Estonian Math Competitions 2005/2006

The Gifted and Talented Development Centre

Tartu 2006

WE THANK:

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Selected Problems from Open Contests

1. A farmer noticed that, during the last year, there were exactly as many calves born as during the two preceding years together. Even better, the number of pigs born during the last year was one larger than the number of pigs born during the two preceding years together. The farmer promised that if such a trend will continue then, after some years, at least twice as many pigs as calves will be born in his cattle, even though this far this target has not yet ever been reached. Will the farmer be able to keep his promise? *(Juniors.)*

Answer: no.
Solution. Let F_n and G_n be the numbers of calves and pigs born during the nth year, respectively. We will prove that if the farmer's promise has been false during the previous years, it will remain so after the nth year as well. From the problem statement, we have $F_n = F_{n-1} + F_{n-2}$ and $G_n = G_{n-1} + G_{n-2} + 1$. If the number of born pigs was less than twice the number of born calves during the previous years, we must have $G_{n-1} \leqslant 2F_{n-1} - 1$ and $G_{n-2} \leqslant 2F_{n-2} - 1$. Consequently,

$$G_n = G_{n-1} + G_{n-2} + 1 \leqslant 2F_{n-1} - 1 + 2F_{n-2} - 1 + 1 = \\ = 2(F_{n-1} + F_{n-2}) - 1 = 2F_n - 1$$

or $G_n \leqslant 2F_n - 1$, i.e. the promise will not be true during the nth year. Since the farmer has kept the cattle for at least two years, we can claim by induction that the promise will never come true.

2. Let $ABCD$ be a parallelogram, M the midpoint of AB and N the intersection of CD and the angle bisector of ABC. Prove that CM and BN are perpendicular iff AN is the angle bisector of DAB. *(Juniors.)*

Solution. The triangle CNB is isosceles since $\angle CNB = \angle MBN = \angle CBN$ (see Figure 1). Thus we have $|NC| = |BC|$.
Assume first $CM \perp BN$. Since BN is both bisector and altitude for triangle BMC, we have $|BM| = |BC|$. Consequently $|BM| = |CN|$, implying that N is the midpoint of CD and MN is a segment joining the midpoints of the sides of the parallelogram. Then we must have $|AM| = |DN| = |NC|$ and $|AD| = |MN| = |BC|$. Thus the sides of $AMND$ are equal and we have a rhombus. Its diagonal AN bisects DAM.
Assume now that AN bisects DAB. Then $\angle DNA = \angle BAN = \angle DAN$, which implies $|DN| = |DA|$. On the other hand, $|DA| = |CB| = |NC|$. Thus N is the midpoint of CD. Since M is the midpoint of AB, we have that $MBCN$ is a rhombus with the diagonals CM and BN being perpendicular.

3. Does there exist a natural number with the sum of digits of its kth power being equal to k, if a) $k = 2004$; b) $k = 2006$? *(Juniors.)*

Answer: a) no; b) no.
Solution. a) Assume there exists such a number a. Since the sum of digits of a^{2004} is

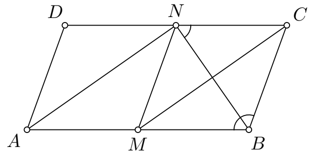
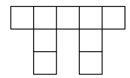


Figure 1

divisible by 3, a^{2004} is divisible by 3 and so is a. But then a^{2004} is also divisible by 9 making the sum of its digits divisible by 9. Since 2004 is not divisible by 9, the sum of the digits of a^{2004} can not be equal to 2004, a contradiction.
b) Assume there exists such a number a. We will use the fact that a natural number and its sum of digits give the same remainder when divided by 3. It follows that $a^{2006} \equiv 2006 \equiv 2 \bmod 3$. On the other hand, the number $a^{2006} = (a^{1003})^2$ is a perfect square that can not give the remainder 2 when divided by 3, a contradiction.

4. A 9×9 square is divided into unit squares. Is it possible to fill each unit square with a number $1, 2, \ldots, 9$ in such a way that, whenever one places the tile so that it fully covers nine unit squares, the tile will cover nine different numbers? *(Juniors.)*



Answer: no.
Solution 1. Assume that the numbers can be written in the required way. Put the tile over the central square; w.l.o.g. we can assume that the numbers are placed like in Figure 2, left. Next move the tile like in Figure 2, middle. Two upper left vacant squares can have neither 6 nor 7. Thus we must have 8 and 9 there, in some order. Now place the tile like in Figure 2, right. We can see that either way we must cover number 8 twice, hence the required configuration of numbers does not exist.

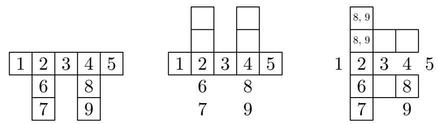


Figure 2

Solution 2. Put the tile over the central square; w.l.o.g. we can assume that the numbers are written like in Figure 3, left. We will analyse which number can be written into the gray central square. Moving the tile one position left, we see that the central square can not contain 1, 2, 3 or 4. Moving the tile one position right, we see that the central square can not contain 5. Moving the tile one position down, we see that the central square can not contain 6, 7, 8 or 9. Thus the required numbering is not possible.

5. Find all real numbers with the following property: the difference of its cube and its square is equal to the square of the difference of its square and the number itself. *(Juniors.)*

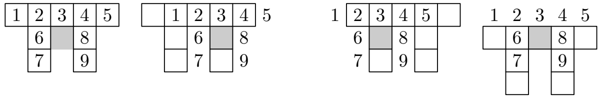


Figure 3

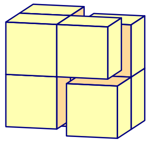


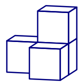
Figure 4

Answer: 0, 1 and 2.

Solution 1. Let x be a number with this property. Then $x^3 - x^2 = (x^2 - x)^2$, which leads to the equation $x^4 - 3x^3 + 2x^2 = 0$ or $x^2(x^2 - 3x + 2) = 0$. Hence $x^2 = 0$ or $x^2 - 3x + 2 = 0$. Solving these quadratic equations, we get $x_0 = 0$, $x_1 = 1$, $x_2 = 2$.

Solution 2. Transform the equation $x^3 - x^2 = (x^2 - x)^2$ to obtain $x^2(x-1) = x^2(x-1)^2$. Therefore $x^2(x-1)^2 - x^2(x-1) = 0$ or $x^2(x-1)(x-2) = 0$. Clearly, the solutions of the last equation are 0, 1, 2.

6. A solid figure consisting of unit cubes is shown in the picture. Is it possible to exactly fill a cube with these figures if the side length of the cube is a) 15; b) 30? *(Juniors.)*



Answer: a) no; b) yes.

Solution. a) Since the figure consists of four unit cubes, the number of unit cubes in every solid composable from these figures is divisible by 4. Since the cube with side length 15 contains an odd number of unit cubes, this cube is not among these solids.

b) From two figures, it is possible to assemble a cube with side length 2 (see Figure 4). From these cubes, it is possible to assemble a cube with side length 30.

7. Two non-intersecting circles, not lying inside each other, are drawn in the plane. Two lines pass through a point P which lies outside each circle. The first line intersects the first circle at A and A' and the second circle at B and B'; here A and B are closer to P than A' and B', respectively, and P lies on segment AB. Analogously, the second line intersects the first circle at C and C' and the second circle at D and D'. Prove that the points A, B, C, D are concyclic if and only if the points A', B', C', D' are concyclic. *(Juniors.)*

Solution 1. Since A, A', C', C are concyclic (see Figure 5), we have $\angle AA'C' + \angle ACC' = 180^\circ$, hence $\angle B'A'C' = \angle ACD$. Analogously $\angle C'D'B' = \angle DBA$. Points A, B, C, D are concyclic if and only if $\angle ACD = \angle DBA$, which is equivalent to $\angle B'A'C' = \angle C'D'B'$, the last equality holds if and only if points A', B', C', D' are concyclic.

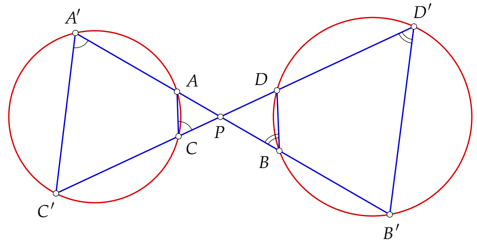


Figure 5

Solution 2. Independently of the location of P, the equalities

$$|PA| \cdot |PA'| = |PC| \cdot |PC'|, \qquad |PB| \cdot |PB'| = |PD| \cdot |PD'|$$

are valid. Multiplying these, we get

$$|PA| \cdot |PB| \cdot |PA'| \cdot |PB'| = |PC| \cdot |PD| \cdot |PC'| \cdot |PD'|.$$

Points A, B, C, D are concyclic if and only if

$$|PA| \cdot |PB| = |PC| \cdot |PD|$$

or, taking into account the previous equality, if and only if

$$|PA'| \cdot |PB'| = |PC'| \cdot |PD'|,$$

which holds if and only if A', B', C', D' are concyclic.

Note. As can be seen from solution 2, the assertion of the problem holds regardless of positions of the circles.

8. A computer outputs the values of the expression $(n+1) \cdot 2^n$ for $n = 1, n = 2, n = 3$, etc. What is the largest number of consecutive values that are perfect squares? *(Juniors.)*

Answer: 2.

Solution. Two consecutive values can be perfect squares, for example, for $n = 7$ and $n = 8$ we get $8 \cdot 2^7 = (2^5)^2$ and $9 \cdot 2^8 = (3 \cdot 2^4)^2$.

Now prove that three consecutive values cannot be perfect squares. Assume that $(n+1) \cdot 2^n$ and $(n+3) \cdot 2^{n+2}$ are both perfect squares. If n is even then both 2^n and 2^{n+2} are perfect squares. Therefore also $n+1$ and $n+3$ must be perfect squares, which is impossible. If n is odd, i.e. $n = 2k+1$ for some $k \geqslant 0$, then $(n+1) \cdot 2^n = (2k+2) \cdot 2^{2k+1} = (k+1) \cdot 2^{2k+2}$ and $(n+3) \cdot 2^{n+2} = (2k+4) \cdot 2^{2k+3} = (k+2) \cdot 2^{2k+4}$. Here, the factors 2^{2k+2} and 2^{2k+4} are perfect squares, therefore also $k+1$ and $k+2$ must be perfect squares, which is impossible for non-negative k.

9. Let a, b, c be positive integers. Prove that the inequality

$$(x-y)^a (x-z)^b (y-z)^c \geqslant 0$$

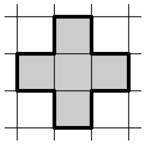


Figure 6

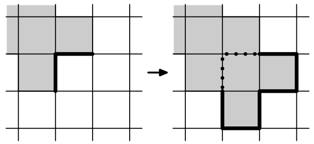


Figure 7

holds for all reals x, y, z if and only if a, b, c are even. *(Juniors.)*

Solution. If a, b, c are positive then the inequality holds. On the other hand, assume that the inequality holds for all reals x, y, z. Choosing $z < x < y$ and dividing the given inequality by a positive number $(x-z)^b(y-z)^c$, we get the inequality $(x-y)^a \geqslant 0$, from which we conclude due to $x - y < 0$ that a is even. Analogously, choosing $y < z < x$, we conclude that c is even. Finally, choosing $x < y < z$, we get after dividing the inequality by a positive number $(x-y)^a(y-z)^c$ that $(x-z)^b \geqslant 0$ from which we conclude due to $x - z < 0$ that b is even.

10. All the streets in a city run in one of two perpendicular directions, forming unit squares. Organizers of a car race want to mark down a closed race track in the city in such a way that it would not go through any of the crossings twice and that the track would turn 90° right or left at every crossing. Find all possible values of the length of the track. *(Seniors.)*

Answer: all positive integers divisible by 4, except for 8.
Solution. Define natural coordinates with the origin at some crossing and consider two consecutive track fragments of length 1. One of them is parallel to x-axis and the other one to y-axis; moving along the first one, parity of the x-coordinate changes, and moving along the second one, parity of the y-coordinate changes. Moving along the track, parities of x- and y-coordinates change alternately, but when returning to the initial point, both parities must be the same as in the beginning. Since the pairs of parities repeat after every four track fragments, the length of the track must be divisible by 4.
There exists a suitable track of length 4 going around one block. There is no track of length 8. If there were one, it would have four fragments parallel to x-axis and four fragments parallel to y-axis. Hence, we could not deviate more than 2 units in either direction and the whole track should fit into 2×2 square. It is easy to see that the track can not contain three corners of the square, but then there will be less that 8 possible turning points left.
All the other positive integers divisible by 4 are attainable. Figure 6 shows a track of length 12 and we can increase this length repeatedly by 4 units using the operation in Figure 7.

11. After the schoolday is over, Juku must attend an extra math class. The teacher writes a quadratic equation $x^2 + p_1x + q_1 = 0$ with integer coefficients on the blackboard and Juku has to find its solutions. If they are not both integers, Juku may go home. If the solutions are integers, then the teacher writes a new equation $x^2 + p_2x + q_2 = 0$, where p_2 and q_2 are the solutions of the previous equation taken in some order, and everything

starts all over. Find all possible values for p_1 and q_1 such that the teacher can hold Juku at school forever. *(Seniors.)*

Answer: either p_1 is any integer and $q_1 = 0$ or $p_1 = 1$ and $q_1 = -2$.
Solution. If $q_1 = 0$ then we have the equation $x^2 + p_1x = 0$ with solutions $-p_1$ and 0. The teacher can write another equation $x^2 - p_1x = 0$ with solutions p_1 ja 0, then again $x^2 + p_1x = 0$, etc. Thus all pairs $(p_1, 0)$ satisfy the conditions of the problem.
If $q_1 = -1$ then the product of the solutions must be -1 and the solutions -1 and 1 in some order. Since the equations $x^2 - x + 1 = 0$ and $x^2 + x - 1 = 0$ have no integer solutions, no pairs of the form $(p_1, -1)$ satisfy the conditions of the problem.
If $q_1 = -2$ then the product of the solutions must be -2 and the solutions are either 2 and -1 or 1 and -2. In the first case, the teacher can choose between the equations $x^2 + 2x - 1 = 0$ and $x^2 - x + 2 = 0$, none of them having integer solutions. In the second case, the teacher can write the equation $x^2 + x - 2 = 0$ with solutions 1 and -2. Thus we see that the pair $(1, -2)$ staisfies the conditions of the problem.
Now let q_1 be an integer not equal to 0, -1 nor -2. If x_1 and x_2 are the solutions of $x^2 + p_1x + q_1 = 0$, Vìète formulae imply that $x_1 + x_2 = -p_1$, $x_1x_2 = q_1$ and

$$x_1^2 + x_2^2 = (x_1 + x_2)^2 - 2x_1x_2 = p_1^2 - 2q_1 < p_1^2 + q_1^2.$$

Thus the sum of squares of the coefficients of the equation strictly decreases for $q_1 \notin [-2; 0]$. Since sum of squares is non-negative, we must sooner or later reach one of the two situations: the solutions are not integers or the constant term belongs to the interval $[-2; 0]$. The latter case is impossible, since every equation $x^2 + px + q = 0$ uniquely determines its predecessor $x^2 - (p+q)x + pq = 0$, hence the pairs $(p_1, 0)$ ja $(1, -2)$ can only arise from equations with constant terms 0 or -2, respectively. Thus there are no other pairs of numbers satisfying the conditions of the problem.

12. Let ABC be an acute triangle and choose points A_1, B_1 and C_1 on sides BC, CA and AB, respectively. Prove that if the quadrilaterals ABA_1B_1, BCB_1C_1 and CAC_1A_1 are cyclic then their circumcentres lie on the sides of ABC. *(Seniors.)*

Solution. Since BCB_1C_1 is cyclic (see Figure 8), we have $\angle BB_1C = \angle BC_1C = \alpha$. Similarily, let $\angle CC_1A = \angle CA_1A = \beta$ and $\angle AA_1B = \angle AB_1B = \gamma$. Considering the angles with vertices at points A_1, B_1 and C_1, we get the following system of equations:

$$\begin{cases} \beta + \gamma &= 180^\circ \\ \gamma + \alpha &= 180^\circ \\ \alpha + \beta &= 180^\circ. \end{cases}$$

Adding the equalities and dividing by 2 gives $\alpha + \beta + \gamma = 270^\circ$, implying $\alpha = \beta = \gamma = 90^\circ$. Thus the segments BC, CA and AB are the diameters of the circles and contain their circumcentres.

13. Martin invented the following algorithm. Let two irreducible fractions $\frac{s_1}{t_1}$ and $\frac{s_2}{t_2}$ be given as inputs, with the numerators and denominators being positive integers. Divide s_1 and s_2 by their greatest common divisor c and obtain a_1 and a_2, respectively. Similarly, divide t_1 and t_2 by their greatest common divisor d and obtain b_1 and b_2,

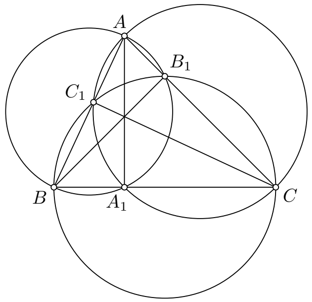


Figure 8

respectively. After that, form a new fraction $\dfrac{a_1b_2 + a_2b_1}{t_1b_2}$, reduce it, and multiply the numerator of the result by c. Martin claims that this algorithm always finds the sum of the original fractions as an irreducible fraction. Is his claim correct? *(Seniors.)*

Answer: yes.
Solution. Since

$$\frac{c\cdot(a_1b_2 + a_2b_1)}{t_1b_2} = \frac{c\cdot\left(\frac{s_1}{c}\cdot\frac{t_2}{d} + \frac{s_2}{c}\cdot\frac{t_1}{d}\right)}{t_1\cdot\frac{t_2}{d}} = \frac{s_1t_2 + s_2t_1}{t_1t_2} = \frac{s_1}{t_1} + \frac{s_2}{t_2},$$

the resulting fraction has correct value. We still need to prove that it is irreducible. For that, it is enough to show that the numbers c and t_1b_2 are relatively prime.
Suppose there exists a prime p dividing both c and t_1b_2. Since $c = \gcd(s_1, s_2)$, we have that p also divides both s_1 and s_2. Consequently, t_1 and t_2 can not be divisible by p, because the fractions $\frac{s_1}{t_1}$ and $\frac{s_2}{t_2}$ are irreducible. Thus p does not divide t_1t_2, implying that p can not divide $t_1b_2 = t_1\cdot\frac{t_2}{d}$ either, a contradiction.

14. Two players A and B play the following game. Initially, there are m equal positive integers n written on a blackboard. A begins and the players move alternately. The player to move chooses one of the non-zero numbers on the board. If this number k is the smallest among all positive integers on the board, the player replaces it with $k-1$; if not, the player replaces it with the smallest positive number on the board. The player who first turns all the numbers into zeroes, wins. Who wins if both players use their best strategies? *(Seniors.)*

Answer: A wins if mn is odd; B wins if mn is even.
Solution. If the quantity m of numbers is even then B has the following winning strategy. B divides all the numbers into pairs and if A makes a move and changes some number, B changes the other number in the pair, ensuring that after his move all pairs contain equal numbers. This is possible, since after A's move the number A wrote must be the

smallest on the board and strictly less than the other number in the pair. The situation with all numbers being equal to zero can this way only occur after B's move.
Now let m be odd and n even. If none of the numbers is yet zero, B can ensure that after his move the following conditions hold: the smallest number on the board is even, the quantity of the smallest numbers is odd and the number of occurrencies of every other number is even. Indeed, if A changes the smallest number then B can change it again, but if A changes some other number then B can change another number equal to the one A changed. It goes on until some number becomes zero, afterwards B can divide all the remaining numbers into pairs and use the strategy described above.
Finally, let m and n be odd. Then after A's first move, there is a position described in the previous paragraph on the board. Thus A can use the strategy of B and win.

15. Kati cut two equal regular n-gons out of paper. To the vertices of both n-gons, she wrote the numbers 1 to n in some order. Then she stabbed a needle through the centres of these n-gons so that they could be rotated with respect to each other. Kati noticed that there is a position where the numbers at each pair of aligned vertices are different. Prove that the n-gons can be rotated to a position where at least two pairs of aligned vertices contain equal numbers. *(Seniors.)*

Solution. Assume that the lower n-gon is fixed and move the upper n-gon. Let the initial position of n-gons be the one found by Kati. For each vertex of the upper n-gon, there is an angle by which rotating clockwise the upper n-gon brings this vertex atop of the vertex of the lower n-gon with the same number. There are n different vertices, but only $n-1$ different rotation angles since the angle 0° is excluded by conditions. Hence for two vertices of the upper n-gon, the rotation angles are equal.

16. A real-valued function f satisfies for all reals x and y the equality

$$f(xy) = f(x)y + xf(y).$$

Prove that this function satisfies for all reals x and $y \neq 0$ the equality

$$f\left(\frac{x}{y}\right) = \frac{f(x)y - xf(y)}{y^2}.$$

(Seniors.)

Solution. From the given expression we obtain $f(x) = \dfrac{f(xy) - xf(y)}{y}$, this equality holds for any reals x and $y \neq 0$. Taking $\frac{x}{y}$ at place of x, we get

$$f\left(\frac{x}{y}\right) = \frac{f\left(\frac{x}{y}\cdot y\right) - \frac{x}{y}f(y)}{y} = \frac{f(x)y - xf(y)}{y^2}.$$

Note. It is possible to prove (for example, using reduction to the Cauchy equation), that the only continuous functions satisfying the given conditions are

$$f(x) = \begin{cases} ax\ln|x|, & \text{if } x \neq 0,\\ 0, & \text{if } x = 0.\end{cases}$$

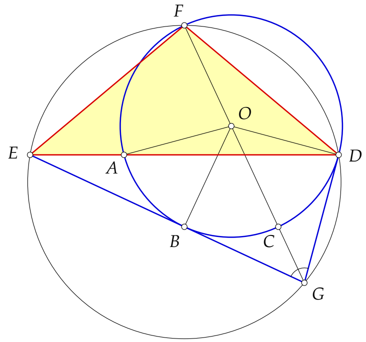


Figure 9

17. Four points A, B, C, D are chosen on a circle in such a way that arcs AB, BC, and CD are of the same length and the arc DA is longer than these three. Line AD and the line tangent to the circle at B intersect at E. Let F be the other endpoint of the diameter starting at C of the circle. Prove that triangle DEF is equilateral. *(Seniors.)*

Solution 1. Let O be the centre of the circle and let G be the point where BE intersects the tangent drawn from D to the circle. Since the total length of arcs DA and AB is larger than the total length of arcs BC and CD, the points G and E lie on different sides from B (see Figure 9). Points B and D are symmetric with respect to the line CF, hence G lies on the line CF and $\angle BGF = \angle DGF$. Now $\angle BED = \frac{1}{2}(\angle BOD - \angle AOB) = \frac{1}{2}\angle COD = \angle CFD$, therefore $\angle GED = \angle GFD$. Consequently G, E, F, D are concyclic. Since $\angle EGF = \angle DGF$, chords EF and DF of the circumcircle of quadrangle $GEFD$ are equal.

Solution 2. Since the lengths of the arcs BA and BC are equal, the lines EB and AC are parallel. Therefore $\angle BED = \angle CAD = \angle BDA$. Thus triangle BED is equilateral. Further, $\angle CBF = 90°$. By symmetry, the lines BC and AD are parallel, hence BF is perpendicular to AD. Segment BF is the altitude of the equilateral triangle BED, it bisects its base ED. This segment is also the altitude of the triangle DEF and it bisects its base. This is possible only when the triangle DEF is equilateral.

Note. The assertion of the problem holds also in the case when the arc DA is shorter than the other three, the point G then lies on the other side.

18. In the sequence (a_n) with general term $a_n = n^3 - (2n+1)^2$, does there exist a term that is divisible by 2006? *(Seniors.)*

Answer: yes.

Solution. First, $a_4 = 4^3 - 9^2 = -17$ and $a_7 = 7^3 - 15^2 = 118$. Since $n^3 - (2n+1)^2$ is a polynomial, a_{4+17k} is divisible by 17 and a_{7+118l} is divisible by 118 for all natural numbers k and l. Since 119 is divisible by 17, adding 118 decreases the remainder by 1 on division by 17. Therefore, when $361 = 7 + 118 \cdot 3$ is divided by 17, the remainder is $7 - 1 \cdot 3 = 4$. This means that the term a_{361} is divisible by both 17 and 118, i.e. it is divisible by 2006.

Note 1. Since 17 and 118 are relatively prime, the existence of the suitable n follows from the Chinese Remainder Theorem: the remainders of n upon division by 17 and 118 must be 4 and 7, respectively.

Note 2. The least number satisfying the given conditions corresponds to $n = 87$, in this case $a_{87} = 627878 = 313 \cdot 2006$.

19. Let $n \geqslant 2$ be a fixed integer and let $a_{i,j}$ $(1 \leqslant i < j \leqslant n)$ be some positive integers. For a sequence $x_1, \ldots, x_n$ of reals, let $K(x_1, \ldots, x_n)$ be the product of all expressions $(x_i - x_j)^{a_{i,j}}$ where $1 \leqslant i < j \leqslant n$. Prove that if the inequality $K(x_1, \ldots, x_n) \geqslant 0$ holds independently of the choice of the sequence $x_1, \ldots, x_n$ then all integers $a_{i,j}$ are even. *(Seniors.)*

Solution 1. Suppose the contrary: some of the numbers $a_{i,j}$ are odd. Let l be the smallest index for which there are odd numbers among the numbers $a_{i,l}$ $(1 \leqslant i < l)$; also let k be the largest index for which $a_{k,l}$ is odd. Then $a_{k,l}$ is the only odd number among the numbers $a_{i,j}$ where $k \leqslant i < j \leqslant l$. Now choose $x_1, \ldots, x_n$ as follows:

$$x_1 > x_2 > \ldots > x_{k-1} > x_l > x_{k+1} > \ldots > x_{l-1} > x_k > x_{l+1} > \ldots > x_n;$$

i.e., choose some n numbers in decreasing order and swap the positions of the k-th and the l-th number. Then in the given expression, the factors $(x_i - x_j)^{a_{i,j}}$, where $i < k$ or $j > l$, are positive, since the bases of the power are positive. All the remaining factors $(x_i - x_j)^{a_{i,j}}$ where $k \leqslant i < j \leqslant l$ have even exponents with the exception of $(x_k - x_l)^{a_{k,l}}$, which has negative base and odd exponent. So the whole product is negative.

Solution 2. Assume that $a_{k,l}$ is odd for some indices k and l. Fix $x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n$ in such a way that they are pairwise different and consider the product $K(x_1, \ldots, x_n)$ as a polynomial of one variable $K(x_k)$. The root x_l of this polynomial has odd multiplicity. Therefore the graph of $K(x_k)$ intersects the x-axis at x_l and we can choose x_k such that $K(x_k)$ is negative, a contradiction.

Selected Problems from the Final Round of National Olympiad

1. Find all pairs of positive integers (a, b) such that

$$ab = \gcd(a, b) + \operatorname{lcm}(a, b).$$

(Grade 9.)

Answer: $(2, 2)$.

Solution. As the left-hand side and summand $\operatorname{lcm}(a, b)$ on the right-hand side are both divisible by a, also $\gcd(a, b)$ has to be divisible by a. On the other hand, $\gcd(a, b) \leqslant a$ as a is positive. Thus $\gcd(a, b) = a$. Analogously we obtain that $\gcd(a, b) = b$. Therefore $a = b$ and the equation has the form $a^2 = a + a$ or $a(a-2) = 0$. The only positive solution of the equation is $a = 2$ and thus also $b = 2$.

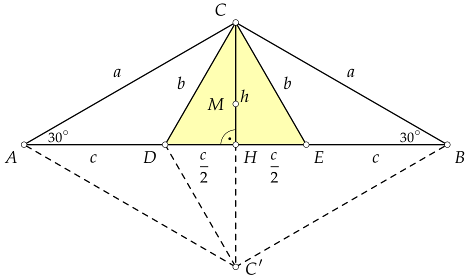


Figure 10

Solution 2. It is known that $ab = \gcd(a,b)\,\mathrm{lcm}(a,b)$ for positive integers a and b. Thus we obtain the equality $\gcd(a,b)\,\mathrm{lcm}(a,b) = \gcd(a,b) + \mathrm{lcm}(a,b)$ which is equivalent to $(\gcd(a,b)-1)\,(\mathrm{lcm}(a,b)-1) = 1$. The last equality expresses 1 as a product of two non-negative integers; this is only possible if both of them are equal to 1. Thus $\gcd(a,b) - 1 = 1$ and $\mathrm{lcm}(a,b) - 1 = 1$ or $\gcd(a,b) = 2$ and $\mathrm{lcm}(a,b) = 2$. Hence $a = b = 2$.

Solution 3. Let $\gcd(a,b) = d$. We represent integers a and b as $a = da'$ and $b = db'$ where a' and b' are relatively prime. Then $\mathrm{lcm}(a,b) = da'b'$. The equation gets the form $d^2a'b' = d + da'b'$, or $a'b' = \dfrac{1}{d-1}$. As $a'b'$ is an integer, we must have $d - 1 = 1$ and $a'b' = 1$. Hence $d = 2$ and $a' = b' = 1$. Therefore the solution of the equation is $a = da' = 2$ and $b = db' = 2$.

2. Let there be $n \geqslant 2$ real numbers such that none of them is greater than the arithmetic mean of the other numbers. Prove that all the numbers are equal. *(Grade 9.)*

Solution. Let a be the greatest among the numbers. Suppose, by contradiction, that not all the numbers are equal. Then there must be some numbers less than a. Considering numbers other than a, we therefore know that their arithmetic mean is less than a. But this contradicts the conditions of the problem. Hence all the numbers are equal.

3. Triangle ABC is isosceles with $AC = BC$ and $\angle C = 120^\circ$. Points D and E are chosen on segment AB so that $|AD| = |DE| = |EB|$. Find the sizes of the angles of triangle CDE. *(Grade 9.)*

Answer: all the angles are 60°.

Solution 1. The base angle of the isosceles triangle ABC is $(180^\circ - 120^\circ) : 2 = 30^\circ$. Let H be the foot of the altitude drawn from vertex C (see Figure 10). Reflect the triangle ABC with respect to side AB, the point C going to C'. As $\angle CAC' = 60^\circ$ and $|AC| = |AC'|$, triangle ACC' is equilateral and AH is its median. Moreover, point D divides the median with ratio $2:1$. Thus the medians of ACC' meet at D, and CD is both a median and an angle bisector to ACC'. We obtain $\angle DCH = 30^\circ$ and $\angle DCE = 60^\circ$. As CDE is isosceles, $\angle CDE = \angle CED = 60^\circ$.

Solution 2. Denote $|CA| = |CB| = a$, $|CD| = |CE| = b$ and $|AD| = |DE| = |EB| = c$ for brevity. Let h be the altitude drawn from vertex C to side AB. In triangle CHA, we have

$h : \dfrac{3c}{2} = \tan 30^\circ = \dfrac{1}{\sqrt{3}}$, therefore $h = \dfrac{\sqrt{3}\,c}{2}$. Hence $b^2 = h^2 + \left(\dfrac{c}{2}\right)^2 = \dfrac{3c^2}{4} + \dfrac{c^2}{4} = c^2$, giving $b = c$. We now have that CDE is an equilateral triangle and its angles are 60°.

Solution 3. As $\dfrac{|CH|}{|CA|} = \sin 30^\circ = \dfrac{1}{2}$ and $\dfrac{|DH|}{|DA|} = \dfrac{1}{2}$, segment CD satisfies the Angle Bisector Property. Hence CD bisects angle ACH whose size is 60° and $\angle DCH = 30^\circ$. Then the vertex angle of the isosceles triangle CDE is 60° and the base angles are also 60°.

Solution 4. Let the medians of triangle CDE meet at M. By the Ray Property, $|DM| = \dfrac{1}{3}|AC| = \dfrac{1}{3}|BC| = |EM|$. We obtain $|CH| = \dfrac{1}{2}|AC|$ as in the previous solution and thus $|CM| = \dfrac{1}{3}|AC|$. So $|CM| = |DM| = |EM|$, i.e., the intersection point M of medians of CDE is also the circumcentre of CDE. It follows that CDE is an equilateral triangle and all its angles are of size 60°.

Solution 5. Assume that $\angle DCE > 60^\circ$. Then $\angle CED < 60^\circ$. Hence $|DE| > |CD|$ giving also $|AD| > |CD|$. Then in triangle ACD, we have $\angle ACD > 30^\circ$, and in triangle BCE by symmetry, $\angle BCE > 30^\circ$. Adding the inequalities, we get $\angle ACB > 120^\circ$ — a contradiction. Analogously we obtain a contradiction by starting with assumption $\angle DCE < 60^\circ$. Hence $\angle DCE = 60^\circ$ and thus $\angle CDE = \angle CED = 60^\circ$.

4. Consider a rectangular grid of 10×10 unit squares. We call a *ship* a figure made up of unit squares connected by common edges. We call a *fleet* a set of ships where no two ships contain squares that share a common vertex (i.e. all ships are vertex-disjoint). Find the least number of squares in a fleet to which no new ship can be added. *(Grade 9.)*

Answer: 16.

Solution. Call a fleet *full* if no new ships can be added. We have to find the least number of squares in a full fleet.

First we show that a full fleet covering 16 unit squares exists. Put on the grid 16 one-square ships as shown in Figure 11. Note that then each square of the grid has a common vertex with one of those ships and thus no ship can be added.

Second we prove that there can not be fewer than 16 unit squares in a full fleet. Suppose a full fleet is fixed. Consider the set of 16 unit squares painted gray in Figure 11. For each of these 16 squares, there is a square of the full fleet that shares (at least) a common

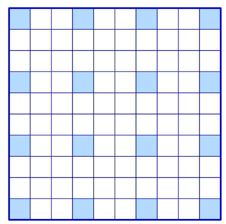


Figure 11

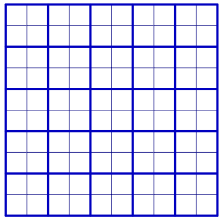


Figure 12

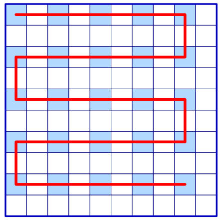


Figure 13

vertex with it. All these squares of the fleet must be different. Hence there are at least 16 squares in the fleet.

5. Consider a rectangular grid of 10×10 unit squares. We call a *ship* a figure made up of unit squares connected by common edges. We call a *fleet* a set of ships where no two ships contain squares that share a common vertex (i.e. all ships are vertex-disjoint). Find the greatest natural number that, for each its representation as a sum of positive integers, there exists a fleet such that the summands are exactly the numbers of squares contained in individual ships. *(Grade 10.)*

Answer: 25.

Solution. First we prove that, for all $n > 25$, we can divide n into summands so that a fleet with respective ship sizes can not be put on the grid. In particular, we prove that one can not put more than 25 ships of size 1 on the grid. Let us divide the grid into squares of 2×2 (as in Figure 12), there are 25 of them. As each 2×2 square can contain at most one ship of size 1, then the total number of such ships is at most 25.

Second we show that, for any representation of 25 as a sum of positive integers, there is a fleet with respective ship sizes. Let us initially put 25 ships of size 1 on the grid as shown in Figure 13. Then, starting from the upper left corner, shift along the line together as many ships as the first summand of the representation tells; then shift together as many ships as the second summand tells etc. The fleet obtained this way satisfies the conditions of the problem.

6. Find the greatest possible value of $\sin(\cos x) + \cos(\sin x)$ and determine all real numbers x, for which this value is achieved. *(Grade 11.)*

Answer: the greatest possible value is $\sin 1 + 1$, which is achieved iff $x = 2k\pi$, where k is an arbitrary integer.

Solution. Since the value of $\cos x$ is in the interval $[-1; 1]$ and since $\sin x$ is increasing in this interval, the greatest possible value of the first addend is $\sin 1$, and the value is achieved iff $\cos x = 1$, or $x = 2k\pi$, where k is an arbitrary integer. The greatest possible value of the second addend is 1, which is achieved iff $\sin x = 0$, or $x = l\pi$, where l is an arbitrary integer. Both terms achieve the maximal value simultaneously iff $x = 2k\pi$, where k is an arbitrary integer; the value of the expression is then $\sin 1 + 1$.

7. In a right triangle, the length of one side is a prime and the lengths of the other side and the hypotenuse are integral. The ratio of the triangle perimeter and the incircle

diameter is also an integer. Find all possible side lengths of the triangle. *(Grade 11.)*

Answer: 3, 4 and 5.

Solution. Let p and m be the lengths of the sides of the triangle and let n be the length of the hypotenuse, where p is a prime (Figure 14). Then $p^2 + m^2 = n^2$, implying $p^2 = (n - m)(n + m)$. Since p is prime, the terms of the right-hand side must satisfy $n - m = 1$, $n + m = p^2$.

The perimeter of the triangle is $p + m + n$. In order to find the incircle diameter d, we note that the total length of the segments tangent to the incircle, originating from the vertex of the right angle, equals the length of the incircle diameter, whereas the total length of the four tangents originating from the other two vertices is $2n$. Thus, $d + 2n = p + m + n$. According to the assumption, $p + m + n$ is divisible by $d = p + m - n$. Substituting $m + n$ and $m - n$ from above, we get that $p + p^2 = p(p + 1)$ is divisible by $p - 1$. Since p and $p - 1$ are coprime, it must be that $p + 1$ is divisible by $p - 1$. Hence, $p - 1 = 1$ or $p - 1 = 2$. We see that $p = 2$ is impossible, since $n - m$ and $n + m$ cannot be of different parity. Thus, $p = 3$, $m = 4$ and $n = 5$. A triangle with side lengths 3, 4, 5 is clearly a right triangle.

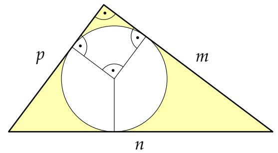


Figure 14

8. The sequence (F_n) of Fibonacci numbers satisfies $F_1 = 1$, $F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for all $n \geqslant 3$. Find all pairs of positive integers (m, n), such that $F_m \cdot F_n = mn$. *(Grade 11.)*

Answer: $(1,1)$, $(1,5)$, $(4,6)$, $(5,1)$, $(5,5)$ and $(6,4)$.

Solution. By induction on n, it is easy to prove that $F_n > n$ for all $n \geqslant 6$ and $F_n > 2n$ for all $n \geqslant 8$. Thus, if $m \geqslant 6$ and $n \geqslant 6$, then $F_m \cdot F_n > m \cdot n$. W.l.o.g., we may now assume $m \leqslant 5$ (the remaining solutions can be obtained by exchanging m and n).

- If $m = 1$, then $1 \cdot F_n = 1 \cdot n$, or $F_n = n$. From above, the only solutions are $n = 1$ and $n = 5$ and the suitable pairs are $(1,1)$, $(1,5)$, $(5,1)$ and $(5,5)$.
- If $m = 2$, we get $1 \cdot F_n = 2 \cdot n$, or $F_n = 2n$. Since there are no solutions for $n < 8$, there are no solutions at all.
- If $m = 3$, we get $2 \cdot F_n = 3 \cdot n$, or $F_n = \frac{3}{2}n$. Since $\frac{3}{2}n < 2n$, there are no solutions, as in the previous case.
- If $m = 4$, then $3 \cdot F_n = 4 \cdot n$, or $F_n = \frac{4}{3}n$. Here, the only solution is $n = 6$, giving pairs $(4,6)$ and $(6,4)$.

- If $m = 5$, then $5 \cdot F_n = 5 \cdot n$, or $F_n = n$; this case is analysed above.

9. In a triangle ABC with circumcentre O and centroid M, lines OM and AM are perpendicular. Let AM intersect the circumcircle of ABC again at A'. Let lines BA' and AC intersect at D and let lines CA' and AB intersect at E. Prove that the circumcentre of triangle ADE lies on the circumcircle of ABC. *(Grade 11.)*

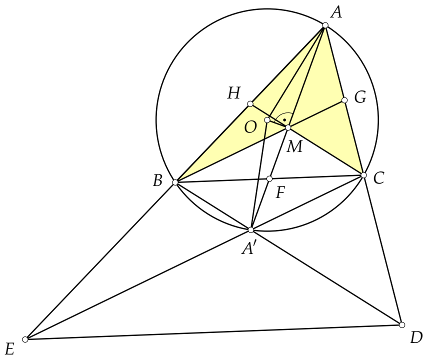


Figure 15

Solution. Let F, G and H be the base points of the medians drawn from vertices A, B and C, respectively (see Figure 15). Then, triangle $A'OA$ is isosceles with height OM and $|A'M| = |MA|$. Since the centroid divides a median in ratio 2 : 1, we get $|FM| = \frac{1}{2}|MA|$, and $|A'F| = |FM|$. On the other hand, $|BF| = |FC|$. Hence, $A'BMC$ is a parallelogram. Parallel sides then imply that triangles ABD and AHC are similar with similarity ratio 2 — the ratio of the lengths of AB and AH. Analogously, triangles ACE and AGB are similar with the same ratio. Homothecy with centre A and ratio 2 brings triangle ABC to triangle AED, while the circumcentre O of triangle ABC is transformed to the second intersection point of AO and the said circumcircle.

Note. The use of homothecy can be avoided by finding the second intersection point P of line AO and the circle and by proving that $|AP| = 2|AO|$, $|DP| = 2|CO|$ and $|EP| = 2|BO|$.

10. A pawn is placed on a square of a $n \times n$ board. There are two types of legal moves: (a) the pawn can be moved to a neighbouring square, which shares a common side with the current square; or (b) the pawn can be moved to a neighbouring square, which shares a common vertex, but not a common side with the current square. Any two consecutive moves must be of different type. Find all integers $n \geqslant 2$, for which it is possible to choose an initial square and a sequence of moves such that the pawn visits each square exactly once (it is not required that the pawn returns to the initial square). *(Grade 11.)*

Answer: $n = 2k$, where k is an arbitrary positive integer.

Solution. First, we prove that a suitable sequence of moves exists for even n. Divide the board into blocks of 2×2 squares (see Figure 16) and place the pawn on the upper left corner square. To move through the first block, take the following moves: down-right, up, down-left, down. Repeat the same combination of moves until the pawn reaches the bottom-most block in a column of blocks. In the bottom-most block, move down-right, left, up-right, and right; the first column of blocks is passed. In the bottom-most block of the second column, move down-right, left, up-right, up, and continue by moving upwards block by block. By passing the columns alternatingly up and down, the pawn visits each square exactly once.

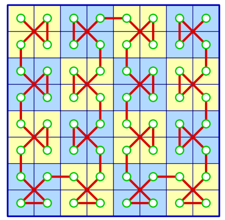


Figure 16

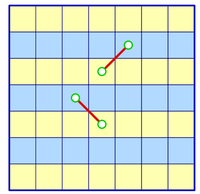


Figure 17

Next, we prove that a sequence does not exist for odd n. Colour the second, fourth, sixth, etc. row dark (see Figure 17), then there are $\frac{n^2 - n}{2}$ dark squares. Note that every diagonal move starts from a dark square or ends on a dark square. Since a suitable sequence does not visit a square twice and does not contain two consecutive diagonal moves, each diagonal move corresponds to a different dark square. Thus, we can make at most $\frac{n^2 - n}{2}$ diagonal moves and, consequently, at most $\frac{n^2 - n}{2} + 1$ non-diagonal moves, or $n^2 - n + 1$ moves altogether. For $n \geqslant 3$, this number is smaller than $n^2 - 1$, the number of moves required to visit all squares.

11. We call a *ship* a figure made up of unit squares connected by common edges. Prove that if there is an odd number of possible different ships consisting of n unit squares on a 10×10 board, then n is divisible by 4. *(Grade 12.)*

Solution. Let n be such that the number of possible different ships of n squares is odd. Divide all ships in classes, such that all ships in the same class are precisely those that can be obtained from one another by shifts, vertical and horizontal reflections. Then there must exist a class with an odd number of ships.

Let L be a ship in such a class. Assume that L is not symmetrical w.r.t. either the vertical or horizontal axis of symmetry of its tight bounding box. Then no ship in this class is symmetrical w.r.t. this axis. Thus, we can divide all ships in this class into pairs: a ship and its reflection from this axis; a contradiction with the odd cardinality of this class. Therefore, L must be symmetrical w.r.t. both the vertical and the horizontal axis.

The side lengths of the rectangle bounding L must be even, for there is an even number of ways to place a rectangle with an odd side on a board with an even side length 10; again a contradiction in parity. Thus, the vertical and horizontal axes of symmetry divide the squares of L into four disjoint reflection symmetrical sets. Since there is an equal number of squares in each set, the total number of squares of the ship is divisible by 4.

12. Find the smallest possible distance of points P and Q on a xy-plane, if P lies on the line $y = x$ and Q lies on the curve $y = 2^x$. *(Grade 12.)*

Answer: $\dfrac{1 + \ln\ln 2}{\sqrt{2}\ln 2}$.

Solution. We find the minimum point of $h(x) = 2^x - x$. Since $h'(x) = 2^x \ln 2 - 1$, we get $2^x \ln 2 - 1 = 0$, and

$$2^x = \frac{1}{\ln 2} \qquad \text{and} \qquad x = -\frac{\ln\ln 2}{\ln 2}.$$

Since $h'(x)$ is increasing, this is indeed a minimum. The value of h at this point is

$$h(x) = \frac{1}{\ln 2} + \frac{\ln\ln 2}{\ln 2} = \frac{1 + \ln\ln 2}{\ln 2}.$$

Here $1 + \ln\ln 2 = \ln(\mathrm{e}\ln 2)$. Since $2 < \mathrm{e} < 4$, we get $\ln(\mathrm{e}\ln 2) > \ln(2\ln 2) = \ln\ln 4 > \ln\ln \mathrm{e} = 0$. Thus, the value of h at the minimum is positive, so the graph of $g(x) = 2^x$ is always higher than the graph of $f(x) = x$. Consider points $A(x, f(x))$ and $B(x, g(x))$ and let C be the projection of B to the graph of f. Then, triangle ABC is a right isosceles triangle, since $\angle BAC = 45^\circ$. Consequently, $|BC| = \dfrac{|AB|}{\sqrt{2}} = \dfrac{h(x)}{\sqrt{2}}$ and the distance $|BC|$ is minimal iff $h(x)$ is minimal. The sought distance is thus

$$\frac{h(x)}{\sqrt{2}} = \frac{1 + \ln\ln 2}{\sqrt{2}\ln 2}.$$

Note. The answer can be expressed in many different ways, e.g., by $\dfrac{\log_2 \mathrm{e} - \log_2 \log_2 \mathrm{e}}{\sqrt{2}}$.

13. Prove or disprove the following statements.

a) For every integer $n \geqslant 3$, there exist n pairwise distinct positive integers such that the product of any two of them is divisible by the sum of the remaining $n - 2$ numbers.

b) For some integer $n \geqslant 3$, there exist n pairwise distinct positive integers, such that the sum of any $n - 2$ of them is divisible by the product of the remaining two numbers.

(Grade 12.)

Answer: a) true; b) false.

Solution 1. a) Take n numbers $(n^2)!$, $2(n^2)!$, $3(n^2)!$, $\ldots$, $n(n^2)!$. The product of any two of these numbers is divisible by $(n^2)!(n^2)!$, whereas the sum of the remaining numbers is $k(n^2)!$, where k is some positive integer smaller than $1 + 2 + \ldots + n$, which in turn is smaller than n^2.

b) Assume that for some n, there exist n suitable integers $a_1 < a_2 < \ldots < a_n$. Then, on the one hand,

$$a_1 + a_2 + \ldots + a_{n-2} < a_{n-2} + a_{n-2} + \ldots + a_{n-2} = (n-2)a_{n-2}$$

but, on the other hand,

$$a_{n-1}a_n \geqslant (n-1)a_n > (n-2)a_{n-2}.$$

Thus, $a_1 + a_2 + \ldots + a_{n-2} < a_{n-1}a_n$, and the sum of $a_1, a_2, \ldots, a_{n-2}$ can not be divisible by the product of a_{n-1} and a_n.

Solution 2. a) Choose arbitrary pairwise distinct numbers b_1, b_2, $\ldots$, b_n and let m be the least common multiple of all sums of $(n-2)$ terms. For each $i = 1, 2, \ldots, n$, take $a_i = mb_i$. Then the product of any two numbers a_k a_l is $a_k a_l = (mb_k) \cdot (mb_l)$, which is divisible by the sum of the remaining numbers, since m^2 is divisible by this sum.

14. Let O be the circumcentre of an acute triangle ABC and let A', B' and C' be the circumcentres of triangles BCO, CAO and ABO, respectively. Prove that the area of triangle ABC does not exceed the area of triangle $A'B'C'$. *(Grade 12.)*

Solution 1. First, we prove that for a fixed circumcircle, a triangle with maximal area is equilateral. Assume that a triangle KLM with maximal area has two sides of unequal lengths, say, KM and LM. Take a point M' on the circumcircle of KLM such that $|KM'| = |LM'|$ (Figure 18). Triangles KLM and KLM' have a common base but the altitude of the first triangle is smaller, a contradiction.

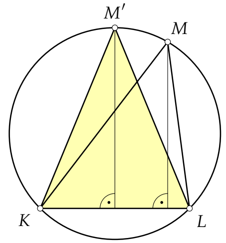


Figure 18

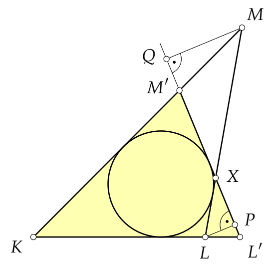


Figure 19

Next, we prove that for a fixed incircle, a triangle with minimal area is equilateral. Assume w.l.o.g. that in a triangle KLM with minimal area, $\angle KLM > \angle KML$. Consider a triangle $KL'M'$, where points L' and M' lie on lines KL and KM such that $\angle KL'M' = \angle KM'L'$ and $L'M'$ is tangent to the incircle (Figure 19). Let X be the intersection point of lines LM and $L'M'$. Draw perpendicular segments LP and MQ from points L and M to line $L'M'$. Then $|XM'| > |XL'|$ and $|MQ| > |LP|$, since $|XQ| > |XP|$ and right triangles MQX and LPX are similar. Thus, the area of triangle $MM'X$ is greater than the area of triangle $LL'X$ and consequently, the area of KLM is greater than the area of $KL'M'$, a contradiction.

Now, let R be the circumradius of ABC. Since the sides of triangle $A'B'C'$ are perpendicular to OA, OB and OC (see Figure 20) and bisect these segments, point O is the

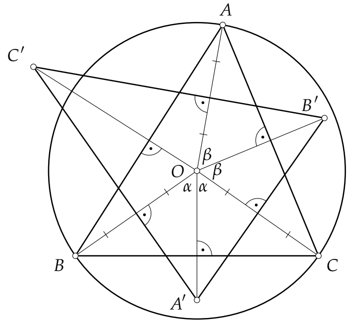


Figure 20

incentre of $A'B'C'$ and the inradius is $\frac{R}{2}$. An equilateral triangle with circumradius R and an equilateral triangle with inradius $\frac{R}{2}$ have equal area S by the property of the centroid. From above, we now get $S_{ABC} \leqslant S \leqslant S_{A'B'C'}$.

Solution 2. Let R be the circumradius of ABC and let α, β and γ be the angles at the vertices of the triangle. Then $\angle BOC = 2\alpha$, $\angle COA = 2\beta$, and $\angle AOB = 2\gamma$. Thus,

$$S_{ABC} = S_{BOC} + S_{COA} + S_{AOB} = \frac{R^2}{2}(\sin 2\alpha + \sin 2\beta + \sin 2\gamma).$$

Since triangle BOC is isosceles, the midperpendicular OA' of side BC is also an angle bisector, and $\angle BOA' = \angle COA' = \alpha$. Similarly, $\angle COB' = \angle AOB' = \beta$ and $\angle AOC' = \angle BOC' = \gamma$. Consider triangle $B'OC'$. The height drawn to side $B'C'$ is $\frac{R}{2}$, so $|B'C'| = \frac{R}{2}(\tan\beta + \tan\gamma)$ and the area of this triangle is $S_{B'OC'} = \frac{R^2}{8}(\tan\beta + \tan\gamma)$. Analogously, $S_{C'OA'} = \frac{R^2}{8}(\tan\gamma + \tan\alpha)$ and $S_{A'OB'} = \frac{R^2}{8}(\tan\alpha + \tan\beta)$. Thus,

$$S_{A'B'C'} = S_{B'OC'} + S_{C'OA'} + S_{A'OC'} = \frac{R^2}{4}(\tan\alpha + \tan\beta + \tan\gamma).$$

Function $f(x) = \tan x - 2\sin 2x$ is concave in the interval $\left[0; \frac{\pi}{2}\right]$, since the second derivative

$$f''(x) = \frac{2\sin x}{\cos^3 x} + 8\sin 2x$$

is non-negative in this interval. Jensen's inequality now gives

$$f(\alpha) + f(\beta) + f(\gamma) \geqslant 3f\left(\frac{\alpha+\beta+\gamma}{3}\right) = 3\left(\tan\frac{\pi}{3} - 2\sin\frac{2\pi}{3}\right) = 0,$$

so $\tan\alpha + \tan\beta + \tan\gamma \geqslant 2\sin 2\alpha + 2\sin 2\beta + 2\sin 2\gamma$. The final equality yields $S_{A'B'C'} \geqslant S_{ABC}$.

15. The Ababi alphabet consists of letters A and B, and the words in the Ababi language are precisely those that can be formed by the following two rules:

1) A is a word.
2) If s is a word, then $s \oplus s$ and $s \oplus \bar{s}$ are words, where $\bar{s}$ denotes a word that is obtained by replacing all letters A in s with letters B, and vice versa; and $x \oplus y$ denotes the concatenation of x and y.

The Ululu alphabet consists also of letters A and B and the words in the Ululu language are precisely those that can be formed by the following two rules:

1) A is a word.
2) If s is a word, then $s \otimes s$ and $s \otimes \bar{s}$ are words, where $\bar{s}$ is defined as above and $x \otimes y$ is a word obtained from words x and y of equal length by writing the letters of x and y alternatingly, starting from the first letter of x.

Prove that the two languages consist of the same words. *(Grade 12.)*

Solution. Since each step doubles the length of a word, both languages contain only words of length 2^n, where each such word has been obtained in exactly n steps.

First, we show that each language contains 2^n words that can be obtained in exactly n steps. Indeed, in 0 steps, we obtain only the word A in both languages. Every k-step word gives two different $(k+1)$-step words and any two different k-step words give different $(k+1)$-step words, since the initial word is always a part of the new word. Thus, the number of $k+1$-step words is twice the number of k-step words. Now, it suffices to prove that every Ababi word is an Ululu word.

Any 0-step Ababi word is clearly an Ululu word. Assume that the claim holds for all k-step Ababi words and consider a $k+1$-step word t. Then, for some word s in the Ababi language, $t = s \oplus s$ or $t = s \oplus \bar{s}$. By the induction hypothesis, there must exist a sequence of k operations by the Ululu rules that allows to construct the word s from the word A.

- If the word t is obtained in the Ababi language by the rule $t = s \oplus s$, apply the aforementioned sequence of Ululu rules to the word AA. It is easy to see that after each step, the new word is of the form $a \oplus a$, where a is the corresponding intermediate word in the construction process of s, since each Ululu operation has the same effect on the two equal halves of a word.
- If the word t is obtained by the rule $t = s \oplus \bar{s}$, apply the aforementioned sequence of Ululu rules to the Ululu word AB. In this case, after each step, the obtained word is $a \oplus \bar{a}$, where a is the corresponding intermediate word in the construction process of s, since if the two halves of a word are "negations" of each other, any Ululu operation preserves this property.

Consequently, t is a word in the Ululu language.