Estonian Math Competitions 2006/2007

The Gifted and Talented Development Centre

Tartu 2007

WE THANK:

Hansapank

skype™

Selected Problems from Open Contests

1. Find all positive integers N with at most 4 digits such that the number obtained by reversing the order of digits of N is divisible by N and differs from N. *(Juniors.)*

Answer: 1089 and 2178.

Solution. Let N be any number satisfying the conditions. Let N' be the number with digits in reversed order. Then $N' = k \cdot N$ for some integer $k \geq 2$. Clearly, N must have at least 2 digits. It cannot end with 0 as otherwise $N' < N$, which is impossible. Let $N = \overline{a \ldots b}$ and $N' = \overline{b \ldots a}$ where a and b are digits. Then $b \geq ka \geq 2a$. On the other hand, a multiple of b ends with a. The pairs (a, b) that meet all conditions imposed by the observations made so far are the following:

$$(1,3),\ (1,7),\ (1,9),\ \ (2,4),\ (2,6),\ (2,7),\ (2,8),\ (2,9),\ \ (3,7),\ (3,9),\ \ (4,8),\ (4,9).$$

For pairs $(1,3)$, $(2,4)$, $(2,7)$, $(2,9)$, $(3,7)$, $(3,9)$, $(4,8)$, $(4,9)$, the conditions $ka \leq b$ and $kb \equiv a \pmod{10}$ are contradictory. Among the remaining pairs, $(1,7)$ gives $k = 3$, while the first digit of $3 \cdot \overline{1 \ldots 7}$ cannot be 7. Analogously, $(2,6)$ gives $k = 2$, while the first digit of $2 \cdot \overline{2 \ldots 6}$ cannot be 6. Hence only the pairs $(1,9)$ and $(2,8)$ are possible.

As 91 is not divisible by 19 and 82 is not divisible by 28, no two-digit numbers satisfy the conditions. For three-digit numbers, there must be either $N = \overline{1c9}$ and $N' = \overline{9c1}$ with $k = 9$ or $N = \overline{2c8}$ and $N' = \overline{8c2}$ with $k = 4$. The first case leads to equality $900 + 10c + 1 = 9 \cdot (100 + 10c + 9)$ which gives $80c + 80 = 0$, so no suitable c exists. The second case fails analogously.

In the case of four-digit numbers, we have either $N = \overline{1cd9}$ and $N' = \overline{9dc1}$ with $k = 9$ or $N = \overline{2cd8}$ and $N' = \overline{8dc2}$ with $k = 4$. In the first case, the equality $9000 + 100d + 10c + 1 = 9 \cdot (1000 + 100c + 10d + 9)$ leads to $d = 89c + 8$. The only solution in digits is $c = 0, d = 8$. In the second case, the equality $8000 + 100d + 10c + 2 = 4 \cdot (2000 + 100c + 10d + 8)$ leads to $2d = 13c + 1$. The only solution in digits is $c = 1, d = 7$.

2. Call a scalene triangle K *disguisable* if there exists a triangle K' similar to K with two shorter sides precisely as long as the two longer sides of K, respectively. Call a disguisable triangle *integral* if the lengths of all its sides are integers.

(a) Find the side lengths of the integral disguisable triangle with the smallest possible perimeter.

(b) Let K be an arbitrary integral disguisable triangle for which no smaller integral disguisable triangle similar to it exists. Prove that at least two side lengths of K are perfect squares.

(Juniors.)

Answer: (a) 4, 6, 9.

Solution. (a) Let K be a triangle with side lengths a, b, c, where $a < b < c$. We show that K is disguisable iff $b^2 = ac$. First let K be disguisable and let K' be a similar triangle to K

with side lengths a', b', c' where $a' = b$ and $b' = c$. Then

$$\frac{b}{a} = \frac{a'}{a} = \frac{b'}{b} = \frac{c}{b},$$

hence $b^2 = ac$. Second let K satisfy $b^2 = ac$. Taking $a' = b$, $b' = c$ and $c' = \frac{c^2}{b}$, we see that

$$\frac{a'}{a} = \frac{b'}{b} = \frac{c'}{c},$$

i.e., triangle K' with side lengths a', b', c' is just like required by the definition of disguisability.

If a, b, c are integers then obviously $b \geq 2$. A case study of possible values of b shows that, for $b = 2, \ldots, 5$, there exist no integers a and c such that $a < b < c$ and $ac = b^2$ and $c < a + b$. For $b = 6$, we can take $a = 4$ and $c = 9$, giving perimeter 19. Thus we have one more condition: $a + b + c \leq 19$. As $a \geq 1$ and $c \geq b + 1$, this implies $2b + 2 \leq 19$ and $b \leq 8$. So it suffices to check that for $b = 7$ and $b = 8$, no integers a and c such that $a < b < c$ and $ac = b^2$ and $c < a + b$ exist.

(b) Let K be a triangle satisfying the conditions of the problem. Then $\gcd(a, b, c) = 1$ and $b^2 = ac$. This implies that $\gcd(a, c) = 1$ (as each common prime divisor of a and c would also divide b). Thus, both a and c are perfect squares.

3. In a school tennis tournament with $m \geq 2$ participants, each match consists of 4 sets. A player who wins more than half of all sets during a match gets 2 points for this match. A player who wins exactly half of all sets during the match gets 1 point, and a player who wins less than half of all sets gets 0 points. During the tournament, each participant plays exactly one match against each remaining player. Find the least number of participants m for which it is possible that some participant wins more sets than any other participant but obtains less points than any other participant. *(Juniors.)*

Answer: 6.

Solution. Let $m = 5$. A participant who wins more sets than any other during the tournament must win more than half of all sets he plays. This implies that he must win more sets than his opponent in at least one match, i.e., he must win at least one match. But in order to obtain less points than anyone else, he must lose more matches than win. As each participant plays 4 matches, this special participant must win exactly one match and lose at least two. Under such conditions, he can win at most 8 sets during the tournament but this is not more than half of the number 16 of all sets.

Thus, for $m = 5$, the described situation is impossible. If it were possible for some m such that $m < 5$, we could obtain a suitable tournament table also for $m = 5$ by adding an appropriate number of players whose matches all end in draw.

The following table shows a situation for $m = 6$ where all conditions are fulfilled:

Player							Marks	Sets won
1.		4 : 0	4 : 0	1 : 3	1 : 3	1 : 3	4	11
2.	0 : 4		2 : 2	3 : 1	2 : 2	3 : 1	6	10

3.	0 : 4	2 : 2		2 : 2	3 : 1	2 : 2	5	9
4.	3 : 1	1 : 3	2 : 2		2 : 2	2 : 2	5	10
5.	3 : 1	2 : 2	1 : 3	2 : 2		2 : 2	5	10
6.	3 : 1	1 : 3	2 : 2	2 : 2	2 : 2		5	10

4. Call a k-digit positive integer a *hyperprime* if all its segments consisting of $1, 2, \ldots, k$ consecutive digits are prime. Find all hyperprimes. *(Juniors.)*

Answer: 2, 3, 5, 7, 23, 37, 53, 73, 373.

Solution. One-digit hyperprimes are precisely the one-digit primes 2, 3, 5, 7.

In a larger hyperprime, all digits must be prime. The last digit can be neither 2 nor 5, and no two consecutive digits can be equal (this would form a composite segment). Adding one digit to all one-digit primes, while following these requirements, we obtain numbers 23, 27, 37, 53, 57, 73. Among these, only 23, 37, 53, 73 are primes.

Note that all segments of any hyperprime are hyperprime. Thus all three-digit hyperprimes can be obtained from two-digit hyperprimes by adding one digit to the end. Following the requirements above, we get hyperprime candidates 237, 373, 537, 737, among which only 373 is really a prime and a hyperprime.

Hyperprimes with more than 3 digits are impossible since their every segment of 3 digits should be 373.

5. In an exam with k questions, n students are taking part. A student fails the exam if he answers correctly less than half of all questions. Call a question *easy* if more than half of all students answer it correctly. For which pairs (k, n) of positive integers is it possible that

(a) all students fail the exam although all questions are easy;

(b) no student fails the exam although no question is easy?

(Juniors.)

Answer: (a) there are no such pairs; (b) all pairs (k, n) with both k and n even.

Solution. Let v be the total number of correct answers given by all students.

(a) If all students fail then each of them gives less than $\frac{k}{2}$ correct answers, i.e., $v < n \cdot \frac{k}{2} = \frac{nk}{2}$. If all questions are easy then, for each of them, more than $\frac{n}{2}$ correct answers are given, i.e., $v > k \cdot \frac{n}{2} = \frac{nk}{2}$, contradiction.

(b) If no student fails then each of them gives at least $\frac{k}{2}$ correct answers, i.e., $v \geq n \cdot \frac{k}{2} = \frac{nk}{2}$; the equality holds iff each student gives exactly $\frac{k}{2}$ correct answers. On the other hand, if no question is easy then no more than $\frac{n}{2}$ correct answers are given to each of them, i.e., $v \leq k \cdot \frac{n}{2} = \frac{nk}{2}$, whereby equality holds iff each question gets exactly $\frac{n}{2}$ correct answers. These two inequalities can both hold only if $v = \frac{nk}{2}$. Consequently, each student answers exactly $\frac{k}{2}$ questions correctly and each question is answered correctly

by exactly $\frac{n}{2}$ students. Thus k and n must be even.

It remains to show that the situation is possible for arbitrary even k and n. For this, enumerate the questions by numbers from 1 to k and the students by numbers 1 to n. Let every student with an odd number answer correctly exactly the questions with an odd number and every student with an even number answer correctly exactly the questions with an even number. The requirements are fulfilled.

6. Let $a_n = 1 + 2 + \ldots + n$ for every $n \geq 1$; the numbers a_n are called *triangular*. Prove that if $2a_m = a_n$ then a_{2m-n} is a perfect square. *(Seniors.)*

Solution. We depict a_n as a set of points organized triangularly as shown in Fig. 1. From two ends of the base, separate two triangles both containing a_m points. For counting $2a_m$ points, we count the points in the intersection of the two triangles twice, while leaving the points in the upper rhomboid uncounted; for counting a_n points, every point is taken into account once. Thus if $2a_m = a_n$ then the intersection contains as many points as the rhomboid. The former contains a_{2m-n} points while the latter contains $(n-m)^2$.

Remark. One can also prove the claim algebraically using the formula of the sum of arithmetic progression.

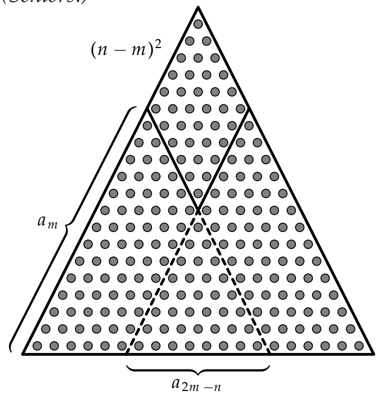


Figure 1

7. Three circles with centres A, B, C touch each other pairwise externally, and touch circle c from inside. Prove that if the centre of c coincides with the orthocentre of triangle ABC, then ABC is equilateral. *(Seniors.)*

Solution 1. Let the tangent point of circles with centres A and B be C', the tangent point of circles with centres B and C be A' and the tangent point of circles with centres C and A be B' (see Fig. **??**). Let A'', B'', C'' be the tangent points of circle c with the circles with centre A, B, C, respectively. Let H and I be the orthocentre and the incentre of ABC, respectively. Assume that H is the centre of c.

We prove first that triangle ABC is acute. Line HA'' passes through A and contains both a radius of c and the altitude of ABC drawn from A. If angle BAC were not acute then the orthocentre of ABC would be on ray AA'' while the centre of c would be outside this ray since $|AA''| < |HA''|$. Analogously, the other angles of ABC must be acute.

Draw tangents to c from points A'', B'', C''. As ABC is acute, point H lies inside it. Hence each of the three arcs of c with endpoints A'', B'', C'' is less than 180°. Consequently, these tangents intersect each other pairwise, forming a triangle DEF whose incircle is c. As both BC and EF are perpendicular to HA'', they are parallel. Analogously, CA and FD are parallel, and AB and DE, too. Thus triangles ABC and DEF are similar.

Prove now that the orthocentre of DEF is I. Points A', B', C' lie on the sides of ABC; it is known that they are also the points where the incircle of ABC touches the sides. Thus IA', $C''D$ and $B''D$ are the radical axes of c and two circles touching it and each other. The radical axes meet at D. Thus ID, IE and IF are perpendicular to BC, CA, AB, respectively, and consequently also to the corresponding sides of DEF.

From all this, we get that the distance between the orthocentre and the incentre is the same in triangles ABC and DEF. As these triangles are similar but not equal, this can happen only if the distance is zero, i.e., the orthocentre and incentre coincide. This implies that ABC is equilateral.

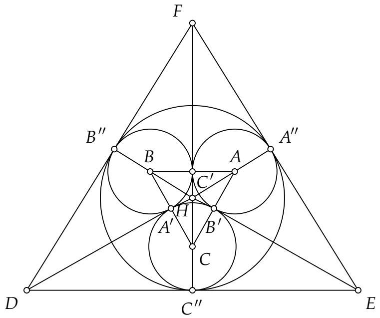


Figure 2

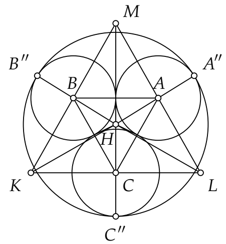


Figure 3

Solution 2. Let r be the radius of c and let r_A, r_B, r_C be the radii of circles with centre A, B, C, respectively. Let h_A be the length of the altitude of ABC drawn from A. Let A', B', C' be the feet of altitudes of triangle ABC drawn from vertices A, B, C, respectively.

From triangle $HA'C$, we get $|HA'|^2 = |CH|^2 - |CA'|^2$; from triangle $AA'C$, we get $|CA'|^2 = |AC|^2 - |AA'|^2$. Here, $|CH| = r - r_C$, $|AC| = r_A + r_C$ and $|AA'| = h_A$. Thus

$$|HA'|^2 = (r - r_C)^2 - (r_A + r_C)^2 + h_A^2 = r^2 - 2(r + r_A)r_C - r_A^2 + h_A^2.$$

Analogously, we obtain

$$|HA'|^2 = r^2 - 2(r + r_A)r_B - r_A^2 + h_A^2.$$

These two equalities together give $r_B = r_C$. Analogously, $r_B = r_A$. Thus the radii of the circles drawn around A, B, C touching pairwise each other are equal. This can be only if the sides of ABC are all equal.

Solution 3. Let r_A, r_B, r_C be defined as in Solution 2. Like in Solution 1, note that H lies inside triangle ABC. As H is the centre of c, radii HA'' and HB'' are equal (see Fig. **??**) which means that $|HA| + r_A = |HB| + r_B$. Adding r_C to both sides of this equality, we get

$$|HA| + |AC| = |HB| + |BC|. \tag{1}$$

Take triangle KLM whose midlines are the sides of ABC; then triangles KLM and ABC are similar. Thereby, AH is the perpendicular bisector of side LM as $LM \parallel BC$ and $|LA| = |AM|$. Also, BH is the perpendicular bisector of side MK. Thus H is the circumcentre of triangle KLM and $\angle MHA = \frac{1}{2}\angle MHL = \angle MKL = \angle CAB$.

From the right triangle MAH, we get $|HA| = r\cos\angle MHA = r\cos\angle CAB$ where r is the radius of the circumcircle of KLM. Analogously, $|HB| = r\cos\angle CBA$. Substituting

these into (**??**), we obtain

$$r\cos\angle CAB + |AC| = r\cos\angle CBA + |BC|. \tag{2}$$

As the opposite angle of a bigger side is bigger in every triangle, $|AC| < |BC|$ would imply $\angle CBA < \angle CAB$ and $\cos\angle CBA > \cos\angle CAB$, leading to $r\cos\angle CAB + |AC| < r\cos\angle CBA + |BC|$ which contradicts (2). Analogously, also $|AC| > |BC|$ cannot be. Consequently, $|AC| = |BC|$. In the same way, we get $|AB| = |AC|$, i.e., triangle ABC is equilateral.

Remark. The claim of the problem holds also without the assumption that the tangency of c with the three circles is inner.

8. Let b be an even positive integer for which there exists a natural number n such that $n > 1$ and $\dfrac{b^n - 1}{b - 1}$ is a perfect square. Prove that b is divisible by 8. *(Seniors.)*

Solution. As b is even, the perfect square $\dfrac{b^n - 1}{b - 1}$ is odd. Hence it is congruent to 8 modulo 1, i.e., the number

$$\frac{b^n - 1}{b - 1} - 1 = b + b^2 + \ldots + b^{n-1} = b\left(1 + b + \ldots + b^{n-2}\right)$$

is divisible by 8. As the factor $1 + b + \ldots + b^{n-2}$ is odd, it is relatively prime to 8 and hence b is divisible by 8.

Remark. One can also prove the claim by considering b modulo 8.

9. The *Fibonacci sequence* is determined by conditions $F_0 = 0$, $F_1 = 1$, and $F_k = F_{k-1} + F_{k-2}$ for all $k \geq 2$. Let n be a positive integer and let $P(x) = a_m x^m + \ldots + a_1 x + a_0$ be a polynomial that satisfies the following two conditions:

(1) $P(F_n) = F_n^2$; (2) $P(F_k) = P(F_{k-1}) + P(F_{k-2})$ for all $k \geq 2$.

Find the sum of the coefficients of P. *(Seniors.)*

Answer. F_n.

Solution. We are asked to find $P(1)$. If $n = 1$ then $P(1) = P(F_1) = F_1^2 = 1$, giving $P(1) = F_1$. If $n \geq 2$ then using condition $P(F_k) = P(F_{k-1}) + P(F_{k-2})$, $2 \leq k \leq n$, repeatedly, we get

$$\begin{aligned} P(F_n) &= P(F_{n-1}) + P(F_{n-2}) = 2P(F_{n-2}) + P(F_{n-3}) = 3P(F_{n-3}) + 2P(F_{n-4}) = \ldots \\ &= F_n P(F_1) + F_{n-1} P(F_0) = F_n P(1) + F_{n-1} P(0). \end{aligned}$$

Using the given condition again for $k = 2$, we obtain

$$P(1) = P(F_2) = P(F_1) + P(F_0) = P(1) + P(0),$$

which gives $P(0) = 0$. Altogether, $F_n^2 = P(F_n) = F_n P(1)$, showing that $P(1) = F_n$.

Remark. We could ask whether there exist polynomials for every n satisfying the conditions of the problem. Using the condition $P(F_k) = P(F_{k-1}) + P(F_{k-2})$, $2 \leq k \leq n$, for finding the other values $P(F_k)$, $2 \le k \le n$, we get $P(F_k) = F_n F_k$, for all k such that $0 \le k \le n$. Elsewhere, the values of the polynomial are not determined. One suitable polynomial is $P(x) = F_n x$.

10. Does there exist a natural number n such that $n > 2$ and the sum of squares of some n consecutive integers is a perfect square? *(Seniors.)*

Answer: yes.
Solution 1. For $n = 11$, we can construct the following example:

$$(-4)^2 + (-3)^2 + (-2)^2 + (-1)^2 + 0^2 + 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 = 11^2.$$

Solution 2. It is easy to prove by induction that $1^2 + 2^2 + \ldots + n^2 = \frac{n}{6}(n+1)(2n+1)$. If $n = 24$ then all three factors in the last product are perfect squares (4, 25 and 49, respectively). Thus the product is also a perfect square.
Remark 1. For $n = 3, \ldots, 10$, no suitable examples exist because there is a number modulo which no sum of squares of n consecutive integers is a quadratic residue.

n	sum	expression	bad modulus
3	$(a-1)^2 + \ldots + (a+1)^2$	$3a^2 + 2$	3
4	$(a-1)^2 + \ldots + (a+2)^2$	$4a^2 + 4a + 6$	4
5	$(a-2)^2 + \ldots + (a+2)^2$	$5a^2 + 10$	25
6	$(a-2)^2 + \ldots + (a+3)^2$	$6a^2 + 6a + 19$	4
7	$(a-3)^2 + \ldots + (a+3)^2$	$7a^2 + 28$	49
8	$(a-3)^2 + \ldots + (a+4)^2$	$8a^2 + 8a + 44$	16
9	$(a-4)^2 + \ldots + (a+4)^2$	$9a^2 + 60$	9
10	$(a-4)^2 + \ldots + (a+5)^2$	$10a^2 + 10a + 85$	25

Remark 2. Using diophantine equation theory, it has been proven that the sum of squares of numbers from 1 to n is a perfect square only for $n = 0$, $n = 1$ and $n = 24$. Thus the choice $n = 24$ in Solution 2 is the only possibility to succeed.
Remark 3. Sloane's *Encyclopedia of Integer Sequences* contains the sequence A001032 with description "Numbers n such that the sum of squares of n consecutive positive integers can be a perfect square" (i.e., in addition to the conditions of our problem, the numbers whose squares are considered must be positive). The sequence starts with numbers 1, 2, 11, 23, 24, 26, 33, 47, 49, 50, 59, 73, 74, 88, 96, 97, 107, 121, 122, 146, 169, 177, 184, 191, 193, 194, 218, 239, 241, 242, 249, 289, 297, 299, 311, 312, 313, 337, 338, 347, 352, 361, 362, 376, 383, 393, 407, 409, 431, 443, 457, 458, 479, 481, 491.

11. Tangents l_1 and l_2 common to circles c_1 and c_2 intersect at point P, whereby tangent points remain to different sides from P on both tangent lines. Through some point T, tangents p_1 and p_2 to circle c_1 and tangents p_3 and p_4 to circle c_2 are drawn. The intersection points of l_1 with lines p_1, p_2, p_3, p_4 are A_1, B_1, C_1, D_1, respectively, whereby the order of points on l_1 is: A_1, B_1, P, C_1, D_1. Analogously, the intersection points of l_2 with lines p_1, p_2, p_3, p_4 are A_2, B_2, C_2, D_2, respectively. Prove that if both quadrangles $A_1A_2D_1D_2$ and $B_1B_2C_1C_2$ are cyclic then radii of c_1 and c_2 are equal. *(Seniors.)*

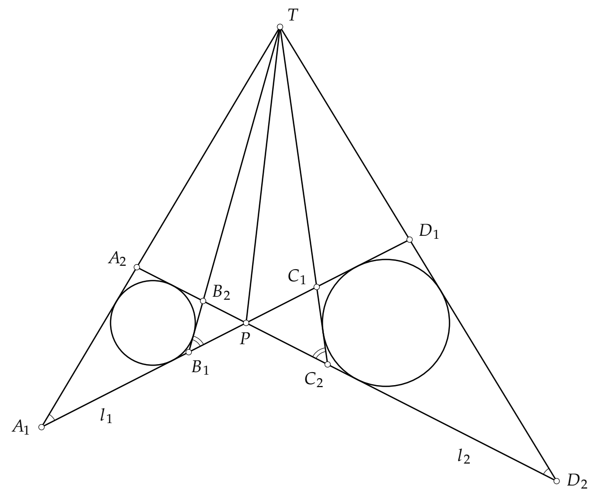


Figure 4

Solution 1. Let r_1 and r_2 be the radii of c_1 and c_2, respectively. Since quadrangle $A_1A_2D_1D_2$ is cyclic, $\angle A_2A_1D_1 = \angle D_1D_2A_2$. In addition, $\angle A_1PA_2 = \angle D_2PD_1$ (see Fig. 4). Thus triangles A_1A_2P and D_2D_1P are similar. Their incircles are c_1 and c_2, hence

$$\frac{|A_1P|}{|D_2P|} = \frac{r_1}{r_2}.$$

Since quadrangle $B_1B_2C_1C_2$ is cyclic, $\angle B_2B_1C_1 = \angle C_1C_2B_2$, whence $\angle TB_1A_1 = \angle TC_2D_2$. As also $\angle TA_1B_1 = \angle TD_2C_2$, triangles A_1B_1T and D_2C_2T are similar. Their incircles are c_1 and c_2 again, hence

$$\frac{|A_1T|}{|D_2T|} = \frac{r_1}{r_2}.$$

Now consider triangles A_1PT and D_2PT. We have $\frac{|A_1P|}{|D_2P|} = \frac{|A_1T|}{|D_2T|}$ and $\angle TA_1P = \angle TD_2P$. Thus these triangles are similar and

$$\frac{r_1}{r_2} = \frac{|A_1P|}{|D_2P|} = \frac{|PT|}{|PT|} = 1.$$

Solution 2. Let r_1 and r_2 be the radii of c_1 and c_2, respectively. Let s be the line that passes through P and is perpendicular to the line joining the centres of c_1 and c_2. Consider the composition of two plane transformations: reflection w.r.t. s and homothety w.r.t. P with factor $\frac{r_2}{r_1}$. This composite transformation takes c_1 to c_2 and P to P.
Denote the image of any point X under this transformation by X'. As the transformation respects all angles between lines, the equality of angles in cyclic quadrilateral $B_1B_2C_1C_2$

implies $\angle B_2'B_1'P = \angle B_2B_1P = \angle C_1C_2P$. As lines $B_1'B_2'$ and C_2C_1 both touch circle c_2 and intersect PC_2 under the same angle, line $B_1'B_2'$ coincides with line C_2C_1. Analogously, $A_1'A_2'$ coincides with line D_2D_1.
Hence the transformation takes the intersection point of lines A_1A_2 and B_1B_2 to the intersection point of lines C_2C_1 and D_2D_1, i.e., T is taken to T. But if $r_1 \neq r_2$ then the transformation obviously can have only one fixpoint. Consequently, $r_1 = r_2$.

12. Find all positive integers n such that one can write an integer 1 to n^2 into each unit square of a $n^2 \times n^2$ table in such a way that, in each row, each column and each $n \times n$ block of unit squares, each number 1 to n^2 occurs exactly once. *(Seniors.)*

Answer: 1.
Solution. In the case $n = 1$, the conditions can be fulfilled trivially. Assume $n \geq 2$.

Denote the unit square in the ith row and jth column by (i, j). Let A be the $n \times n$ block containing both $(1, 1)$ and (n, n). Let B and C be the $n \times n$ blocks obtained from A by shifting it by one unit down and right, respectively (see Fig. 5).

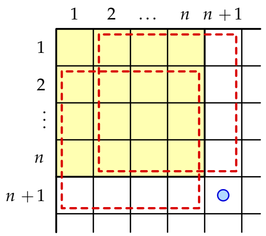


Figure 5

The numbers in the bottommost row of B must be the same as the numbers in the topmost row of A in some order since both blocks must contain each number $1, \ldots, n^2$ exactly once. Analogously, the rightmost column of C must contain the same numbers as the leftmost column of A in some order. Now, the number in $(n+1, n+1)$ cannot occur in the topmost row of A since all these occur already in the row number $n+1$ left from the square under consideration. Analogously, this number cannot occur in the leftmost column of A. As A contains all numbers 1 to n^2, this number must occur elsewhere in A. But then it occurs twice in the $n \times n$ block that contains squares $(2, 2)$ and $(n+1, n+1)$ which is prohibited.

13. Consider triangles whose each side length squared is a rational number. Is it true that

(a) the square of the circumradius of every such triangle is rational;
(b) the square of the inradius of every such triangle is rational?

(Seniors.)

Answer: (a) yes; (b) no.
Solution 1. (a) Fix a triangle from the family under consideration. Let a, b, c be its side lengths, γ the size of the angle opposite to the last side and R the circumradius. The cosine law gives $\cos\gamma = \dfrac{a^2+b^2-c^2}{2ab}$, whence $\cos^2\gamma = \dfrac{(a^2+b^2-c^2)^2}{4a^2b^2}$. As a^2, b^2, c^2 are rational, also $\cos^2\gamma$ is rational. Therefore $1 - \cos^2\gamma$, i.e., $\sin^2\gamma$, is rational. The sine law gives $R = \dfrac{c}{2\sin\gamma}$, hence $R^2 = \dfrac{c^2}{4\sin^2\gamma}$. Thus R^2 is rational.
(b) Consider the right isosceles triangle with side lengths 1, 1, $\sqrt{2}$. Let r be its inradius. The area of this triangle, computed via leg lengths, is $\dfrac{1}{2}$; computing the area via the inradius and the perimeter gives $\dfrac{2+\sqrt{2}}{2}r$. Thus $\dfrac{1}{2} = \dfrac{2+\sqrt{2}}{2}r$, whence $r = \dfrac{1}{2+\sqrt{2}} = \dfrac{2-\sqrt{2}}{2} = 1 - \dfrac{\sqrt{2}}{2}$. Therefore $r^2 = 1 + \dfrac{1}{2} - \sqrt{2} = \dfrac{3}{2} - \sqrt{2}$, so r^2 is not rational.
Solution 2. (a) By the formula $S = \dfrac{abc}{4R}$, we get $R^2 = \dfrac{a^2b^2c^2}{16S^2}$. Thus it suffices to prove that S^2 is rational. By Heron's formula,

$$S^2 = p(p-a)(p-b)(p-c) = \frac{1}{16}\Big(\big(2a^2(b^2+c^2) - (b^2-c^2)^2 - (a^2)^2\big)\Big).$$

The last expression clearly evaluates to a rational number.
(b) In part (a), it was proven that S^2 is rational. For the triangle with side lengths 1, 1 and $\sqrt{2}$, we get $p^2 = \left(\dfrac{2+\sqrt{2}}{2}\right)^2 = \dfrac{6+4\sqrt{2}}{4} = \dfrac{3}{2} + \sqrt{2}$, hence p^2 is not rational and neither is $r^2 = \dfrac{S^2}{p^2}$.

Selected Problems from the Final Round of National Olympiad

1. Two medians drawn from vertices A and B of triangle ABC are perpendicular. Prove that side AB is the shortest side of ABC. *(Grade 9.)*

Solution. Let the medians intersect in point M and let the median drawn from vertex C intersect AB in point F (see Fig. 6). Then, F is the midpoint of the hypotenuse of the right triangle ABM and thus the midpoint of the circumcircle of ABM, so we get $|AB| = 2|FM|$. Since M divides median CF in ratio $2:1$, we have $|AB| = |CM|$. The largest angle of triangle AMC is the obtuse angle AMC, therefore AC is the longest side of this triangle. We get $|AC| > |MC| = |AB|$. The proof of $|BC| > |AB|$ is analogous.
Remark. One can also solve the problem using the Pythagorean theorem and the fact that the centroid divides the medians in ratio $2:1$.

2. Juhan wants to order by weight five balls of pairwise different weight, using only a balance scale. First, he labels the balls with numbers 1 to 5 and creates a list of weighings, such that each element in the list is a pair of two balls. Then, for every pair in the list, he weighs the two balls against each other. Can Juhan sort the balls by weight, using a list with less than 10 pairs? *(Grade 9.)*

Answer: no.
Solution. There are 10 possible pairs of 5 balls. Suppose w.l.o.g. that Juhan does not weigh the pair $(1, 2)$. Then, it is not possible to distinguish orderings $1, 2, 3, 4, 5$ and $2, 1, 3, 4, 5$, since the remaining 9 weighings give the same result. Thus, Juhan's list must contain all 10 pairs.

3. Two radii OA and OB of a circle c with midpoint O are perpendicular. Another circle touches c in point Q and the radii in points C and D, respectively. Determine $\angle AQC$. *(Grade 10.)*

Answer: 45°.

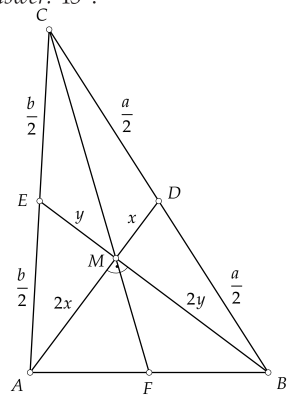


Figure 6

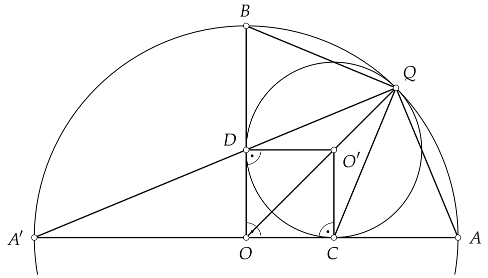


Figure 7

Solution. By symmetry, $\angle AQC = \angle BQD$ (see Fig. 7). Since AQB is an internal angle of a regular octagon, we have $\angle AQB = 135^\circ$. Let now O' be the midpoint of the circle through C, D and Q. The quadrilateral $OCO'D$ has three right angles: $\angle COD = 90^\circ$ by assumption, while angles OCO' and ODO' are angles between a radius and a tangent. Thus, $CO'D$ is also a right angle, so $\angle CQD = \frac{1}{2}\angle CO'D = 45^\circ$ and

$$\angle AQC = \frac{1}{2}(\angle AQB - \angle CQD) = \frac{1}{2}(135^\circ - 45^\circ) = 45^\circ.$$

Remark. One can also make use of the similarity of triangles $QO'D$ and QOA' (where A' is the other endpoint of the diameter of c drawn through A), yielding that angle AQA' subtends the diameter. Another approach is to apply the tangent chord property on tangent AC and secant CD to obtain that triangles AQC and CQD are similar.

4. Prove that the sum of the squares of any three pairwise different positive odd integers can be represented as the sum of the squares of six (not necessarily different) positive integers. *(Grade 10.)*

Solution. Let $a > b > c$ be positive integers. Then

$$a^2 + b^2 + c^2 = \frac{a^2}{2} + \frac{b^2}{2} + \frac{b^2}{2} + \frac{c^2}{2} + \frac{c^2}{2} + \frac{a^2}{2} =$$
$$= \left(\frac{a+b}{2}\right)^2 + \left(\frac{a-b}{2}\right)^2 + \left(\frac{b+c}{2}\right)^2 + \left(\frac{b-c}{2}\right)^2 + \left(\frac{a+c}{2}\right)^2 + \left(\frac{a-c}{2}\right)^2.$$

Since a, b and c are all odd, the latter is a sum of squares of six positive integers.

5. Two triangles are drawn on a plane in such a way that the area covered by their

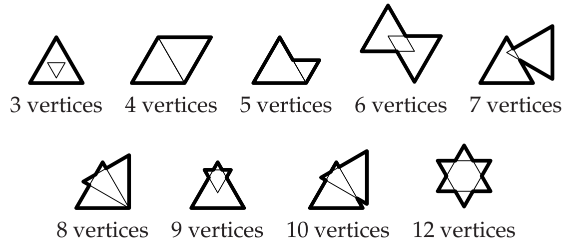


Figure 8

union is an n-gon (not necessarily convex). Find all possible values of the number of vertices n. *(Grade 10.)*

Answer: all integers from 3 to 12, except 11.

Solution. The n-gon must have at least 3 vertices. We show first that the number of vertices is at most 12. Indeed, each vertex of the n-gon is either a vertex of one of the triangles or an intersection point of some two sides of the two triangles. There are 6 triangle vertices and 6 possible intersection points, since every side of the first triangle can intersect at most two sides of the second triangle. Thus, $n \leq 12$.

Next, suppose that the n-gon has 11 vertices. If 6 of those vertices are vertices of the two triangles, then neither triangle can contain a vertex of the other triangle. Thus, each side of each triangle intersects the other triangle either never or twice, so we cannot have an odd number of intersection points. On the contrary, if the n-gon has 6 intersection points as vertices, every side of each triangle must intersect the second triangle twice, and thus all vertices of one triangle must be outside the other triangle. Thus, all 6 triangle vertices are also vertices of the n-gon, and $n = 12$.

The remaining configurations from 3 to 12 are all possible (see Fig. 8).

6. The identifier of a book is an n-tuple of numbers $0, 1, \ldots, 9$, followed by a checksum. The checksum is computed by a fixed rule that satisfies the following property: whenever one increases a single number in the n-tuple (without modifying the other numbers), the checksum also increases. Find the smallest possible number of required checksums if all possible n-tuples are in use. *(Grade 10.)*

Answer: $9n + 1$.

Solution. Consider the checksum of $(0, 0, \ldots, 0)$. Increasing the first number, we get n-tuples $(1, 0, \ldots, 0)$, $\ldots$, $(9, 0, \ldots, 0)$ with 9 new checksums. Increasing the second number in the last n-tuple, we get $(9, 1, \ldots, 0)$, $\ldots$, $(9, 9, \ldots, 0)$, and again obtain 9 new values. Continuing like this, we see that the number of different check values is at least $9n + 1$.

It is easy to see that the sum of all numbers in the n-tuple is a valid checksum. On the other hand, the sum of n numbers $0, 1, \ldots, 9$ is at least 0 and at most $9n$, so with this rule, we have exactly $9n + 1$ different checksums.

7. Find all real numbers a such that all solutions to the quadratic equation $x^2 - ax + a = 0$ are integers. *(Grade 11.)*

Answer: $a = 0$ and $a = 4$.
Solution. Let x and y be the solutions of the quadratic equation. Viète formulae give $x + y = xy = a$. The case $y = 1$ gives a contradiction $1 + x = x$, while $y \neq 1$ gives $x = \dfrac{y}{y-1}$. Thus, x is an integer if and only if $y = 2$ or $y = 0$. Now, $y = 2$ gives $x = 2$ and $a = x + y = 4$; $y = 0$ gives $x = 0$ and $a = x + y = 0$.
Remark. One can also proceed from the fact that the sum a as well as the difference $\sqrt{a^2 - 4a} = \sqrt{(a-2)^2 - 4}$ of the solutions is integral.

8. A 3-dimensional chess board consists of $4 \times 4 \times 4$ unit cubes. A rook can step from any unit cube K to any other unit cube that has a common face with K. A bishop can step from any unit cube K to any other unit cube that has a common edge with K, but does not have a common face. One *move* of both a rook and a bishop consists of an arbitrary positive number of consecutive steps in the same direction. Find the average number of possible moves for either piece, where the average is taken over all possible starting cubes K. *(Grade 11.)*

Answer: the rook has on average 9 moves, the bishop has 10.5.

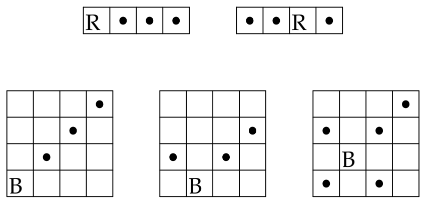


Figure 9

Solution. The rook has always 3 possible moves in the direction of one axis, regardless of the choice of the starting cube K (see Fig. 9), and thus 9 possible moves on average.
In any of the 8 "middle" cubes of the chess board, the bishop has 5 possible moves in the direction of every axis, and thus 15 moves in total. In any of the 24 cubes that have a common face with a middle cube, the bishop has 5 moves on the plane parallel to the common face, and 3 moves on each of the remaining two planes, or 11 moves in total. For the remaining 32 cubes on the edge of the board, the bishop has 3 moves in the direction of every axis, and 9 moves in total. Averaging over all cubes, we get the result.
Remark. The number of possible moves shows to some extent the strength of every piece. The solution to this problem implies that in 3-dimensional chess, a bishop is perhaps stronger than a rook (as opposed to regular chess). On the other hand, the bishop is weakened by the fact that it can always reach only half of the cubes or squares.
While a 3-dimensional $4 \times 4 \times 4$ board and a regular 8×8 board have the same number of cells, a regular rook and bishop have 14 and 8,75 moves on average, respectively.

9. A circle passing through the endpoints of the leg AB of an isosceles triangle ABC intersects the base BC in point P. A line tangent to the circle in point B intersects the circumcircle of ABC in point Q. Prove that P lies on line AQ if and only if AQ and BC are perpendicular. *(Grade 11.)*

Solution 1. Let line AQ intersect base BC in point R (see Fig. 10). On one hand, ABC is isosceles, so $\angle ACR = \angle ABP$. On the other hand, the tangent chord property gives

$\angle CAR = \angle CBQ = \angle PAB$, so triangles ACR and ABP are similar. If $AQ \perp BC$, then AR is the height of ABC and $\angle ARC = \angle APB = 90^\circ$, so $P = R$ and thus P lies on AQ. If P lies on AQ, then again $P = R$ and $\angle CRA = \angle CPA = 180^\circ - \angle BPA = 180^\circ - \angle CRA$, or $\angle CRA = 90^\circ$, so $AQ \perp BC$.

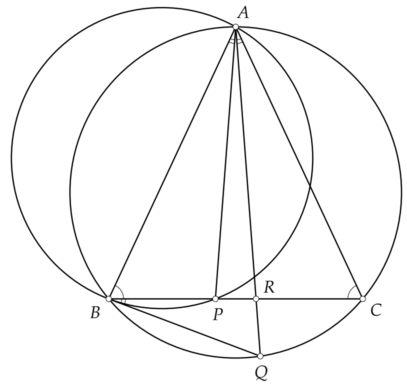


Figure 10

Solution 2. First consider the case where AB is the diameter of the circle through A, B and P. Then $\angle APB = 90^\circ$ and $\angle QBA = 90^\circ$ giving that AQ is the diameter of circumcircle of ABC. Thus $AQ \perp BC$ and P lies on line AQ.
On the other hand, when the centre of the circle passing through points A and B moves away from the leg AB on the halfplane containing point C, points Q and P will move toward points B and C, respectively. When the centre of the circle moves away from AB on the other halfplane, points Q and P will move toward points C and B, respectively. Therefore in either case point P neither lies on line AQ nor AQ is perpendicular to BC.

10. Find all pairs (m, n) of positive integers such that $m^n - n^m = 3$. *(Grade 11.)*

Answer: $(4, 1)$.
Solution. First, m and n must have different parity, for otherwise the lhs is even.

- If m is odd and n even, then $m^n \equiv 1 \pmod 4$, so $n^m \equiv 2 \pmod 4$, which is possible only for $m = 1$. But then $m^n = 1$ and the lhs is smaller than 3. Thus, in this case there are no solutions.
- If now m is even and n odd, then $n^m \equiv 1 \pmod 8$, so $m^n \equiv 4 \pmod 8$. Thus, $n \leq 2$, and n odd gives $n = 1$, $m = 4$ as the only solution.

11. Some circles of radius 2 are drawn on the plane. Prove that the numerical value of the total area covered by these circles is at least as big as the total length of arcs bounding the area. *(Grade 11.)*

Solution 1. The boundary line of the area consists of circular arcs, each corresponding to a circular sector. According to the formula $S = \dfrac{rl}{2}$, where r is the radius and l the length of arc, we obtain that in the case $r = 2$ the area of circular sector equals numerically the length of the circular arc on its boundary.
We prove that no two such sectors have common interior points. First, the sectors from the same circle share only the centre of the circle. Now let one sector be part of circle w_1 with centre O_1 and the other be part of circle w_2 with centre O_2. Assuming that the sectors have a common interior point C, we draw the radii O_1A and O_2B of circles w_1 and w_2 through C, respectively. Obviously A and B lie on the boundary of the area covered by circles, meaning that B and A do not lie in the interior of w_1 and w_2, respectively.

Therefore $|O_1B| \geq |O_1A|$ and $|O_2A| \geq |O_2B|$. We draw the mediator of segment AB (see Fig. 11). The inequalities imply that O_1 and A lie on the one side of the mediator and O_2 and B lie on the other side. Hence segments O_1A and O_2B cannot have common points, a contradiction.

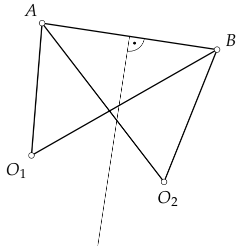


Figure 11

We have obtained that the total length of the boundary line of the area equals the sum of the lengths of circular arcs. But the total area is greater or equal than the sum of the areas of the sectors.

Solution 2. We shall use induction on the number of circles. For one circle, the circumference and the area are equal (4π). We prove that whenever we add a circle, the area a covered by the intersection of the new circle with the old area is at most as big as the perimeter p of that intersection.

First, notice that $a \leq 4\pi$. Assume now that also $p \leq 4\pi$. We write $p = 2\pi r$, so $r \leq 2$. Since a circle maximizes the area for a fixed perimeter, we see that the area of the intersection is $a \leq \pi r^2 \leq 2\pi r = p$ as desired.

12. Consider a cylinder and a cone with a common base such that the volume of the part of the cylinder enclosed in the cone equals the volume of the part of the cylinder outside the cone. Find the ratio of the height of the cone to the height of the cylinder. *(Grade 12.)*

Answer: $1 + \dfrac{1}{\sqrt{3}}$.

Solution. Denote by v and V, h and H the volume and the height of the cylinder and the cone, respectively, and denote by S the area of the common base. The vertex of the cone must lie outside the cylinder, for otherwise the volume of the intersection would be at most $\dfrac{1}{3}$ of the total volume of the cylinder.

Denote $\dfrac{h}{H} = x$. The part of the cone that lies outside the cylinder is a cone similar to the original cone with scale factor $\dfrac{H-h}{H} = 1 - x$ and volume $(1-x)^3V$. The volume of the part of the cylinder inside the cone is thus $V - (1-x)^3V = \dfrac{v}{2}$. From $v = Sh$ and $V = \dfrac{1}{3}SH$ we get $(x^3 - 3x^2 + 3x) = \dfrac{3}{2}x$, so $x = \dfrac{3 \pm \sqrt{3}}{2}$. Since $h < H$ implies $x < 1$, the only possible solution is $x = \dfrac{3-\sqrt{3}}{2}$ and we get the desired ratio.

13. Let x, y, z be positive real numbers such that x^n, y^n and z^n are side lengths of some triangle for all positive integers n. Prove that at least two of x, y and z are equal. *(Grade 12.)*

Solution. Assume that x, y, z are all different and assume w.l.o.g. $x < y < z$. For any n, the triangle inequality implies $x^n + y^n > z^n$, or

$$\left(\frac{x}{y}\right)^n + 1 > \left(\frac{z}{y}\right)^n.$$

Since $\dfrac{x}{y} < 1$, $\left(\dfrac{x}{y}\right)^n + 1 < 2$ holds for all n. On the other hand, since $\dfrac{z}{y} > 1$, there exists an integer N such that $\left(\dfrac{z}{y}\right)^N > 2$, contradiction.

14. Does there exist an equilateral triangle

(a) on a plane; (b) in a 3-dimensional space;

such that all its three vertices have integral coordinates? *(Grade 12.)*

Answer: (a) no; (b) yes.

Solution 1. (a) Suppose that such a triangle ABC exists. Then vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ have integral coordinates. Denote $\overrightarrow{AB} = (x, y)$, then the vector of the height drawn from vertex C is $\dfrac{\sqrt{3}}{2}(y, -x)$. The coordinates of $\overrightarrow{AC}$ are then either

$$\overrightarrow{AC} = \frac{1}{2}(x,y) + \frac{\sqrt{3}}{2}(y,-x) = \left(\frac{x+\sqrt{3}y}{2}, \frac{y-\sqrt{3}x}{2}\right)$$

or

$$\overrightarrow{AC} = \frac{1}{2}(x,y) - \frac{\sqrt{3}}{2}(y,-x) = \left(\frac{x-\sqrt{3}y}{2}, \frac{y+\sqrt{3}x}{2}\right).$$

In either case, the coordinates are integral only for $x = y = 0$.

(b) Triangle ABC with $A = (1,0,0)$, $B = (0,1,0)$ and $C = (0,0,1)$ is equilateral.

Solution 2. (a) W.l.o.g. assume that one vertex of the triangle is $A(0,0)$. We also assume w.l.o.g. that the ordinates of B and C are non-negative and these vertices do not lie on the y-axis.

Since all the coordinates are integral, the slopes of AB and AC are rational. Denote the slope angles of lines AB and AC by β and γ, respectively, then $\gamma = \beta \pm 60^\circ$. Denote $\tan\beta = k$. We have

$$\tan\gamma = \frac{\tan\beta \pm \tan 60^\circ}{1 \mp \tan\beta \tan 60^\circ} = \frac{k \pm \sqrt{3}}{1 \mp k\sqrt{3}} = \frac{(k \pm \sqrt{3})(1 \pm k\sqrt{3})}{1 - 3k^2}.$$

But $(k \pm \sqrt{3})(1 \pm k\sqrt{3}) = k \pm k^2\sqrt{3} \pm \sqrt{3} + 3k = 4k \pm (k^2+1)\sqrt{3}$. As $k^2 + 1 > 0$, it is impossible that $\tan\gamma$ and k would be simultaneously rational.

Remark. There are several other solutions to this problem. One may w.l.o.g. denote the vertices $A(0,0)$, $B(x_1, y_1)$ and $C(x_1, y_1)$, where x_1, y_1, x_2 and y_2 are relatively prime, derive the equalities $x_1^2 + y_1^2 = x_2^2 + y_2^2 = 2(x_1x_2 + y_1y_2)$ and consider all cases of remainders of x_1, y_1, x_2 and y_2 modulo 2.

Another approach would be to notice that $(x_1 + x_2)^2 + (y_1 + y_2)^2$ is divisible by 3 and use the fact that perfect squares have only remainders 0 and 1 modulo 3.

The most straightforward way is to compute the area S of the triangle with vertices $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ in two different ways: $2S = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix}$ and $2S =$

$\dfrac{a^2\sqrt{3}}{2}$ (where a is the side of the triangle). Since all the coordinates as well as the side length are integral, such a triangle cannot exist.

15. Let a, b, c be positive integers such that $\gcd(a, b, c) = 1$ and the product of every two of these integers is divisible by the third one.

(a) Prove that every one of these integers equals the least common multiple of the remaining two integers divided by the greatest common divisor of these two integers.
(b) Give an example of such integers $a > 1$, $b > 1$ and $c > 1$.

(Grade 12.)

Answer: (b) For example, $a = 6, b = 10, c = 15$.
Solution 1. (a) Let $d = \gcd(a, b)$, $a = a'd$ and $b = b'd$, where $\gcd(a', b') = 1$. Then $\operatorname{lcm}(a, b) = a'b'd$ and $\dfrac{\operatorname{lcm}(a,b)}{\gcd(a,b)} = a'b'$. We prove that $a'b' = c$.
On the one hand, since $ab = a'b'd^2$ is divisible by c and $\gcd(d, c) = 1$ because of $\gcd(a, b, c) = 1$, $a'b'$ must be divisible by c. On the other hand, the conditions of the problem imply that $ca = ca'd$ is divisible by $b = b'd$, i.e. ca' is divisible by b'. As $\gcd(a', b') = 1$, c is divisible by b'. Analogously, c is divisible by a'. Since a' and b' are coprime, c is divisibly by $a'b'$. Altogether we have $c = a'b'$ as desired. The claims about a and b are proven analogously.
(b) Let x, y and z be different numbers that are pairwise coprime, e.g. different primes. Having $a = xy$, $b = yz$ and $c = xz$, the numbers a, b and c satisfy the conditions of the problem. Indeed,

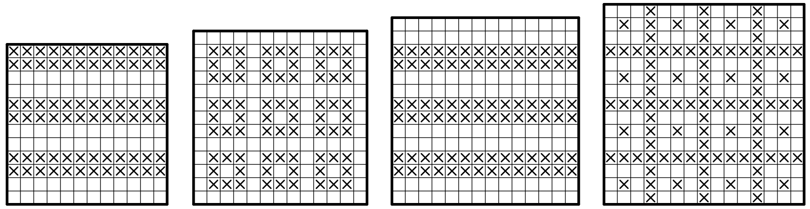
$$\frac{\operatorname{lcm}(a,b)}{\gcd(a,b)} = \frac{xyz}{y} = xz = c,$$

analogously for other cases. For an example we may take $x = 2, y = 3, z = 5$ that yields 6, 15 and 10.
Solution 2. (a) Let p be some prime factor of at least one of a, b and c. Since $\gcd(a, b, c) = 1$, we may assume w.l.o.g. that p does not divide c. At the same time, since ca is divisible by b and cb is divisible by a, the factor p must have the same exponent α in both a and b. Similarly, the exponents of any prime factor p' in the prime factorization of a, b and c are α', α' and 0 in some order. Since every exponent in such a triple equals the difference of the maximum and the minimum of the remaining two, the result follows.

16. Some squares of an $n \times n$ grid are marked in such a way that in every 4×4 square, at least half of the squares are marked. Find the smallest possible number of marked squares in the grid. *(Grade 12.)*

Answer: $8q^2$ for $n = 4q$ or $n = 4q + 1$; $8q^2 + 4q$ for $n = 4q + 2$; $8q^2 + 8q + 1$ for $n = 4q + 3$ and $q \geq 1$; 0 for $n = 3$.
Solution. Let $A(n)$ be the smallest possible number of marked squares. Write $n = 4q + r$, where $0 \leq r < 4$. First, we bound $A(n)$ from above.

- If $r = 0$, we mark half of the squares in groups of two rows as shown in Fig. 12, so $A(4q) \leq 8q^2$.



$n = 4q$ $\quad$ $n = 4q + 1$ $\quad$ $n = 4q + 2$ $\quad$ $n = 4q + 3$

Figure 12

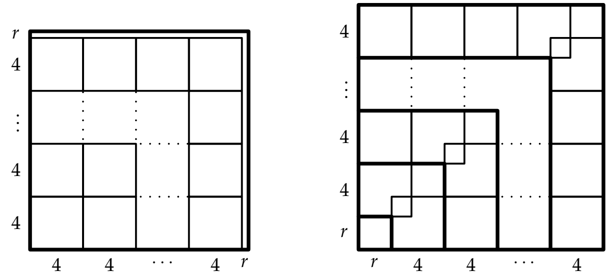


Figure 13

- If $r = 1$, we mark groups of 8 squares. Since there are q^2 such groups, we again get $A(4q + 1) \leq 8q^2$.
- If $r = 2$, we mark as in the first case, but leave the last two rows empty. This way, we mark $2q$ rows with n squares, or $2q(4q + 2)$ squares in total. Thus, $A(4q + 2) \leq 8q^2 + 4q$.
- If $r = 3$ and $q > 0$, we mark every fourth row and column and the middle squares of the remaining 3×3 groups. There are $(q + 1)^2$ marked middle squares and q marked rows and columns, giving $(q + 1)^2 + 2(4q + 3)q - q^2$ marked squares in total. Thus, $A(4q + 3) \leq 8q^2 + 8q + 1$. For $q = 0$, we get $n = 3$ and $A(3) = 0$.

We now show that these bounds are tight.

- For $r = 0$ or $r = 1$, we can divide a $4q \times 4q$ grid into q^2 squares of size 4×4. In every such square, at least 8 squares must be marked, so $A(n) \geq 8q^2$.
- For $r = 2$ or $r = 3$, cut an $r \times r$ square from the lower left corner and divide the remaining squares into q L-shaped strips of width 4 (see Fig. 13, right). The ith strip then contains $2i$ 4×4 squares. The last two squares intersect in the corner of the L and the intersection is a square of size $(4 - r) \times (4 - r)$. Every strip also contains an $r \times r$ square in the corner.

  In order to bound $A(n)$ from below, we add the smallest possible number of marked squares in the 4×4 squares and the $r \times r$ squares and subtract the largest possible number of marked squares in the $(4 - r) \times (4 - r)$ squares.

The case $r = 2$ gives

$$A(n) \geq 8 \cdot (2+4+\ldots+2q) + 0 \cdot (q+1) - 4 \cdot q = 8 \cdot \frac{(2q+2)q}{2} - 4q = 8q^2 + 4q.$$

If $r = 3$ every 3×3 square contains at least one marked square, so

$$A(n) \geq 8 \cdot (2+4+\ldots+2q) + 1 \cdot (q+1) - 1 \cdot q = 8 \cdot \frac{(2q+2)q}{2} + 1 = 8q^2 + 8q + 1.$$

IMO team selection contest

First day

1. On the control board of a nuclear station, there are n electric switches ($n > 0$), all in one row. Each switch has two possible positions: up and down. The switches are connected to each other in such a way that, whenever a switch moves down from its upper position, its right neighbour (if it exists) automatically changes position. At the beginning, all switches are down. The operator of the board first changes the position of the leftmost switch once, then the position of the second leftmost switch twice etc., until eventually he changes the position of the rightmost switch n times. How many switches are up after all these operations?

Answer: 1.

Solution 1. Enumerate the switches with numbers 1 to n from left to right. We prove first that the result of two consecutive changes does not depend on the order of the changes. Let x and y be the numbers of the switches changed, $x < y$.

- If there exists a number z such that $x \leq z < y$ and switch number z is down then changing the position of x can influence only switches from x to z, changing the position of switch y can influence only this switch and switches right from y. Thus the results of the changes are independent of each other.
- If no such z exists then changing switch number x causes a change of switch number y. After that, switches x to $y-1$ are all down while all switches in the right from them are in the same position as if switch number y were changed. Thus after moving both x and y in either order, switches from x to $y-1$ are down and the switches with larger number are in the position as when switch y were moved twice.

We prove now that, after all operations, precisely the leftmost switch is up. This claim holds trivially for $n = 1$. Assume the claim holding for n switches and consider a board with $n+1$ switches. According to what was proven above, the moves can be performed in arbitrary order. Therefore, first change switch number 2 once, then switch number 3 twice etc., until the last switch n times. By the induction hypothesis, switch number 2 is up and all the others are down. Each switch has to be moved once more; if we do it from right to left then switches $n+1$ to 3 go up, then moving switch 2 down brings all them down and finally switch 1 is moved up. Thus 1 is the only switch remaining up.

Solution 2. Let a_i be the number of times the ith switch changes its position during the whole process. According to the conditions of the problem, each switch moves either when it is moved directly by the operator or its left neighbour moves down. As all switches are down at the beginning, the ith switch moves down $\left\lfloor \frac{a_i}{2} \right\rfloor$ times. Thus $a_1 = 1$ and $a_i = i + \left\lfloor \frac{a_{i-1}}{2} \right\rfloor$ for all $i \geq 2$.

We prove by induction that $a_i = 2(i-1)$ for all $i \geq 2$. As $a_2 = 2 + \left\lfloor \frac{a_1}{2} \right\rfloor = 2 + \left\lfloor \frac{1}{2} \right\rfloor = 2$, this claim holds for $i = 2$. Assuming that it holds for i, we obtain

$$a_{i+1} = i + 1 + \left\lfloor \frac{a_i}{2} \right\rfloor = i + 1 + \left\lfloor \frac{2(i-1)}{2} \right\rfloor = 2i,$$

i.e., the claim holds also for $i+1$.
Altogehter, this shows that a_1 is odd and a_i is even for all $i \geq 2$. Hence, after the process, the first switch is up and all the others are down.

Solution 3. Interpret the position of switches on the board as binary numbers so that the ith switch from the left corresponds to the ith lowest binary digit: being down encodes 0 and being up encodes 1. Changing the ith switch then works like addition of 2^{i-1} modulo 2^n. The initial position encodes number 0 and the final position encodes $1 \cdot 2^0 + 2 \cdot 2^1 + \ldots + n \cdot 2^{n-1}$ modulo 2^n.
We prove by induction that $1 \cdot 2^0 + 2 \cdot 2^1 + \ldots + n \cdot 2^{n-1} \equiv 1 \pmod{2^n}$. If $n = 1$ then this holds. Assume that the claim holds for $n = k$. Multiplying this congruence by 2 gives

$$1 \cdot 2^1 + 2 \cdot 2^2 + \ldots + k \cdot 2^k \equiv 2 \pmod{2^{k+1}}.$$

Adding $2^0 + 2^1 + \ldots + 2^k$ to both sides gives

$$1 \cdot 2^0 + 2 \cdot 2^1 + 3 \cdot 2^2 + \ldots + (k+1) \cdot 2^k \equiv 2 + 2^{k+1} - 1 \equiv 1 \pmod{2^{k+1}},$$

i.e., the claim holds for $n = k+1$.

Remark. In Solution 3, one could prove by induction a stronger claim: $1 \cdot 2^0 + 2 \cdot 2^1 + \ldots + n \cdot 2^{n-1} = (n-1) \cdot 2^n + 1$.

2. Let D be the foot of the altitude of triangle ABC drawn from vertex A. Let E and F be points symmetric to D w.r.t. lines AB and AC, respectively. Let R_1 and R_2 be the circumradii of triangles BDE and CDF, respectively, and let r_1 and r_2 be the inradii of the same triangles. Prove that

$$|S_{ABD} - S_{ACD}| \geq |R_1 r_1 - R_2 r_2|$$

where S_K denotes the area of figure K.

Solution 1. Consider first the case where D lies between points B and C (see Fig. 14). As $S_{ABD} = \frac{1}{2} \cdot |AD| \cdot |BD|$ and $S_{ACD} = \frac{1}{2} \cdot |AD| \cdot |CD|$, we have

$$S_{ABD} - S_{ACD} = \frac{1}{2} \cdot |AD| \cdot (|BD| - |CD|).$$

Let G be the incentre of triangle BDE and let G' be the projection of G to line BD. Then $|GG'| = r_1$. By symmetry, $\angle BEA = \angle BDA = 90°$, hence quadrangle $BEAD$ is cyclic and line segment AB is its circumdiameter. Thus $|AB| = 2R_1$. As triangles ADB and $GG'B$ are similar, we have $\dfrac{|AB|}{|AD|} = \dfrac{|GB|}{|GG'|}$, implying $2R_1r_1 = |AD| \cdot |GB|$. Let H be the incentre of triangle CDF; then analogously $2R_2r_2 = |AD| \cdot |HC|$. Hence

$$R_1r_1 - R_2r_2 = \frac{1}{2} \cdot |AD| \cdot (|GB| - |HC|).$$

Triangle ADG is isosceles because $\angle ADG = 90° - \frac{1}{2}\angle BDE = 90° - \frac{1}{2}\angle DAG$. Thus $|AD| = |AG|$. Analogously, $|AD| = |AH|$. Thus $|AG| = |AH|$.
Subtracting equality $|AD|^2 + |CD|^2 = |AC|^2$ from $|AD|^2 + |BD|^2 = |AB|^2$ gives $|BD|^2 - |CD|^2 = |AB|^2 - |AC|^2$ which is equivalent to $\big(|BD| - |CD|\big) \cdot \big(|BD| + |CD|\big) = \big(|AB| - |AC|\big) \cdot \big(|AB| + |AC|\big)$. Consequently,

$$\big||BD| - |CD|\big| \cdot |BC| = \big||GB| - |HC|\big| \cdot \big(|AB| + |AC|\big).$$

As $|BC| < |AB| + |AC|$, we must have $\big||BD| - |CD|\big| \geq \big||GB| - |HC|\big|$, which gives the desired inequality.

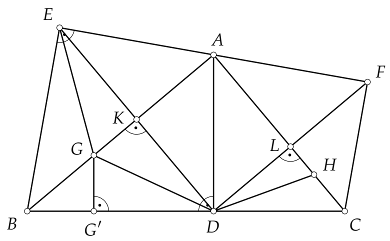


Figure 14

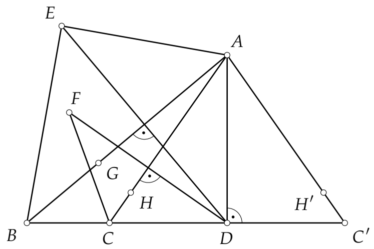


Figure 15

If D does not lie between B and C (see Fig. 15) then assume w.l.o.g. that it is on ray BC. Reflect line segment AC w.r.t. line AD; points C and H transform to some points C' and H', respectively. Now apply the solution above for triangle ABC'. The desired claim follows then by using $|C'D| = |CD|$ and $|H'C'| = |HC|$.
Solution 2. Denote $\angle BAD = \beta$ and $\angle CAD = \gamma$. Then

$$S_{ABD} = \frac{1}{2} \cdot |AD| \cdot |BD| = \frac{1}{2}|AD|^2 \tan\beta.$$

As in Solution 1, show that quadrangle $BEAD$ is cyclic. Let K be the point of intersection of its diagonals. As $R_1 = \dfrac{|AB|}{2}$, we get $R_1 = \dfrac{|AD|}{2\cos\beta}$. Furthermore, $r_1 = |GK|$ and $\angle GDK = \dfrac{\angle BDE}{2} = \dfrac{\angle BAD}{2} = \dfrac{\beta}{2}$. Thus $r_1 = |DK| \tan\dfrac{\beta}{2} = |AD| \sin\beta \tan\dfrac{\beta}{2}$. Consequently,

$$R_1r_1 = \frac{|AD|}{2\cos\beta} \cdot |AD| \, \sin\beta \, \tan\frac{\beta}{2} = \frac{1}{2} \, |AD|^2 \tan\beta \, \tan\frac{\beta}{2}.$$

Analogously we obtain

$$S_{ACD} = \frac{1}{2} \, |AD|^2 \tan\gamma, \qquad R_2r_2 = \frac{1}{2} \, |AD|^2 \tan\gamma \, \tan\frac{\gamma}{2}.$$

From these equalities, we can conclude that $S_{ABD} - S_{ACD}$ and $R_1r_1 - R_2r_2$ have the same sign since β and γ belong to the first quarter where tan is increasing. W.l.o.g., assume that both are non-negative (otherwise interchange B and C). Then $\beta \geq \gamma$ and the desired inequality is equivalent to $S_{ABD} - R_1r_1 \geq S_{ACD} - R_2r_2$. Now

$$S_{ABD} - R_1r_1 = \frac{1}{2} \, |AD|^2 \tan\beta \left(1 - \tan\frac{\beta}{2}\right) =$$
$$= \frac{1}{2} \, |AD|^2 \, \frac{2\tan\frac{\beta}{2}}{1 - \tan^2\frac{\beta}{2}} \left(1 - \tan\frac{\beta}{2}\right) = |AD|^2 \, \frac{\tan\frac{\beta}{2}}{1 + \tan\frac{\beta}{2}},$$

whence

$$S_{ABD} - R_1r_1 = |AD|^2 \left(1 - \frac{1}{1 + \tan\frac{\beta}{2}}\right)$$

and, analogously,

$$S_{ACD} - R_2r_2 = |AD|^2 \left(1 - \frac{1}{1 + \tan\frac{\gamma}{2}}\right).$$

By $\beta \geq \gamma$ and tan being increasing, the inequality $S_{ABD} - R_1r_1 \geq S_{ACD} - R_2r_2$ follows.

3. Let n be a natural number, $n \geq 2$. Prove that if $\dfrac{b^n - 1}{b - 1}$ is a prime power for some positive integer b then n is prime.

Solution. Clearly $b \geq 2$. Assume that $\dfrac{b^n - 1}{b - 1} = p^l$ where p is prime, then $n \geq 2$ implies $l \geq 1$. If $n = xy$ where both x and y are greater than 1 then consider the representation

$$\frac{b^{xy} - 1}{b - 1} = \frac{b^{xy} - 1}{b^y - 1} \cdot \frac{b^y - 1}{b - 1} = \big(1 + b^y + \ldots + b^{y(x-1)}\big) \cdot \frac{b^y - 1}{b - 1}.$$

As the product is a power of p, both factors must be powers of p. As $x > 1$ and $y > 1$, both factors are multiples of p. Then $b^y - 1$ is a multiple of p. Thus all addends in the first factor are congruent to 1 modulo p which implies that the first factor is congruent to x modulo p. Hence x is divisible by p. As x was an arbitrary non-trivial factor of n, this shows that $n = p^m$ for a positive integer m.
Now consider the representation

$$\frac{b^{p^m} - 1}{b - 1} = \frac{b^{p^m} - 1}{b^{p^{m-1}} - 1} \cdot \ldots \cdot \frac{b^{p^2} - 1}{b^p - 1} \cdot \frac{b^p - 1}{b - 1}.$$

Each factor is both greater than 1 and a power of p. As $\dfrac{b^p-1}{b-1}$ is a positive integral power of p, the numerator is divisible by p, i.e., $b^p \equiv 1 \pmod p$. By Fermat's little theorem, $b^p \equiv b \pmod p$. Thus $b \equiv 1 \pmod p$ and $b-1$ is divisible by p. But then the numerator b^p-1 must be divisible by p^2, i.e., $b^p \equiv 1 \pmod{p^2}$. If $m \geq 2$ then the representation above contains factor $\dfrac{b^{p^2}-1}{b^p-1} = 1 + b^p + \ldots + b^{p(p-1)}$. On one hand, this is congruent to p modulo p^2 as all addends are congruent to 1. On the other hand, this factor is a power of p while being greater than p, hence it is a multiple of p^2. This contradiction shows that $m=1$, qed.

Remark 1. Fermat's little theorem can easily be avoided in the solution. Cutting this out from the solution above, it still shows that if $m \geq 2$ then b^p-1 is not divisible by p^2. Continuing from this, we see that $\dfrac{b^p-1}{b-1}$ is divisible by p but not by p^2. Hence this factor must be p. Now

$$\frac{b^p-1}{b-1} = 1 + b + \ldots + b^{p-1} > b^{p-1} \geq 2^{p-1} \geq p$$

gives a contradiction.

Remark 2. In the special case $b=2$, $l=1$, the problem reduces to the well-known fact that a Mersenne's number M_n can be prime only if n is prime.

Second day

4. In square $ABCD$, points E and F are chosen in the interior of sides BC and CD, respectively. The line drawn from F perpendicular to AE passes through the intersection point G of AE and diagonal BD. A point K is chosen on FG such that $|AK| = |EF|$. Find $\angle EKF$.

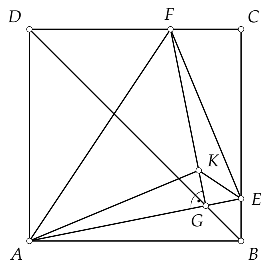


Figure 16

Answer: 135°.

Solution. Since $AGFD$ is a cyclic quadrilateral (see Fig. 16), $\angle GAF = \angle GDF = 45^\circ$ and $\angle GFA = \angle GDA = 45^\circ$, so triangle AGF is isosceles and $|GA| = |GF|$. Now, right triangles AGK and FGE are congruent, and $|GK| = |GE|$, so triangle GKE is also isosceles. Finally, $\angle GKE = 45^\circ$ and $\angle EKF = 180^\circ - \angle GKE = 135^\circ$.

5. Find all continuous functions $f\colon \mathbb{R} \to \mathbb{R}$ such that for all reals x and y

$$f(x + f(y)) = y + f(x+1).$$

Answer: $f(x) = 1 + x$ and $f(x) = 1 - x$.

Solution. Taking $y = -f(x+1)$, we see that there is a value a such that $f(a) = 0$. We consider two cases.

Let first $a \neq 1$. Taking $y = x+1$, we get $f(x + f(x+1)) = x + 1 + f(x+1)$. Let $g(x) = x + f(x+1)$, then $f(g(x)) = 1 + g(x)$ for all x. Since f is continuous, so is g.

Taking $y = a$ in the initial relation, we get $f(x) = a + f(x+1)$, and so $g(x-1) - g(x) = a - 1$ for all x. Since $a \neq 1$, g is unbounded and by continuity, takes all real values, so $f(z) = 1 + z$ for all z.

Let now $a = 1$, i.e., $f(1) = 0$. Then $x = 0$ yields $f(f(y)) = y$ for all reals y. Taking now $y = f(1-x)$ in the initial relation, we get $f(x + f(f(1-x))) = f(1-x) + f(x+1)$, or $0 = f(1-x) + f(x+1)$. Finally, taking $y = 1-x$ yields $f(x + f(1-x)) = 1 - x + f(x+1)$, so $f(x + f(1-x)) = 1 - x - f(1-x)$. Let $h(x) = x + f(1-x)$, then $f(h(x)) = 1 - h(x)$ holds for all x. Replacing x with $-x$ and taking $y = 1$ in the initial relation, we get $f(-x) = 1 + f(1-x)$, so $h(x+1) - h(x) = 2$. Again, h is continuous and must take all real values, so $f(z) = 1 - z$ for all z.

It is straightforward to verify that both solutions indeed satisfy the initial relation.

6. Consider a 10×10 grid. On every move, we colour 4 unit squares that lie in the intersection of some two rows and two columns. A move is allowed if at least one of the 4 squares is previously uncoloured. What is the *largest* possible number of moves that can be taken to colour the whole grid?

Answer: 81.

Solution. By always choosing the first line, the first column and a square of the remaining 9×9 grid as the lower right square, the whole grid can be coloured in 81 moves.

We now prove that it is not possible to make more than 81 moves. Consider a sequence of moves. Select for each move one square that is chosen for the first time during this move and colour the remaining squares already before starting the sequence. Then, take all squares that were *not* selected and colour them in advance, i.e., already before starting the sequence of moves. Since all selected squares must be different, every move in the sequence now colours exactly one square.

Next, consider a bipartite graph with the 10 rows and 10 columns as vertices. Every time a square is coloured, draw an edge between the row and the column corresponding to this square. We claim that the graph is connected *before* we start the sequence of moves. Indeed, suppose that during some move, we pick rows (a, b) and columns (c, d), such that only the square (b, d) is coloured for the first time, i.e., we add the edge (b, d). But then b is already connected with d through $b - c - a - d$, so the number of connected components does not decrease. Since the graph of a fully coloured grid is connected, it must also be connected in the beginning. But a connected graph with 20 vertices must have at least 19 edges, so we can add only $100 - 19 = 81$ new edges, and hence any sequence can have at most 81 moves.