle QPC$.

4. Define $a_1, a_2, \ldots, a_n, \ldots$ as follows:

   $$a_1=0,\quad a_2=1,\quad a_n=5a_{n-1}-a_{n-2},\ \text{for } n>2.$$

   For which n is a_n divisible by: a) 5; b) 15?

5. For which positive integers n is it possible to write n real numbers, not all equal to 0, on a circle so that each of these numbers is equal to the absolute value of the difference of its two neighbouring numbers?

Seniors (11th and 12th grade)

1. The sides a, b and c of a right triangle form a geometric progression, and $abc=1$. Find a, b, c.

2. Let a, b be any real numbers such that $|a|\neq|b|$. Prove that

   $$\left|\frac{a+b}{a-b}\right|^{ab}\geqslant 1 .$$

3. Let $ABCD$ be a rhombus with $\angle DAB = 60^\circ$. Let K, L be points on its sides AD and DC and M a point on the diagonal AC such that $KDLM$ is a parallelogram. Prove that triangle BKL is equilateral.

4. Call a 10-digit natural number *magic* if it consists of 10 distinct digits and is divisible by 99999. How many such magic numbers are there (not starting with digit 0)?

5. Find the maximum number of distinct four-digit positive integers consisting only of digits 1, 2 and 3 such that any two of these numbers have equal digits in at most one position?

Solutions of Spring Open Contest

J1. *Answer:* a) yes; b) no.

Since $4^2 = 16 < 16+1$ and $6^2 = 36 > 16+15$ then only 9 can be adjacent to 16 (giving $16+9 = 25 = 5^2$). Hence it is impossible to arrange the numbers 1 to 16 on a circle in the required manner. A suitable arrangement on a straight line is:

$$16,\ 9,\ 7,\ 2,\ 14,\ 11,\ 5,\ 4,\ 12,\ 13,\ 3,\ 6,\ 10,\ 15,\ 1,\ 8\ .$$

J2. *Answer:* no.

We need to determine whether there exist positive integers a and b such that $a^2+b^2 = 2002$. Note that 2002 is divisible by 7 and the square of any integer is congruent to 0, 1, 2 or 4 modulo 7. Hence a and b must both be divisible by 7, but then a^2+b^2 is divisible by 49, and 2002 is not divisible by 49.

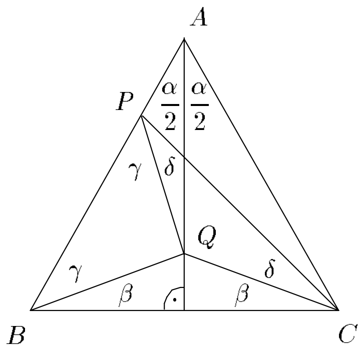


Figure 3

J3. *Answer:* $\angle QPC = \dfrac{\alpha}{2}$.

Since $|AB| = |AC|$ then the altitude drawn from A is also an angle bisector. Note that $|QB| = |QC| = |PQ|$ (see Figure 3), i.e. the triangles BQC, BQP and PQC are isosceles. Denote $\angle QBC = \angle QCB = \beta$, $\angle QBP = \angle QPB = \gamma$ and $\angle QPC = \angle QCP = \delta$, then $\angle QCA = \gamma$. From triangle ABC we now have $\alpha + 2\beta + 2\gamma = 180^\circ$ and from triangle PBC we have $2\delta + 2\beta + 2\gamma = 180^\circ$. Hence $\angle QPC = \delta = \dfrac{\alpha}{2}$.

J4. *Answer:* a) for any odd n; b) for $n = 6k+1$.

a) From the equality $a_n = 5a_{n-1} - a_{n-2}$ we see that a_n is divisible by 5 if and only if a_{n-2} is divisible by 5. Since $a_1 = 0$ is divisible by 5 but $a_2 = 1$ is not divisible by 5, then a_n is divisible by 5 if and only if n is odd.

b) Taking $n+1$ instead of n in the given equality we have

$$a_{n+1} = 5a_n - a_{n-1} = 5\cdot(5a_{n-1} - a_{n-2}) - a_{n-1} = 24a_{n-1} - 5a_{n-2}\ .$$

From here we see that a_{n+1} is divisible by 3 if and only if a_{n-2} is divisible by 3. Since $a_1 = 0$ is divisible by 3 but $a_2 = 1$ and $a_3 = 5\cdot 1 - 0 = 5$ are not divisible by 3, then a_n is divisible by 3 if and only if $n = 3m+1$ for some m. Hence a_n is divisible by 15 if and only if n is both odd and of the form $n = 3m+1$, i.e. $n = 6k+1$.

J5. *Answer:* if and only if n is divisible by 3.

Since each number written on the circle is equal to the absolute value of the difference of two others, then all these numbers are non-negative. Let a be maximal among these numbers (then $a > 0$) and let b and c be the numbers adjacent to it, with $b \geqslant c \geqslant 0$. Since $b \leqslant a$ then also $b - c \leqslant a$, and the equality $a = b - c$ holds only if $b = a$ and $c = 0$. Hence any number a on the circle must have a and 0 as its neighbours, and any number 0 must have its both neighbours equal. We see that the numbers on the circle must be a, a, 0, a, a, 0, ... (see Figure 4) and hence n must be a multiple of 3.

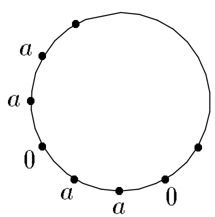


Figure 4

On the other hand it is easy to check that for any $n = 3k$ and $a > 0$ the numbers $\underbrace{a,\ a,\ 0,\ a,\ a,\ 0,\ \ldots,\ a,\ a,\ 0}_{k \text{ triples } a,a,0}$ satisfy the required conditions.

S1. *Answer:* $\sqrt{\frac{\sqrt{5}-1}{2}}$, 1 and $\sqrt{\frac{\sqrt{5}+1}{2}}$.

Assume w.l.o.g. that $a < b < c$, then $a = \frac{b}{q}$ and $c = bq$ for some $q > 1$. Hence from $abc = 1$ we have $b^3 = 1$ and $b = 1$. From the Pythagorean Theorem we now have $\left(\frac{1}{q}\right)^2 + 1 = q^2$, or $q^4 - q^2 - 1 = 0$. Since the equation $x^2 - x - 1$ has $\frac{\sqrt{5}+1}{2}$ as its only positive solution, then $q = \sqrt{\frac{\sqrt{5}+1}{2}}$ and $\frac{1}{q} = \sqrt{\frac{\sqrt{5}-1}{2}}$.

S2. For any real y and $x > 0$ we have:

a) $x^y > 1$, if $x > 1$ and $y > 0$ or $x < 1$ and $y < 0$;
b) $x^y < 1$, if $x > 1$ and $y < 0$ or $x < 1$ and $y > 0$;
c) $x^y = 1$, if $x = 1$ or $y = 0$.

We shall now consider the cases where ab is positive, negative or equal to zero.

If $ab > 0$, then a and b have the same sign and $|a+b| > |a-b| > 0$, hence $\left|\frac{a+b}{a-b}\right| > 1$ and $\left|\frac{a+b}{a-b}\right|^{ab} > 1$.

If $ab < 0$, then a and b have opposite signs and $|a-b| > |a+b| > 0$, hence $0 < \left|\frac{a+b}{a-b}\right| < 1$ and $\left|\frac{a+b}{a-b}\right|^{ab} > 1$.

If $ab = 0$, then $\left|\frac{a+b}{a-b}\right|^{ab} = 1$ since $\left|\frac{a+b}{a-b}\right| \neq 0$.

S3. The rhombus $ABCD$ consists of two equilateral triangles ABD and BCD. We shall prove that $|KD| = |LC|$ (see Figure 5) — then triangles KBD and LBC are congruent and $|KB| = |LB|$, $\angle KBD = \angle LBC$. Hence $\angle KBL = \angle DBC = 60°$, i.e. the triangle BKL is equilateral.

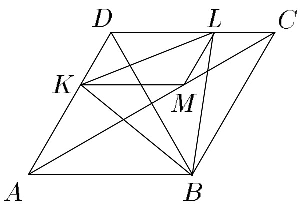


Figure 5

To prove the equality $|KD| = |LC|$ note that LM is parallel to AD and $\angle LMC = \angle DAC = \angle DCA = \angle LCM$. Hence the triangle MLC is isosceles, i.e. $|LC| = |LM| = |KD|$.

S4. *Answer:* 3456.

We can write any ten-digit number $\overline{abcdefghij}$ as

$$\begin{aligned}\overline{abcdefghij} &= 100000 \cdot \overline{abcde} + \overline{fghij} = \\ &= 99999 \cdot \overline{abcde} + \overline{abcde} + \overline{fghij}\,.\end{aligned}$$

Hence $\overline{abcdefghij}$ is divisible by 99999 if and only if the sum $\overline{abcde} + \overline{fghij}$ is divisible by 99999. Since each summand here is positive and less than 99999, we must have $\overline{abcde} + \overline{fghij} = 99999$, or equivalently

$$a + f = b + g = c + h = d + i = e + j = 9$$

(since the sum contains only digits 9, no carries can occur on addition). We see that magic numbers are in one-to-one correspondence with numbers of the form $\overline{abcde}$ where a, b, c, d, e are five distinct digits such that $a \neq 0$ and the sum of no two of them is 9. There are $9 \cdot 8 \cdot 6 \cdot 4 \cdot 2 = 3456$ such numbers $\overline{abcde}$.

S5. *Answer:* 9.

Note that we cannot have more than $3 \cdot 3 = 9$ integers with the required property since the pairs of first two digits of any two of them must be distinct. A suitable set of 9 integers is 1111, 1222, 1333, 2123, 2231, 2312, 3132, 3213, 3321.

Final Round of National Olympiad: March 2002

9th grade

1. Points K and L are taken on the sides BC and CD of a square $ABCD$ so that $\angle AKB = \angle AKL$. Find $\angle KAL$.

2. Do there exist distinct non-zero digits a, b and c such that the two-digit number $\overline{ab}$ is divisible by c, the number $\overline{bc}$ is divisible by a and $\overline{ca}$ is divisible by b?

3. Let $a_1, a_2, \ldots, a_n$ be pairwise distinct real numbers and m be the number of distinct sums $a_i + a_j$ (where $i \neq j$). Find the least possible value of m.

4. Mary writes 5 numbers on the blackboard. On each step John replaces one of the numbers on the blackboard by the number $x + y - z$, where x, y and z are three of the four other numbers on the blackboard. Can John make all five numbers on the blackboard equal, regardless of the numbers initially written by Mary?

5. There were $n > 1$ aborigines living on an island, each of them telling only the truth or only lying, and each having at least one friend among the others. The

new governor asked each aborigine whether there are more truthful aborigines or liars among his friends, or an equal number of both. Each aborigine answered that there are more liars than truthful aborigines among his friends. The governor then ordered one of the aborigines to be executed for being a liar and asked each of the remaining $n-1$ aborigines the same question again. This time each aborigine answered that there are more truthful aborigines than liars among his friends.

Determine whether the executed aborigine was truthful or a liar, and whether there are more truthful aborigines or liars remaining on the island.

10th grade

1. The greatest common divisor d and the least common multiple v of positive integers m and n satisfy the equality $3m+n=3v+d$. Prove that m is divisible by n.

2. Let ABC be a non-right triangle with its altitudes intersecting in point H. Prove that ABH is an acute triangle if and only if $\angle ACB$ is obtuse.

3. John takes seven positive integers $a_1, a_2, \ldots, a_7$ and writes the numbers $a_i a_j$, $a_i + a_j$ and $|a_i - a_j|$ for all $i \neq j$ on the blackboard. Find the greatest possible number of distinct odd integers on the blackboard.

4. Find the maximum length of a broken line on the surface of a unit cube, such that its links are the cube's edges and diagonals of faces, the line does not intersect itself and passes no more than once through any vertex of the cube, and its endpoints are in two opposite vertices of the cube.

5. The teacher writes numbers 1 at both ends of the blackboard. The first student adds a 2 in the middle between them; each next student adds the sum of each two adjacent numbers already on the blackboard between them (hence there are numbers 1, 3, 2, 3, 1 on the blackboard after the second student; 1, 4, 3, 5, 2, 5, 3, 4, 1 after the third student etc.) Find the sum of all numbers on the blackboard after the n-th student.

11th grade

1. Determine all real numbers a such that the equation $x^8 + ax^4 + 1 = 0$ has four real roots forming an arithmetic progression.

2. Inside an equilateral triangle there is a point such that the distances from it to the sides of the triangle are 3, 4 and 5. Find the area of the triangle.

3. The teacher writes a 2002-digit number consisting only of digits 9 on the blackboard. The first student factors this number as ab with $a>1$ and $b>1$ and replaces it on the blackboard by two numbers a' and b' such that $|a-a'|=2$ and $|b-b'|=2$. The second student chooses one of the numbers on the blackboard, factors it as cd with $c>1$ and $d>1$ and replaces the chosen number by two numbers c' and d' such that $|c-c'|=2$ and $|d-d'|=2$. The third student again chooses one of the numbers on the blackboard and replaces it by two numbers following a similar procedure, etc. Is it possible that after a certain number of students have been to the blackboard all numbers written there are equal to 9?

4. Let a_1, a_2, a_3, a_4, a_5 be real numbers such that at least N of the sums $a_i + a_j$, where $i<j$, are integers. Find the greatest value of N for which it is possible that not all of the sums $a_i + a_j$ are integers.

5. John built a robot that moves along the border of a regular octagon, passing each side of the octagon in exactly 1 minute. The robot begins its movement in some vertex A of the octagon, and further on reaching each vertex can either continue movement in the same direction, or turn around and continue in the opposite direction. In how many different ways can the robot move so that after n minutes it will be in the vertex B opposite to A?

12th grade

1. Peter, John, Kate and Mary are standing at the entrance of a dark tunnel. They have one torch and none of them dares to be in the tunnel without it, but the tunnel is so narrow that at most two people can move together. It takes 1 minute for Peter, 2 minutes for John, 5 minutes for Kate and 10 minutes for Mary to pass the tunnel. Find the minimum time in which they can all get through the tunnel.

2. Does there exist an integer containing only digits 2 and 0, which is a k-th power of a positive integer with $k \geqslant 2$?

3. Prove that for positive real numbers a, b and c the inequality
$$2(a^4+b^4+c^4) < (a^2+b^2+c^2)^2$$
holds if and only if there exists a triangle with side lengths a, b and c.

4. All vertices of a convex quadrilateral $ABCD$ lie on a circle ω. The rays AD, BC intersect in point K and the rays AB, DC intersect in point L. Prove that the circumcircle of triangle AKL is tangent to ω if and only if the circumcircle of triangle CKL is tangent to ω.

5. There is a lottery at John's birthday party with a certain number of identical prizes, whereas each of the guests can win at most one prize. It is known that if there was one prize less than there actually is, then the number of possible distributions of the prizes among the guests would be 50% less than it actually is, while if there was one prize more than there actually is, then the number of possible distributions of the prizes would be 50% more than it actually is. Find the number of possible distributions of the prizes.

Solutions of Final Round

9-1. *Answer:* 45°.

Let AM be the perpendicular drawn from A to KL (see Figure 6). Since ABK and AMK are congruent right triangles then $|AM| = |AB| = |AD|$, i.e. AML and ADL are also congruent right triangles, and

$$\angle KAL = \angle KAM + \angle LAM = \angle KAB + \angle LAD\,,$$

whence

$$2\angle KAL = \angle KAM + \angle LAM + \angle KAB + \angle LAD = 90^\circ$$

and $\angle KAL = 45^\circ$.

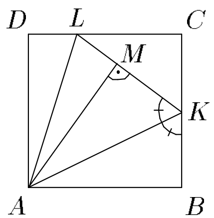


Figure 6

9-2. *Answer:* no.

Note that if a, b and c satisfy the required conditions and one of them is even, then all three are even. Then $\frac{a}{2}$, $\frac{b}{2}$ and $\frac{c}{2}$ also satisfy the required conditions. Hence we can assume w.l.o.g. that a, b and c are all odd. Also note that none of these numbers can be 5, since then the other two should also be 5. Hence it suffices to consider 1, 3, 7 and 9 and one of a, b and c must be 3 or 9 — let this be a. Then $\overline{bc}$ is divisible by 3, which gives $\{b, c\} = \{3, 9\}$, a contradiction.

9-3. *Answer:* $2n - 3$.

We can assume w.l.o.g. that $a_1 < a_2 < \ldots < a_n$. Then

$$a_1 + a_2 < a_1 + a_3 < \ldots < a_1 + a_n < a_2 + a_n < \ldots < a_{n-1} + a_n\,,$$

i.e. there are at least $2n-3$ distinct sums. Taking $a_i = i$ we have $1 + 2 = 3$ as the minimal sum and $(n-1) + n = 2n - 1$ as the maximal sum, so there are exactly $2n - 3$ distinct sums in this case.

9-4. *Answer:* yes.

Denote the numbers written by Mary by a, b, c, d and e (not necessarily distinct). First John can replace each of a and b by $x = c + d - e$. Then he can replace each of c and d by $e + x - x = e$ and finally replace both numbers x by $e + e - e = e$:

$$(a, b, c, d, e) \to (x, x, c, d, e) \to (x, x, e, e, e) \to (e, e, e, e, e)\,.$$

9-5. *Answer:* the executed aborigine was truthful and after the execution only liars remained on the island.

First note that there was a truthful aborigine on the island before the execution, since otherwise all friends of each aborigine would have been liars, and hence their answers would have been true — a contradiction.

Suppose now there was a truthful aborigine on the island after the execution. Then both his answers must have been true — but this is impossible since the execution of one aborigine could not change the difference of the numbers of liars and truthful aborigines among his friends from positive to negative.

10-1. Let $m = dm'$ and $n = dn'$ where $\gcd(m', n') = 1$. Then $v = m'n'd$ and we have $3m'd + n'd = 3m'n'd + d$, yielding $3m' + n' = 3m'n' + 1$ and $(3m' - 1)(n' - 1) = 0$. Since $3m' - 1 \neq 0$, then $n' - 1 = 0$ and hence $n = d$ divides m.

10-2. If H is the orthocenter of triangle ABC, then C is the orthocenter of triangle ABH. We'll consider three possible cases.

(1) If ABC is an acute triangle (see Figure 7), then H lies inside triangle ABC and C lies outside triangle ABH, whence ABH is an obtuse triangle.

(2) If ABC is an obtuse triangle with $\angle ACB$ acute, then assume w.l.o.g. that $\angle B$ is obtuse (see Figure 8). Then C and H lie on opposite sides of AB. Hence C is outside triangle ABH, and ABH is an obtuse triangle.

(3) If $\angle ACB$ is obtuse (see Figure 9), then C lies inside triangle ABH, whence ABH is an acute triangle.

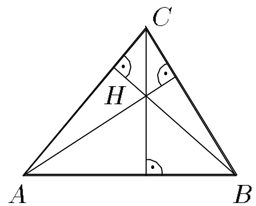


Figure 7

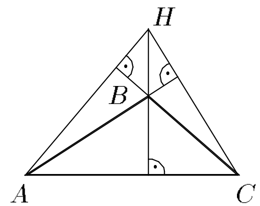


Figure 8

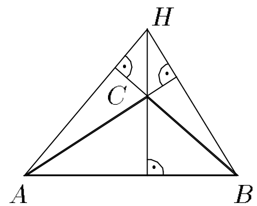


Figure 9

10-3. *Answer:* 30.

If there are m odd integers among a_i, then the maximum number of odd integers written on the blackboard is

$$f(m) = \frac{m(m-1)}{2} + 2\cdot m(7-m) = \frac{1}{2}(m^2 - m + 28m - 4m^2) =$$
$$= \frac{3}{2}m(9-m) = \frac{3}{2}\cdot\left(\frac{81}{4} - \left(\frac{9}{2} - m\right)^2\right)$$

and the maximum value of $f(m)$ is $f(4) = f(5) = 30$.

It remains to check that these numbers can all be distinct, e.g. for $a_1 = 2$, $a_2 = 4$, $a_3 = 6$ and $a_4 = 25 = 5^2$, $a_5 = 125 = 5^3$, $a_6 = 15625 = 5^6$, $a_7 = 9765625 = 5^{10}$.

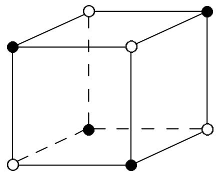


Figure 10

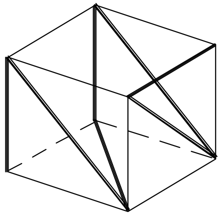


Figure 11

10-4. *Answer:* $3 + 4\sqrt{2}$.

The links of such a broken line are the edges of the cube (of length 1) and the diagonals of its faces (of length $\sqrt{2}$). Since the line passes each vertex at most once, it can have at most 7 links. Coloring the vertices as shown on Figure 10 we see that opposite vertices are of different colour and the endpoints of the diagonal of each face are of the same colour — hence an odd number of links have to be edges of the cube. Also, it is clear from this colouring that no more than three consecutive links can be diagonals (since the cube has only 4 vertices of each colour). It is now easy to check that a broken line with 1 edge and 6 diagonals is impossible, hence its length cannot exceed $3 + 4\sqrt{2}$; a suitable broken line of this length is shown on Figure 11.

10-5. *Answer:* $3^n + 1$.

Let S_n be the sum of the numbers on the blackboard after the n-th student. We shall prove by induction that $S_n = 3^n + 1$. Indeed, $S_0 = 2 = 3^0 + 1$ and each number that is on the blackboard after the k-th student is counted in exactly two of the sums written by the $(k+1)$-th student, with the exception of the two 1-s written by the teacher which are counted only once — hence

$$S_{k+1} = S_k + 2S_k - 2 = 3(3^k + 1) - 2 = 3^{k+1} + 1\,.$$

11-1. *Answer:* $a = -\dfrac{82}{9}$.

Substituting $t = x^4$ we get a quadratic equation in t, and the equation $x^4 = t_0$ has at most two roots which in this case have equal absolute values. Hence the four roots of the equation $x^8 + ax^4 + 1 = 0$ have to be of the form $\pm x_0$ and $\pm x_1$. Assuming w.l.o.g. that $x_1 > x_0$ we see that these roots form an arithmetic progression if and only if $x_1 = 3x_0$. Since the roots of the equation $t^2 + at + 1$ are then x_0^4 and $81x_0^4$, we have $81x_0^8 = 1$, yielding $x_0^4 = \dfrac{1}{9}$ and $a = -82x_0^4 = -\dfrac{82}{9}$.

11-2. *Answer:* $\dfrac{36 + 25\sqrt{3}}{4}$.

Consider a triangle ABC with a point P inside such that $|PA| = 3$, $|PB| = 4$ and $|PC| = 5$. Rotating the triangle by $60°$ around C, we map A to B and B to some B' (see Figure 12). Then P maps to P', where $|P'B| = |PA| = 3$, $|P'B'| = |PB| = 4$ and $|P'C| = |PC| = 5$. Since CPP' is an equilateral triangle then $|PP'| = 5$. Hence $|PB|^2 + |P'B|^2 = 4^2 + 3^2 = 5^2 = |PP'|^2$, and $\angle PBP' = 90°$. From triangles APB and $BP'B'$ we have

$$\angle ABP + \angle BAP = \angle ABP + \angle B'BP' = 120° - 90° = 30°$$

and $\angle APB = 180° - 30° = 150°$. The cosine law in APB now gives

$$|AB|^2 = |AP|^2 + |BP|^2 - 2\cdot|AP|\cdot|BP|\cdot\cos\angle APB = 25 + 12\sqrt{3}$$

and the area is $S = \dfrac{\sqrt{3}}{4}\cdot|AB|^2 = \dfrac{36 + 25\sqrt{3}}{4}$.

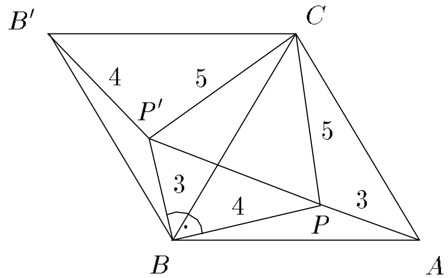


Figure 12

11-3. *Answer:* no.

The initial 2002-digit number $999\ldots9$ is congruent to 3 modulo 4. If $N = ab$ and N is congruent to 3 modulo 4 then one of a and b is congruent to 3 and the other is congruent to 1 modulo 4 and the same is true for a' and b'. Hence at all times there is a number on the blackboard which is congruent to 3 modulo 4, while 9 is congruent to 1 modulo 4.

11-4. *Answer:* 6.

If there are four integers and one non-integer among a_i then $N = 6$. To prove the maximality we denote the fractional part of x by $\{x\}$ and note that:

(a) if $\{a\} \neq \{b\}$ and c is any real number then at most one of $c + a$ and $c + b$ is an integer;

(b) if $a = b$ then $a + b$ is an integer if and only if $\{a\} = 0$ or $\{a\} = 0{,}5$;

(c) if $\{a\} \neq \{b\}$ and $a + b$ is an integer then neither $\{a\}$ nor $\{b\}$ is 0 or 0,5.

Considering now the possible partitions of the set $\{a_1, a_2, a_3, a_4, a_5\}$ to subsets (of elements with equal fractional parts) and keeping in mind the above remarks (a), (b) and (c) we see that the only case when there can be more than 6 integer sums $a_i + a_j$ is when $\{a_1\} = \{a_2\} = \{a_3\} = \{a_4\} = \{a_5\}$, and in this case all these sums are integers.

11-5. *Answer:* $2^{k-1}(2^{k-1} - 1)$ for $n = 2k$, and 0 for $n = 2k + 1$.

Colour the vertices of the octagon alternately black and white. Since A and B are of the same colour and in each minute the robot moves from a vertex of one colour to a vertex of the opposite colour, then n must be even.

We now label the vertices by 1 to 8 so that $A = 1$ and $B = 5$, and denote by $a^{(k)} = (a_1^{(k)}, a_2^{(k)}, \ldots, a_8^{(k)})$ the numbers of possibilities, for the robot to reach vertices $1, 2, \ldots, 8$ from $A = 1$ in k minutes. Using induction by m we shall prove that for any $m \geqslant 1$

$$a^{(2m)} = (2^{2m-2} + 2^{m-1},\ 0,\ 2^{2m-2},\ 0,\ 2^{2m-2} - 2^{m-1},\ 0,\ 2^{2m-2},\ 0)\ .$$

Obviously we have $a^{(2)} = (2, 0, 1, 0, 0, 0, 1, 0)$. Suppose now that the claim is true for $m = k$ and denote $2^{k-1} = s$, then

$$a^{(2k)} = (s^2 + s,\ 0,\ s^2,\ 0,\ s^2 - s,\ 0,\ s^2,\ 0)\ .$$

Since the robot can move to any vertex M from either of its neighbouring vertices, we have

$$a^{(2k+1)} = (0,\ 2s^2 + s,\ 0,\ 2s^2 - s,\ 0,\ 2s^2 - s,\ 0,\ 2s^2 + s)$$

and similarly

$$a^{(2k+2)} = (4s^2 + 2s,\ 0,\ 4s^2,\ 0,\ 4s^2 - 2s,\ 0,\ 4s^2,\ 0)\ .$$

Since $4s^2 = 2^{2k}$ and $2s = 2^k$, we have proved the claim for $m = k + 1$. Hence the number of possibilities to reach $B = 5$ from $A = 1$ in $n = 2k$ minutes is $2^{2k-2} - 2^{k-1} = 2^{k-1}(2^{k-1} - 1)$.

12-1. *Answer:* 17 minutes.

Note that it is possible to get everyone through the tunnel in 17 minutes:

1) Peter and John go through the tunnel (2 minutes);
2) Peter brings back the torch (1 minute);
3) Kate and Mary go through the tunnel (10 minutes);
4) John brings back the torch (2 minutes);
5) Peter and John go through the tunnel (2 minutes).

It remains to show the minimality of this total. Clearly they have to go through the tunnel an odd number of times and bring back the torch at least twice, hence they have to go through the tunnel at least 3 times in one direction and 2 times in another direction. If they go through the tunnel 7 or more times then the total time cannot be less than $10+2+5\cdot 1 = 17$ minutes. If they go through the tunnel 5 times then each pass in the "initial" direction takes at least 2 minutes and one of these (with Mary) takes 10 minutes. If Peter brings back the torch both times then Kate and Mary cannot go through the tunnel together and they need at least $10 + 5 + 2 + 2 \cdot 1 = 17$ minutes. If someone else brings back the torch at one time then they need at least $10 + 2 + 2 + 2 + 1 = 17$ minutes.

12-2. *Answer:* no.

Consider an integer N containing only digits 2 and 0 and ending in t zeroes ($t \geqslant 0$), then

$$N = \overline{2\ldots 2} \cdot 10^t = \overline{1\ldots 1} \cdot 2^{t+1} \cdot 5^t\ ,$$

where the dotted part in $2\ldots 2$ can contain both 2-s and 0-s (in $1\ldots 1$ both 1-s and 0-s respectively). Since $\overline{1\ldots 1}$ is not divisible by 2 or 5 then in the case when $N = n^k$ both $t + 1$ and t have to be multiples of k, yielding $k = 1$.

12-3. The given inequality is equivalent to

$$a^4 + b^4 + c^4 - 2a^2b^2 - 2b^2c^2 - 2c^2a^2 < 0\ .$$

Transforming the left side of this inequality we have

$$\begin{aligned} a^4 + b^4 + c^4 - 2a^2b^2 - 2b^2c^2 - 2c^2a^2 &= (a^2 + b^2 - c^2)^2 - 4a^2b^2 = \\ &= (a^2 + b^2 - c^2 - 2ab)(a^2 + b^2 - c^2 + 2ab) = \\ &= \big((a - b)^2 - c^2)\big)\big((a + b)^2 - c^2\big) = \\ &= (a - b + c)(a - b - c)(a + b + c)(a + b - c)\ . \end{aligned}$$

Hence the given inequality is equivalent to

$$(a + b + c)(a + b - c)(b + c - a)(c + a - b) > 0\ . \tag{1}$$

Here the first term is positive and at most one of the other three can be negative (e.g. $a+b-c < 0$ and $b+c-a < 0$ would give $2b < 0$ — a contradiction). Hence (1) holds if and only if a, b and c satisfy the triangle inequalities.

12-4. Let ω_1 and ω_2 be the circumcircles of triangles AKL and CKL (see Figure 13). Suppose that ω and ω_2 are tangent to each other in point C, and let l_2 be their common tangent. Then

$$\angle KLC = \angle KCl_2 = \angle BCl_2 = \angle BDC .$$

Hence $KL \parallel BD$ and $\angle ADB = \angle AKL$ — therefore then angle between AB and the tangent to ω in A is equal to the angle between AL and the tangent to ω_1 in A. Since the points A, B, L are collinear then the tangents to ω and ω_1 in A coincide, i.e. these circles are tangent to each other.

This argument can be reversed to show that ω and ω_1 being tangent to each other in A implies ω and ω_2 being tangent to each other in C.

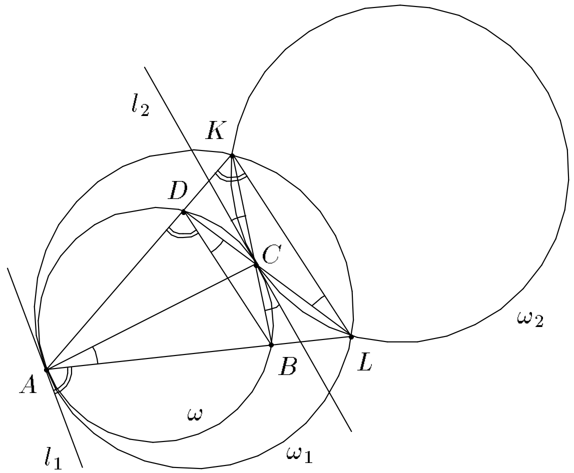


Figure 13

Alternative solution. Let ω_1 and ω_2 be the circumcircles of triangles AKL and CKL. If ω and ω_1 are tangent to each other in A then some homothety relative to A maps ω to ω_1. Since K is the intersection point of AD with ω_1 and L is the intersection point of AB with ω_1, and points B and D lie on ω, then this homothety takes D to K and B to L, whence $KL \parallel BD$. Since BK and DL intersect in C then some homothety relative to C maps B to K and D to L. This homothety then maps the circumcircle ω of triangle CDB to the circumcircle ω_2 of triangle CKL. Hence ω and ω_2 are tangent to each other in C.

Similarly we can show that ω and ω_2 being tangent to each other in C implies ω and ω_1 being tangent to each other in A.

12-5. *Answer:* 2002.

We have $\binom{n}{k}$ possible distributions of k prizes among n guests, and

$$\binom{n}{k+1} = \frac{n-k}{k+1} \cdot \binom{n}{k} . \qquad (2)$$

Let n be the number of guests and m the actual number of prizes, then we have $\binom{n}{m} = 2 \cdot \binom{n}{m-1}$ and $\binom{n}{m+1} = \frac{3}{2} \cdot \binom{n}{m}$. Substituting from (2) we have $\frac{n-m+1}{m} \cdot \binom{n}{m-1} = 2 \cdot \binom{n}{m-1}$ and $\frac{n-m}{m+1} \cdot \binom{n}{m} = \frac{3}{2} \cdot \binom{n}{m}$. Hence $n - m + 1 = 2m$, yielding $n = 3m - 1$, and $2(n - m) = 3(m + 1)$. Plugging in $n = 3m - 1$ here we have $4m - 2 = 3m + 3$, whence $m = 5$ and $n = 14$. It remains to calculate $\binom{14}{5} = 2002$.

IMO Team Selection Test: May 2002

First Day

1. The princess wishes to have a bracelet with r rubies and s emeralds arranged in such order that there exist two jewels on the bracelet such that starting with these and enumerating the jewels in the same direction she would obtain identical sequences of jewels. Prove that it is possible to fulfill the princess's wish if and only if r and s have a common divisor.

2. Consider an isosceles triangle KL_1L_2 with $|KL_1| = |KL_2|$, and let KA, L_1B_1, L_2B_2 be its angle bisectors. Prove that $\cos \angle B_1AB_2 < \frac{3}{5}$.

3. In a certain country there are 10 cities connected by a network of *one-way* nonstop flights so that it is possible to fly (using one or more flights) from any city to any other. Let n be the least number of flights needed to complete a trip starting from one of the cities, visiting all others and returning to the starting point. Find the greatest possible value of n.

Second Day

4. Let $ABCD$ be a cyclic quadrilateral such that $\angle ACB = 2\angle CAD$ and $\angle ACD = 2\angle BAC$. Prove that $|CA| = |CB| + |CD|$.

5. Let $0 < \alpha < \dfrac{\pi}{2}$ and $x_1, x_2, \ldots, x_n$ be real numbers such that

$$\sin x_1 + \sin x_2 + \ldots + \sin x_n \geqslant n \cdot \sin \alpha\ .$$

Prove that

$$\sin(x_1 - \alpha) + \sin(x_2 - \alpha) + \ldots + \sin(x_n - \alpha) \geqslant 0\ .$$

6. Place a pebble at each *non-positive* integer point on the real line, and let n be a fixed positive integer. At each step we choose some n consecutive integer points, remove one of the pebbles located at these points and rearrange all others arbitrarily within these points (placing at most one pebble at each point).

Determine whether there exists a positive integer n such that for any given $N > 0$ we can place a pebble at a point with coordinate greater than N in a finite number of steps described above.

Solutions of Selection Test

1. Note that if $\gcd(r, s) = d > 1$ then the princess's wish can be fulfilled. Let $r' = \dfrac{r}{d}$ and $s' = \dfrac{s}{d}$ — we place on the bracelet r' rubies and s' emeralds, then again r' rubies and s' emeralds, etc. (d times) — now any two jewels at distance $r' + s'$ have the required property.

Suppose now that the required ordering exists. Label the positions on the bracelet by $0, \ldots, n–1$ where $n = r+s$ (thinking of them as modulo n) and denote by $P(i)$ the jewel at position i for an ordering P. It suffices to show that $\gcd(r, n) > 1$.

Let P be the required ordering, and let a, $a+i$ (where $0 < i < n$) be the positions of the two jewels mentioned in the condition. Then $P(a + j) = P(a + i + j)$ for any $j \geqslant 0$ and hence $P(b) = P(b + i) = P(b + 2i) = \ldots$ for any position b. Let k be the smallest positive integer such that n divides ki, and let R be the set of all positions with rubies. For any b in R we have k distinct rubies at positions $b, b + i, \ldots, b + (k - 1)i$: denote the set of these rubies $O(b)$ and call the *orbit* of b. A standard argument shows that the set of all rubies on the bracelet is the disjoint union of some number of orbits, with each orbit containing k rubies. Hence k divides r and since n divides ki with $0 \leqslant i < n$ then $\gcd(r, n) > 1$.

2. Denote $\alpha = \angle L_1KL_2$, $\beta = \angle KL_1L_2 = \angle KL_2L_1$ and $\xi = \angle B_1AB_2$ (see Figure 14). Since B_1 and B_2 are symmetric relative to KA, we have $B_1B_2 \perp KA$ and $B_1B_2 \parallel L_1L_2$, i.e. triangle AB_1B_2 is isosceles and AK is its angle bisector. Since $\angle B_2B_1L_1 = \angle L_2L_1B_1 = \angle B_2L_1B_1$, then $B_2L_1B_1$ is also isosceles. Denote $s = |B_1B_2| = |B_2L_1|$ and $t = |AB_1| = |AB_2|$.

The sine rule in triangle AL_1B_2 yields

$$\frac{s}{t} = \frac{\sin \angle B_2AL_1}{\sin\beta} = \frac{\cos\xi/2}{\cos\alpha/2}$$

and $\left(\dfrac{s}{t}\right)^2 = \dfrac{1+\cos\xi}{1+\cos\alpha}$. The cosine rule in triangle AB_1B_2 yields

$$s^2 = t^2 + t^2 - 2t^2\cos\xi = 2t^2(1-\cos\xi),$$

and $\left(\dfrac{s}{t}\right)^2 = 2(1-\cos\xi)$. Hence $\dfrac{1+\cos\xi}{1+\cos\alpha} = 2(1-\cos\xi)$ and

$$\cos\xi = \frac{1+2\cos\alpha}{3+2\cos\alpha} = 1 - \frac{2}{3+2\cos\alpha} < 1 - \frac{2}{5} = \frac{3}{5}.$$

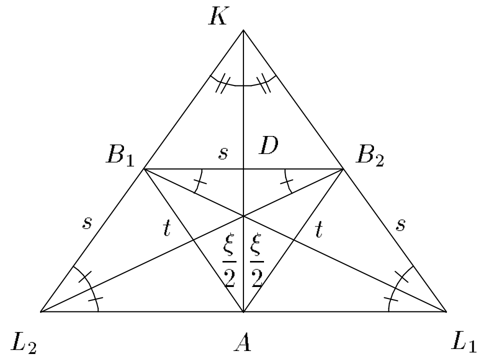


Figure 14

3. *Answer:* 30.

Let $L_1, \ldots, L_{10}$ be the cities and denote by x_{ij} the minimum number of flights required to reach L_j from L_i. Let

$$m = \max_{i \neq j} x_{ij}\ ;$$

we can assume w.l.o.g. that $i = 1$, $j = m + 1$ and the shortest path from L_1 to L_{m+1} is

$$L_1,\ L_2,\ \ldots,\ L_m,\ L_{m+1}\ .$$

We continue this path, flying from L_{m+1} to L_{m+2}, then to L_{m+3} etc. and finally from L_n back to L_1:

$$L_1,\ L_2,\ \ldots,\ L_m,\ L_{m+1} \to L_{m+2} \to \ldots \to L_n \to L_1\ .$$

Here each of the $10-(m+1)+1=10-m$ sections denoted by arrows contains at most m flights, hence the entire round-trip contains at most

$$m+m\cdot(10-m)=m\cdot(11-m)\leqslant\Big(\frac{m+11-m}{2}\Big)^2=\frac{121}{4}$$

flights, i.e. no more than 30 flights.

An example of a network requiring exactly 30 flights is shown on Figure 15.

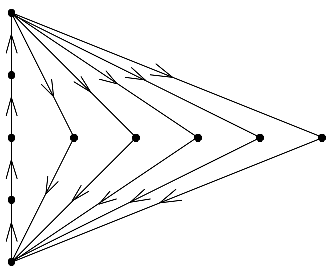


Figure 15

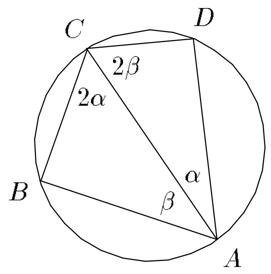


Figure 16

4. Denote $\angle CAD=\alpha$ and $\angle BAC=\beta$, then $\angle ACB=2\alpha$ and $\angle ACD=2\beta$ (see Figure 16). Since $ABCD$ is cyclic then $3\alpha+3\beta=\angle BCD+\angle BAD=180^\circ$ and $\alpha+\beta=60^\circ$. Applying the sine rule to triangles ABC and ACD gives

$$|CB|=2R\cdot\sin\beta\,,\quad |CD|=2R\cdot\sin\alpha\,,\quad |CA|=2R\cdot\sin(\alpha+2\beta)\,,$$

where R is the circumradius of $ABCD$. Hence it is sufficient to show that $\sin\alpha+\sin\beta=\sin(\alpha+2\beta)$ if $\alpha+\beta=60^\circ$. Indeed:

$$\begin{aligned}\sin\alpha+\sin\beta &= 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}=2\cdot\frac{1}{2}\cdot\cos\Big(\frac{\alpha+\beta}{2}-\beta\Big)=\\ &=\cos(30^\circ-\beta)=\sin(60^\circ+\beta)=\sin(\alpha+2\beta)\,.\end{aligned}$$

5. Suppose the claim does not hold, i.e.

$$\sin(x_1-\alpha)+\sin(x_2-\alpha)+\ldots+\sin(x_n-\alpha)<0$$

which gives

$$\cos x_1+\ldots+\cos x_n>\frac{\cos\alpha}{\sin\alpha}\cdot(\sin x_1+\ldots+\sin x_n)\geqslant n\cdot\cos\alpha$$

and hence

$$(\sin x_1+\ldots+\sin x_n)^2+(\cos x_1+\ldots+\cos x_n)^2>n^2\,.$$

On the other hand,

$$(\sin x_1+\ldots+\sin x_n)^2+(\cos x_1+\ldots+\cos x_n)^2=\sum_{i=1}^{n}\sum_{j=1}^{n}\cos(x_i-x_j)\leqslant n^2\,,$$

a contradiction.

6. *Answer:* there is no such n.

For $n=1$ and $n=2$ we cannot place a pebble in any point with a positive coordinate — hence let $n\geqslant 3$. Consider the infinite sum

$$S=a^{x_1}+a^{x_2}+a^{x_3}+\ldots$$

where x_1, x_2, x_3, ... are the coordinates of the points with pebbles at some given moment and a a positive real number to be determined later. We show that it is possible to choose a according to n so that the initial sum $S_0=a^0+a^{-1}+a^{-2}+a^{-3}+\ldots$ converges (for this it suffices to have $a>1$) and at each step the sum S can only decrease. Hence we always have $S\leqslant S_0$ and it is impossible to place a pebble at a point with an arbitrarily large positive coordinate N, since for sufficiently large N we have $a^N>S_0$.

To show this consider for $n=2k-1$ the equation

$$1+x+x^2+\ldots+x^{k-1}=x^k+\ldots+x^{2k-2} \tag{3}$$

and for $n=2k$ the equation

$$1+x+x^2+\ldots+x^{k-1}=x^{k+1}+\ldots+x^{2k-1}\,. \tag{4}$$

For $0\leqslant x\leqslant 1$ the left side exceeds the right side but for sufficiently large positive x the right side exceeds the left side. Hence the equation has a root $a>1$. It remains to show that for any integers m and t such that $1\leqslant t\leqslant\left[\frac{n+1}{2}\right]$ the sum of any $t-1$ elements of $A=\{a^m, a^{m+1}, \ldots, a^{m+n-1}\}$ does not exceed the sum of any t elements of A (here m, ..., $m+n-1$ are the chosen n consecutive integer points and t is the number of points having a pebble before this step and no pebble after this step — hence $t-1$ points have no pebble before this step and a pebble after this step). Note that it suffices to have $m=0$ and prove that the sum of $t-1$ largest elements of A does not exceed the sum of t smallest elements, i.e.

$$1+a+a^2+\ldots+a^{t-1}\geqslant a^{n-t+1}+\ldots+a^{n-1}\,,$$

where $1\leqslant t\leqslant\left[\frac{n+1}{2}\right]=k$. This directly follows from (3) or (4) for $x=a$ and from the fact that since $a>1$ then deleting an equal number of terms from each side makes the left side larger than the right side.