Estonian Math Competitions 2009/2010

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Estonian Mathematical Olympiad

Mathematics Contests in Estonia

The Estonian Mathematical Olympiad is held annually in three rounds – at the school, town/regional and national levels. The best students of each round (except the final) are invited to participate in the next round. Every year, about 110 students altogether reach the final round.

In each round of the Olympiad, separate problem sets are given to the students of each grade. Students of grade 9 to 12 compete in all rounds, students of grade 7 to 8 participate at school and regional levels only. Some towns, regions and schools also organise olympiads for even younger students. The school round usually takes place in December, the regional round in January or February and the final round in March or April in Tartu. The problems for every grade are usually in compliance with the school curriculum of that grade but, in the final round, also problems requiring additional knowledge may be given.

The first problem solving contest in Estonia took place already in 1950. The next one, which was held in 1954, is considered as the first Estonian Mathematical Olympiad.

Apart from the Olympiad, open contests are held twice a year, usually in October and in December. In these contests, anybody who has never been enrolled in a university or other higher education institution is allowed to participate. The contestants compete in two separate categories: the Juniors and the Seniors. In the first category, students up to the 10th grade can participate; the other category has no restriction. Being successful in the open contests generally assumes knowledge outside the school curriculum.

According to the results of all competitions during the year, about 20 IMO team candidates are selected. IMO team selection contest for them is held in April or May. This contest lasts two days; each day, the contestants have 4.5 hours to solve 3 problems, similarly to the IMO. All participants are given the same problems. Some problems in our selection contest are at the level of difficulty of the IMO but somewhat easier problems are usually also included.

The problems of previous competitions are available at the Estonian Mathematical Olympiad's website `http://www.math.olympiaadid.ut.ee/eng`.

Besides the above-mentioned contests and the quiz "Kangaroo" other regional competitions and matches between schools are held as well.

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This booklet contains problems that occurred in the open contests, the final round of national olympiad and the team selection contest. For the open contests and the final round, selection has been made to include only problems that have not been taken from other competitions or problem sources and seem interesting enough. The team selection contest is presented entirely.

Selected Problems from Open Contests

OC-1. Find all positive integers n such that $1 + 2^2 + 3^3 + 4^n$ is a perfect square. *(Juniors.)*

Answer: $n = 1$.

Solution 1. Let $1 + 2^2 + 3^3 + 4^n = x^2$. This implies $32 = x^2 - 4^n$ or, equivalently, $2^5 = (x - 2^n)(x + 2^n)$. As the l.h.s. is a power of 2, the factors in the r.h.s. are of the form $x - 2^n = 2^a$ and $x + 2^n = 2^{5-a}$ where a is 0, 1 or 2. Subtracting the first of the two equalities from the second gives $2^{n+1} = 2^{5-a} - 2^a$. This leads to an integral n only if $a = 2$; then $n = 1$. A check shows that $1 + 2^2 + 3^3 + 4^1 = 6^2$ indeed.

Solution 2. Observe that $1 + 2^2 + 3^3 + 4^n = 32 + 4^n = 2^5 + 2^{2n} = 2^5 \cdot (1 + 2^{2n-5})$. If $n \geqslant 3$, then $2n - 5 \geqslant 1$; hence 2^{2n-5} is an even integer and $1 + 2^{2n-5}$ is therefore odd. Thus in the prime factorization of the number given in the problem, the exponent of 2 is 5. As this is odd, the number cannot be a perfect square. If $n = 2$ or $n = 1$, then $32 + 4^n = 48$ or $32 + 4^n = 36$, respectively, where only the latter is a perfect square. Consequently, only $n = 1$ is possible.

Solution 3. If $n = 1, 2, 3$, then the given number is 36, 48, 96, respectively, where only the first is a perfect square. If $n \geqslant 4$, then $2 \cdot 2^n + 1 \geqslant 2 \cdot 16 + 1 > 32$, implying $(2^n)^2 < (2^n)^2 + 32 < (2^n)^2 + 2 \cdot 2^n + 1 = (2^n + 1)^2$. As the number under question is equal to $(2^n)^2 + 32$, it falls between two consecutive perfect squares, hence cannot be a perfect square itself.

OC-2. Given a convex quadrangle $ABCD$ with $|AD| = |BD| = |CD|$ and $\angle ADB = \angle DCA$, $\angle CBD = \angle BAC$, find the sizes of the angles of the quadrangle. *(Juniors.)*

Answer: $75°, 120°, 45°$, and $120°$.

Solution 1. Denote $\angle ADB = \angle DCA = \alpha$ and $\angle CBD = \angle BAC = \beta$ (Fig. 1). In triangle DAC we have $|DA| = |DC|$ and therefore $\angle DAC = \angle DCA = \alpha$; analogously in triangles DAB and DBC, we have $\angle DBA = \angle DAB = \alpha + \beta$ and $\angle DCB = \angle DBC = \beta$, respectively. So $\angle BCA = \beta - \alpha$. From triangle ABC now $\beta + \alpha + \beta + \beta + \beta - \alpha = 180°$ or, equivalently, $4\beta = 180°$, giving $\beta = 45°$. From triangle ADB we get $\alpha + \beta + \alpha + \beta + \alpha = 180°$ or, equivalently, $3\alpha = 180° - 2\beta = 90°$ and $\alpha = 30°$. Therefore, the sizes of the angles of quadrangle $ABCD$ are $\angle DAB = \alpha + \beta = 75°$, $\angle ABC = \alpha + 2\beta = 120°$, $\angle BCD = \beta = 45°$, and $\angle CDA = 360° - 75° - 120° - 45° = 120°$.

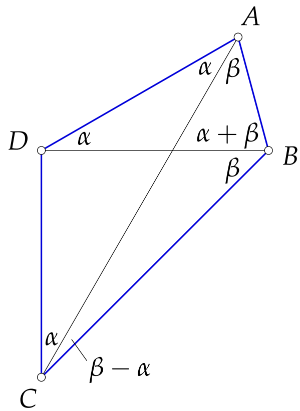


Fig. 1

Solution 2. We use the same notation as in the Solution 1. Triangle BCD is isosceles, hence $\angle DCB = \angle DBC = \beta$. As D is the circumcenter of ABC, we have $\angle BDC = 2\angle BAC = 2\beta$. The sizes of the angles of triangle BCD are therefore β, β, and 2β; thus $\beta + \beta + 2\beta = 180^\circ$, whence $\beta = 45^\circ$. As $\angle BCA = \frac{\angle BDA}{2}$, we have $\angle BCD = \frac{\alpha}{2} + \alpha = \beta$, whence $\alpha = \frac{2}{3}\beta = 30^\circ$. Consequently, the sizes of the angles of quadrangle $ABCD$ are $\angle DAB = \angle ABD = \frac{180^\circ - \alpha}{2} = 75^\circ$, $\angle ABC = \angle ABD + \angle CBD = 75^\circ + 45^\circ = 120^\circ$, $\angle BCD = \beta = 45^\circ$, and $\angle CDA = \angle CDB + \angle BDA = 90^\circ + 30^\circ = 120^\circ$.

Remark. The convexity of the quadrangle actually follows from the other constraints of the problem. Namely, consider the circle with center D, passing through points A and B, and a point C on it. If A and B were on different sides from line CD, we would have $\angle ADB > \angle BCA > \angle DCA$, hence $\angle ADB$ and $\angle DCA$ could not be equal.

OC-3. In the buffet of the kitchen, there are three candy boxes, each containing the same number of candies. Every time when Juku goes into the kitchen, he takes either three candies from one box or one candy from every box. Prove that irrespectively of how Juku takes the candies, he always retains the possibility to completely clean out all candy boxes. *(Juniors.)*

Solution. The difference of the numbers of candies in any two boxes can only be a multiple of 3 because it is 0 in the beginning and, with every move, it changes by either 0 or 3. Hence, starting from an arbitrary intermediate state, Juku can clean out the boxes as follows: he takes one candy from each box as many times as possible, after which one box is empty and the number of candies in each of the other two is divisible by 3, and then empties the other boxes by taking three candies from one box every time.

OC-4. Four musketeers together bought a plot of rectangular shape and paid for it equally. They divided the plot by two cuts into four pieces of rectangular shape, from which every musketeer got one. It turned out that one musketeer obtained as much land as the other three in total. Prove that the price per acre of one musketeer's piece turned out as large as the sum of the prices per acre of the other three musketeers' pieces. *(Juniors.)*

Solution. Let a and b be the side lengths of the plot. Assume that the cuts divided the side of length a to parts of length x and $a - x$ where x being the greater part, and the side of length b to parts of length y and $b - y$ where y being the greater part. Then the area of the largest piece was xy. The condition that this area equals the sum of the areas of the other three pieces can be written as follows:

$$xy = (a - x)y + x(b - y) + (a - x)(b - y).$$

Dividing both sides by $x(a-x)y(b-y)$, one obtains

$$\frac{1}{(a-x)(b-y)} = \frac{1}{x(b-y)} + \frac{1}{(a-x)y} + \frac{1}{xy}.$$

If the price that every musketeer paid for the plot was 1, then the l.h.s. of the last equality is precisely the price per area unit of the piece with area $(a-x)\cdot(b-y)$. Analogously, the r.h.s. equals the sum of the prices per area unit of the other three pieces. Hence multiplying the sides of this equality by the number of area units per acre, the claim of the problem follows.

OC-5. Let a be a fixed real number. Find all real numbers b such that, for every real number x, at least one of the numbers x^2+ax+b and x^2-ax+b is non-negative. *(Juniors.)*

Answer: $b \geqslant 0$.

Solution. Note that x^2+ax+b and x^2-ax+b sum up to $2x^2+2b$. If $b \geqslant 0$, then it is non-negative for arbitrary real number x, implying that at least one of the numbers added was non-negative. If $b<0$, then taking $x=0$ turns both summands negative.

Remark. This problem can be solved also in technical ways, by calculating the negative and non-negative domains of the quadratic polynomials.

OC-6. Call a positive integer n *prime-prone* if there exist at least three prime numbers from which we can get n by removing the last digit. Prove that every two prime-prone positive integers differ from each other by at least 3. *(Juniors.)*

Solution. As the prime numbers under consideration have at least two digits, the last digit can be only 1, 3, 7, or 9. Thus n is prime-prone if and only if, among numbers $10n+1$, $10n+3$, $10n+7$, and $10n+9$, at least three are primes.

If $n=3k$, then $10n+3=30k+3$ and $10n+9=30k+9$ are divisible by 3 and hence composite. If $n=3k+2$, then $10n+1=30k+21$ and $10n+7=30k+27$ are divisible by 3 and hence composite again. Consequently, all prime-prone integers are congruent to 1, and hence to each other, modulo 3. Thus they differ by a multiple of 3, i.e., by at least 3.

OC-7. Does there exist a prime number p such that both p^3+2008 and p^3+2010 are primes as well? *(Seniors.)*

Answer: no.

Solution. Let p be any prime number. If p is not divisible by 7, then p^3 is congruent to either 1 or -1 modulo 7. Since $2008 \equiv -1 \pmod 7$ and $2010 \equiv 1 \pmod 7$, either of the numbers p^3+2008 and p^3+2010 is divisible by 7 and hence composite. If p is divisible by 7, then $p=7$ and $p^3+2010 = 7^3+2010 = 2353 = 13\cdot 181$ is composite, too.

OC-8. In a regular n-gon, either 0 or 1 is written at each vertex. Using non-intersecting diagonals, Juku divides this polygon into triangles. Then he writes into each triangle the sum of the numbers at its vertices. Prove that Juku can choose the diagonals in such a way that the maximal and minimal number written into the triangles differ by at most 1. *(Seniors.)*

Solution. If all numbers written at the vertices of the polygon are equal, then the claim holds trivially. Hence assume that there are both zeros and ones among the numbers at the vertices. We prove by induction that, for every convex polygon, the partition into triangles can be chosen in such a way that Juku writes either 1 or 2 to each triangle.

If $n = 3$, then this claim holds since the sum of the numbers at the vertices of a triangle can be neither 0 nor 3. If $n = 4$ (Fig. 2), then draw the diagonal that connects the vertices where 0 and 1 are written, respectively, or, if such a diagonal does not exist, then an arbitrary diagonal. In both cases, only sums 1 and 2 can arise. If $n \geqslant 5$, then choose two consecutive vertices with different labels and a third vertex P that is not neighbour to either of them (Fig. 3). Irrespective of whether the label of P is 0 or 1, we can draw the diagonal from it to one of the two consecutive vertices chosen before so that the labels of its endpoints are different. Now the polygon is divided into two convex polygons with smaller number of vertices so that both 0 and 1 occur among their vertex labels. By the induction hypothesis, both polygons can be partitioned into triangles with sum of labels of vertices either 1 or 2.

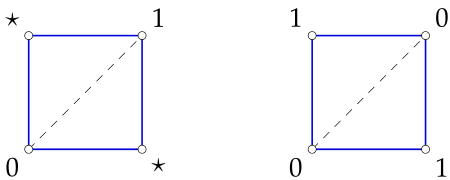


Fig. 2

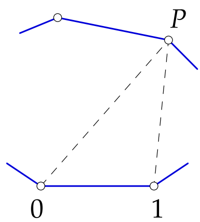


Fig. 3

OC-9. Circle c passes through vertices A and B of an isosceles triangle ABC, whereby line AC is tangent to it. Prove that circle c passes through the circumcenter or the incenter or the orthocenter of triangle ABC. *(Seniors.)*

Solution. Consider three cases $|AB| = |AC|$, $|BC| = |BA|$, and $|CA| = |CB|$.

1. We show that if $|AB| = |AC|$ (Fig. 4), then circle c passes through the circumcenter of ABC. Let O be the point at the same side from AB as C that is the intersection of the perpendicular bisector of side AB and circle c. Then $\angle OAB = \angle OBA$ and, by inscribed angles theorem,

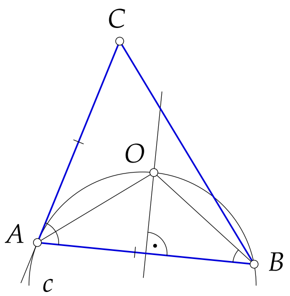


Fig. 4

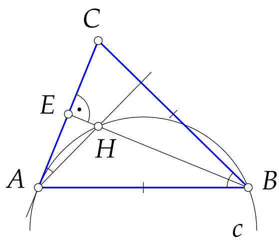


Fig. 5

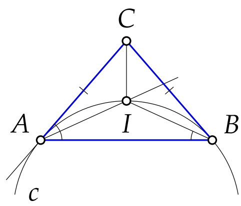


Fig. 6

$\angle OBA = \angle OAC$. Hence O lies on the bisector of angle CAB. Since $|AB| = |AC|$, this angle bisector is also the perpendicular bisector of side BC. Consequently, O is the intersection point of the perpendicular bisectors of the sides of triangle ABC.

2. For the case $|BC| = |BA|$ (Fig. 5), we show that circle c passes through the orthocenter of triangle ABC. Let E be the foot of the altitude of triangle ABC drawn from B and let H be the second intersection point of this altitude with circle c (in the special case with tangency and no intersection, take $H = B$). By the inscribed angles theorem, $\angle EBA = \angle EAH$. Thus $\angle ACB + \angle CAH = \angle CAB + \angle EBA = 90°$ whence $AH \perp BC$. Consequently, H is the orthocenter.
3. Finally, we show that if $|CA| = |CB|$ (Fig. 6), then circle c passes through the incenter of triangle ABC. Let I be the intersection point of the bisector of angle CAB with circle c. By the inscribed angles theorem, $\angle CAI = \angle IBA$. Hence $\angle BAI = \angle IBA$ whence I lies on the perpendicular bisector of side AB. As $|CA| = |CB|$, this perpendicular bisector is also the bisector of angle ACB. Consequently, I is the intersection point of the angle bisectors.

OC-10. Let $n \geqslant 2$. Positive integers $a_1, a_2, \ldots, a_n$ whose sum is even and which satisfy $a_i \leqslant i$ for every $i = 1, 2, \ldots, n$, are given. Prove that it is possible to choose signs in the expression $a_1 \pm a_2 \pm \ldots \pm a_n$ in such a way that its value becomes 0. *(Seniors.)*

Solution 1. Prove the claim by induction on n. If $n = 2$, then the only way to choose integers that satisfy the conditions of the problem is $a_1 = 1$ and $a_2 = 1$. In this case, $a_1 - a_2 = 0$. Assume now that the claim holds whenever $2 \leqslant n \leqslant k$ and show that it holds also for $n = k + 1$. Consider two cases.

1. If $a_{k+1} = a_k$, then $a_1 + a_2 + \ldots + a_{k-1}$ is even. As this case is possible only for $k > 2$, the induction hypothesis is applicable for $n = k - 1$. Thus it is possible to choose signs in the expression $a_1 \pm a_2 \pm \ldots \pm a_{k-1}$

in such a way that it evaluates to 0. Adding $a_k - a_{k+1}$ to it, the desired expression for $n = k + 1$ is obtained.

2. If $a_k \neq a_{k+1}$, then consider integers $a_1, \ldots, a_{k-1}, |a_k - a_{k+1}|$. As $|a_k - a_{k+1}|$ and $a_k + a_{k+1}$ have the same parity, the sum of these k numbers is even. Also note that $1 \leqslant |a_k - a_{k+1}| \leqslant k$. Thus these numbers satisfy the conditions of the problem, so it is possible to choose signs in the expression $a_1 \pm a_2 \pm \ldots \pm a_{k-1} \pm |a_k - a_{k+1}|$ in such a way that it evaluates to 0. As either $|a_k - a_{k+1}| = a_k - a_{k+1}$ or $|a_k - a_{k+1}| = a_{k+1} - a_k$, this also leads to a corresponding expression for numbers $a_1, a_2, \ldots, a_k, a_{k+1}$.

Solution 2. Prove by induction on i that, for each i and s such that $1 \leq i \leq n$ and $1 \leq s \leq a_1 + \ldots + a_i$, it is possible to choose some of the numbers $a_1, \ldots, a_i$ that sum up to s. If $i = 1$, then this claim holds since $a_1 = 1$. Assume that the claim holds for $i = k - 1$ and consider the case $i = k$. Let $S = a_1 + \ldots + a_k$ and $S' = a_1 + \ldots + a_{k-1}$. If $1 \leq s \leq S'$, then the desired statement holds by the induction hypothesis. If $S' < s \leq S$, then $0 \leq s - a_k \leq S'$ (the first inequality holds because $s - a_k \geq s - S' - 1 \geqslant 0$, implied by $a_k \leq k$ and $S' \geq k - 1$; the second inequality follows from $S = S' + a_k$). Therefore, to get the sum s, we can choose the number a_k, and if $s - a_k > 0$, then add to it those numbers among $a_1, \ldots, a_{k-1}$ whose sum is $s - a_k$, using the induction hypothesis.

Let now $a_1 + a_2 + \ldots + a_n = 2T$. Choose the numbers among $a_1, a_2, \ldots, a_n$ that sum up to T. This divides all the numbers into two groups with equal sum. It remains to write minuses in front of every term of the group that does not contain a_1.

Solution 3. Start choosing signs from right to left. Denote $S_1 = a_n$ and define S_{k+1}, $k = 1, \ldots, n - 1$, as follows: if $S_k \geqslant 0$, then $S_{k+1} = S_k - a_{n-k}$, otherwise $S_{k+1} = S_k + a_{n-k}$. We show that then always $|S_k| \leqslant n - k + 1$. This holds if $k = 1$. Assume therefore that it holds for $k = m$ and prove it for $k = m + 1$. If $S_m \geqslant 0$, then $S_{m+1} = S_m - a_{n-m} \leqslant (n - m + 1) - 1 = n - m$ and $S_{m+1} = S_m - a_{n-m} \geqslant 0 - (n - m)$, hence $|S_{m+1}| \leqslant n - m$. If $S_m < 0$, then $S_{m+1} = S_m + a_{n-m} < 0 + n - m$ and $S_{m+1} = S_m + a_{n-m} \geqslant -(n - m + 1) + 1 = -(n - m)$, hence $|S_{m+1}| \leqslant n - m$ again.

Now $|S_n| \leqslant 1$ since $|S_k| \leqslant n - k + 1$ for every $k = 1, \ldots, n$. Thus $S_n = 0$ as the sum of all terms is even. If in this formal sum, the term a_1 has minus sign, turn all signs to the opposite one.

Remark. Solution 2 shows that the assumption $a_k \leqslant k$ for all $k = 1, \ldots, n$ could be replaced with the more general assumption $a_k \leqslant 1 + a_1 + \ldots + a_{k-1}$ for all $k = 1, \ldots, n$.

One can also note that the lemma proved at the beginning of Solution 2 does not need the assumption that the sum of all numbers is even.

OC-11. The diagonals of trapezoid $ABCD$ with bases AB and CD meet at P. Prove the inequality $S_{PAB} + S_{PCD} > S_{PBC} + S_{PDA}$, where S_{XYZ} denotes the area of triangle XYZ. *(Seniors.)*

Solution 1. Let $a = |AB|$ and $b = |CD|$ and let h_a and h_b be the altitudes of triangles PAB and PCD drawn from P (Fig. 7). Denote $S_1 = S_{PAB} + S_{PCD}$ and $S_2 = S_{PBC} + S_{PDA}$. Then $S_1 = \frac{1}{2}(ah_a + bh_b)$ and $S_1 + S_2 = \frac{1}{2}(a+b)(h_a + h_b)$, whence $S_2 = \frac{1}{2}(ah_b + bh_a)$. Since triangles PAB and PCD are similar, $a > b$ implies $h_a > h_b$ and also $a < b$ implies $h_a < h_b$ ($a \neq b$ because a and b are the lengths of the bases of the trapezoid). Hence

$$S_1 - S_2 = \frac{1}{2}(ah_a + bh_b - ah_b - bh_a) = \frac{1}{2}(a-b)(h_a - h_b) > 0,$$

i.e., $S_1 > S_2$.

Solution 2. Let M and N be the intersection points of the arms BC and DA of the trapezoid with the line being parallel to the bases of the trapezoid and passing through point P. Let l be the length of MN, let d be the length of the midline of the trapezoid, and let h and S be the height and the area of the trapezoid, respectively. Let $S' = S_{PBC} + S_{PDA}$. Then $S = dh$ and $S' = \frac{1}{2}lh$ whence it suffices to show that $l < d$.

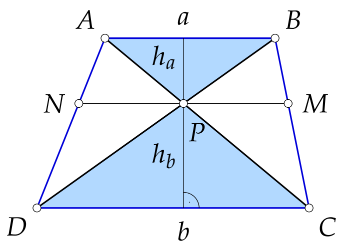


Fig. 7

W.l.o.g., assume $|AB| < |CD|$. Comparing the heights of similar triangles PAB and PCD shows that MN is closer to base AB than to base CD. Thus MN is situated between the midline and the shorter base AB. Consequently, MN is shorter than the midline.

Solution 3. Let $a = |AB|$ and $b = |CD|$. Let h and S be the height and the area of the trapezoid, respectively, and let h_a and h_b be the heights corresponding to vertex P of triangles PAB and PCD, respectively. Similar triangles PAB and PCD imply $h_a : h_b = a : b$. As $h_a + h_b = h$ we get

$$h_a = \frac{a}{a+b} \cdot h, \qquad h_b = \frac{b}{a+b} \cdot h.$$

Now

$$S_{PAB} + S_{PCD} = \frac{1}{2}(ah_a + bh_b) = \frac{1}{2} \cdot \frac{a^2 + b^2}{a+b} \cdot h.$$

It suffices to show that $S_{PAB} + S_{PCD} > \frac{S}{2}$ or, equivalently,

$$\frac{1}{2} \cdot \frac{a^2 + b^2}{a+b} \cdot h > \frac{1}{2} \cdot \frac{a+b}{2} \cdot h,$$

or, equivalently, $2(a^2 + b^2) > (a+b)^2$. But the last inequality is equivalent to $(a-b)^2 > 0$ ($a \neq b$ since a and b are the lengths of the bases of the trapezoid).

OC-12. Call *pure* any positive integer n that does not occur in any integer sequence $c_0, c_1, c_2, \ldots$, where $0 < c_0 < n$ and

$$c_i = \begin{cases} \frac{1}{2}c_{i-1} & \text{if } c_{i-1} \text{ is even,} \\ 3c_{i-1} - 1 & \text{if } c_{i-1} \text{ is odd,} \end{cases}$$

for every $i \geqslant 1$. (For instance, 10 is not pure since it occurs in the sequence 5, 14, 7, 20, 10, ...)

a) Is every positive multiple of 3 pure?
b) Prove that if an integer $n > 1$ is pure but not divisible by 3, then $n+1$ is divisible by 6.

(Seniors.)

Answer: a) yes.

Solution. a) Note that $3c_{i-1} - 1$ is never divisible by 3 and if $\frac{1}{2}c_{i-1}$ is divisible by 3, then also c_{i-1} is divisible by 3. Thus, if some term $c_k = n$ is divisible by 3, then, up to it, only dividing by 2 is used to build the terms (i.e., $c_i = \frac{1}{2}c_{i-1}$ for every i such that $1 \leqslant i \leqslant k$) and, consequently, $c_0 > c_1 > \ldots > c_k = n$. But this contradicts the condition $c_0 < n$. Hence every positive multiple of 3 is pure.

b) If n is not divisible by 3, then $n = 3k+1$ or $n = 6k+2$ or $n = 6k+5$. If $n = 3k+1$, then taking $c_0 = 2k+1$ gives $c_1 = 6k+2$ and $c_2 = 3k+1 = n$. Thereby $k > 0$ since $n > 1$, therefore $c_0 < n$. Hence none of such numbers n is pure. If $n = 6k+2$, then taking $c_0 = 2k+1$ gives $c_1 = 6k+2 = n$, whereby $c_0 < n$. Hence also none of such numbers n is pure. Hence, among the positive integers $n > 1$ not divisible by 3, only those of the form $n = 6k+5$ can be pure.

Remark. Not every integer of the form $n = 6k+5$ is pure. For example, $23 = 6 \cdot 3 + 5$ occurs in the sequence 21, 62, 31, 92, 46, 23, ...

OC-13. Let a and b the lengths of the legs of a given right triangle. Prove that angle φ, where $0 < \varphi < 90°$, is an acute angle of this triangle if and only if $(a\cos\varphi + b\sin\varphi)(a\sin\varphi + b\cos\varphi) = 2ab$. *(Seniors.)*

Solution 1. The equality given in the problem is equivalent to

$$(a^2+b^2)\sin\varphi\cos\varphi + ab(\sin^2\varphi + \cos^2\varphi) = 2ab$$

and hence also to

$$(a^2+b^2)\sin\varphi\cos\varphi = ab. \tag{1}$$

Let α and β be the angles opposite to legs with length a and b, respectively. Then $\sin\alpha = a/\sqrt{a^2+b^2}$, $\sin\beta = \cos\alpha = b/\sqrt{a^2+b^2}$, implying

$$(a^2+b^2)\sin\alpha\cos\alpha = ab.$$

Comparing this to (1) shows the equivalence of the equality of the problem and the equality $\sin\varphi\cos\varphi = \sin\alpha\cos\alpha$, i.e., equality $\sin 2\varphi = \sin 2\alpha$. As

$0 < \alpha, \beta < 90^\circ$, this implies $2\varphi = 2\alpha$ or $2\varphi = 180^\circ - 2\alpha$, whence $\varphi = \alpha$ or $\varphi = 90^\circ - \alpha = \beta$. Hence, φ satisfies the equality if and only if it equals one of the acute angles of the right triangle.

Solution 2. Let ABC be the given triangle with right angle at vertex C. Let c be the length of its hypothenuse and h be the height corresponding to the hypothenuse. Let C' be a point on the circumcircle of ABC such that one acute angle of triangle ABC' is φ (Fig. 8). Let a' and b' be the lengths of the legs of triangle ABC' and h' be the height of the triangle ABC' corresponding to its hypothenuse. Then $(a^2 + b^2)\sin\varphi\cos\varphi = c^2\sin\varphi\cos\varphi = (c\sin\varphi)(c\cos\varphi) = a'b' = ch'$. Since the equality in the problem is equivalent to the equality (1) from the Solution 1, it is also equivalent to $ch' = ab$. But $ab = ch$, hence it is also equivalent to $h = h'$. This condition holds if and only if $ABC \sim ABC'$ or $ABC \sim BAC'$, i.e., φ equals one of the acute angles of triangle ABC.

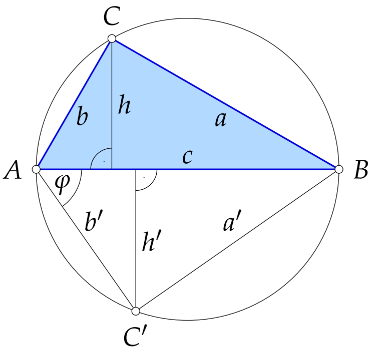


Fig. 8

Selected Problems from the Final Round of National Olympiad

FR-1. Let a, b and c be positive integers such that ab is divisible by $2c$, bc is divisible by $3a$ and ca is divisible by $5b$. Find the least possible value of abc. *(Grade 9.)*

Answer: 900.

Solution. Since ab is divisible by $2c$ and ca is divisible by $5b$, $ab \cdot ca$ must be divisible by $2c \cdot 5b$, hence a^2 is divisible by $2 \cdot 5$. Therefore a^2 is divisible by 2 and 5, hence a is divisible by 2 and 5. Similarly b is divisible by 2 and 3, and c is divisible by 3 and 5. Consequently abc is divisible by $2 \cdot 5 \cdot 2 \cdot 3 \cdot 3 \cdot 5 = 900$. On the other hand, $a = 10$, $b = 6$ and $c = 15$ satisfy the conditions and $abc = 900$.

FR-2. Prove the inequality

$$2010 < \frac{2^2+1}{2^2-1} + \frac{3^2+1}{3^2-1} + \ldots + \frac{2010^2+1}{2010^2-1} < 2010\,\frac{1}{2}.$$

(Grade 9.)

Solution. Since $\dfrac{n^2+1}{(n-1)(n+1)} = 1 + \dfrac{1}{n-1} - \dfrac{1}{n+1}$, the given sum can be rewritten in the form $1 + \dfrac{1}{1} - \dfrac{1}{3} + 1 + \dfrac{1}{2} - \dfrac{1}{4} + \ldots + 1 + \dfrac{1}{2009} - \dfrac{1}{2011} =$

$2010 + \dfrac{1}{2} - \dfrac{1}{2010} - \dfrac{1}{2011}$. Because $0 < \dfrac{1}{2} - \dfrac{1}{2010} - \dfrac{1}{2011} < \dfrac{1}{2}$, the inequality is proved.

FR-3. Juku drew a regular hexagon and chose three triangles with different areas whose vertices were among the vertices of the hexagon. Prove that the sum of the areas of the triangles is equal to the area of the hexagon. *(Grade 9.)*

Solution. Any triangle whose vertices are among the vertices of a regular hexagon is one of the following:

- a triangle Δ_1 whose vertices are three consecutive vertices of the hexagon;
- a triangle Δ_2 whose two vertices are adjacent vertices of the hexagon and the third one is adjacent to none of the first two;
- a triangle Δ_3 where any two vertices are not adjacent vertices of the hexagon.

Since the areas of the chosen triangles are different, the triangles must be equal to the triangles Δ_1, Δ_2, Δ_3. The hexagon can be divided into four parts (Fig. 9): the triangle Δ_3 surrounded by three triangles Δ_1. The area of the triangle Δ_2 (marked by a dotted line in Fig. 9) is twice the area of the triangle Δ_1 beacuse they have the same base but the height of Δ_2 is twice the height of Δ_1.

FR-4. Points A', B' and C' are chosen correspondingly on the sides AB, BC, and CA of an equilateral triangle ABC so that $\frac{|A'B|}{|AB|} = \frac{|B'C|}{|BC|} = \frac{|C'A|}{|CA|} = k$. Find all positive real numbers k for which the area of the triangle $A'B'C'$ is exactly half of the area of the triangle ABC. *(Grade 10.)*

Answer: $k = \frac{1}{2} \pm \frac{\sqrt{3}}{6}$.

Solution. Let α be the angle at the vertex A (Fig. 10). The area of the triangle $AA'C'$ is $S_{AA'C'} = \frac{1}{2} \cdot |AA'| \cdot |AC'| \cdot \sin\alpha = \frac{1}{2} \cdot (1-k)|AB| \cdot k|AC| \cdot \sin\alpha = (1-k)k\, S_{ABC}$. Similarly $S_{BB'A'} = (1-k)k\, S_{ABC}$ and $S_{CC'B'} = (1-k)k\, S_{ABC}$. Hence the triangles $AA'C'$, $BB'A'$ and $CC'B'$ are of equal area. Therefore

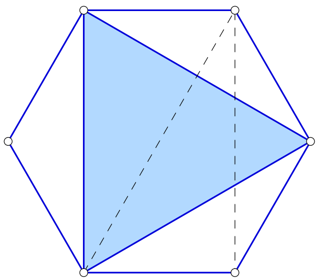


Fig. 9

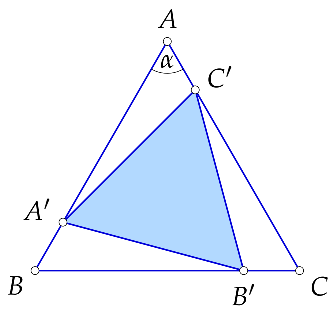


Fig. 10

the area of the triangle $A'B'C'$ is half of the area of the triangle ABC iff the area of the triangle $AA'C'$ is one sixth of the area of the triangle ABC, i.e. $(1-k)k = \frac{1}{6}$. The solutions of $k^2 - k + \frac{1}{6} = 0$ are $k_{1,2} = \frac{1}{2} \pm \frac{\sqrt{3}}{6}$, both of them are positive.

Remark. As seen from the solution, the result actually holds for an arbitrary triangle.

FR-5. Three players A, B and C play the following game. At the beginning of the game, each player has a sheet of paper with the name of the player written on it. Player A chooses one of the other players and replaces the name on this player's sheet with the name on his own sheet. Then player B makes a similar move, then player C and after that the turn to move goes to player A again. The game ends when all the sheets have the same name written on them and the winner is the player whose name it is. Does any of the players have a winning strategy (i.e., a strategy that allows a player to win no matter what his opponents play)? *(Grade 10.)*

Answer: no.

Solution 1. Player B does not have a winning strategy, since on the first move player A can write the name A on his sheet, after that the name B is not on any of the sheets. Similarly player C does not have a winning strategy.

To prove that even player A does not have a winning strategy, we show that players B and C have a joint strategy which guarantees that among the names written on the sheets there are at least two different names. Namely, if player A on his move writes a name on the sheet of player B, then B writes a name on the sheet of player A, otherwise on the sheet of player C. Player C always writes a name on the sheet of player B.

In the beginning both players B and C have names different from the name on the sheet of player A. Hence A cannot win in one move. Independent of which name A changes on his move, after B moves, the name on the sheet of C differs from the name on the sheet of A, and after C moves, both B and C have names on their sheets different from the one on the sheet of A, as in the beginning. So the cycle repeats.

Solution 2. Denote the players starting from any player in the order of their turns by X, Y, and Z. Show that the players Y and Z can together always keep X from winning. Indeed, X can win only on his move because Y and Z can always play so that their move does not result immediately in X winning. X can win on his turn only if before his move he and somebody else have his name on their sheets. The player Z cannot prevent this situation only if the same situation occurred already before his move and his sheet has the name of X on it. But after Y moves, then either X or Z has the same name on their sheets as Y has, and so Y can always prevent both X and Z having the same name on their sheets. Thus none of the three players has a winning strategy.

FR-6. A regular 2010-gon is divided into pieces of triangular shape. Find the least possible number of pieces. *(Grade 10.)*

Answer: 2008.

Solution. All the interior angles of the 2010-gon can be built from the inner angles of the triangular pieces. As the sum of the inner angles of the 2010-gon is $2008 \cdot 180^\circ$ and that of every triangle is 180°, there must be at least 2008 triangles. On the other hand, each convex 2010-gon can be divided into exactly 2008 triangles by choosing one vertex and cutting the figure into pieces along the diagonals that start from this vertex.

FR-7. Let x, y and z be positive integers satisfying $\gcd(x,y,z) = 1$. Prove that if $(y^2 - x^2) - (z^2 - y^2) = ((y-x) - (z-y))^2$, then x and z are perfect squares. *(Grade 11.)*

Solution. Remove the parentheses, collect the terms and divide both sides by 2 to get $x^2 + y^2 + z^2 - 2xy - 2yz + xz = 0$. This equality can be written as $(x - y + z)^2 = xz$. Hence xz is a square of an integer. If x and z have a common divisor d, then xz is divisible by d^2, and by the previous equality $(x - y + z)^2$ is divisible by d^2, therefore $x - y + z$ is divisible by d. Since x and z are divisible by d, y must be divisible by d, hence $d = 1$, i.e. x and z do not have common divisors. Since xz is a square of an integer, it follows that both x and z are squares of integers.

FR-8. Find all pairs of integers (m, n) such that for all positive real numbers x and y the inequality $x^m + y^n \geqslant x^n y^m$ holds. *(Grade 11.)*

Answer: $(0, 0)$.

Solution. If $m = 0$, then the inequality is $1 + y^n \geqslant x^n$. This holds for all positive real numbers x and y iff $n = 0$. Hence $(0, 0)$ is a solution. Let now both m and n be different from zero. If the pair (m, n) satisfies the condition, then substituting x and y by $\frac{1}{x}$ and $\frac{1}{y}$ we see that the pair $(-m, -n)$ also satisfies the condition. Hence we can assume without loss of generality that $m \geqslant n$ and $m \geqslant 0$. If $m > n$, then by taking $x = 1$ we get $1 + y^n \geqslant y^m$, which does not hold for y large enough. Hence $m = n$. By taking $x = y = 4$ we get $2 \cdot 4^m \geqslant 4^{2m}$ which does not hold for any positive integer m. Therefore there are no more suitable pairs.

FR-9. Let D be the midpoint of side BC of triangle ABC. Prove that the intersection point of medians of triangle ABD and that of triangle ACD are equidistant from line AD. *(Grade 11.)*

Solution. Triangles ABD and ACD have equal area since $|BD| = |CD|$ and the altitudes drawn from A coincide (Fig. 11). As these triangles have a common side AD, also the altitudes drawn from vertices B and C, respectively, must be equal. Thus B and C are equidistant from line AD. Since the point

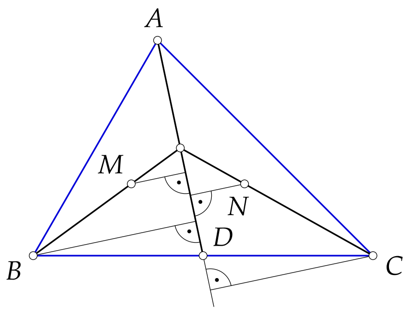
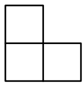


Fig. 11

of intersection of medians cuts $\frac{1}{3}$ part of every median, the distance of the point of intersection of medians of ABD from line AD is thrice shorter than the distance of point B from line AD. An analogous relation holds also for triangle ACD. Hence the claim follows.

FR-10. A unit L-shape consists of three unit squares as shown in the picture. Prove that for any positive integer k it is possible to cut a similar L-shape with k times larger side lengths into unit L-shapes. *(Grade 11.)*



Solution. Let the L-shape be placed so that the two longer sides meet at the top left corner. Starting from the top left we place on it k unit L-shapes diagonally with the same orientation as the large L-shape (Fig. 12). The rest consists of two equal staircase-like parts; it is enough to show that one of them, e.g. the lower part, can be covered. The staircase has k stairs, the lowest one at the height $k-1$ and the highest at the height $2k-2$.

In case $k = 1$ the staircase is empty, in case $k = 2$ it can be covered with one unit L-shape. Assume that the claim holds for the staircase with k stairs and consider the staircase with $k+2$ stairs. Separate a strip of width 2 from the left and bottom. The rest can be covered by the induction assumption. The topmost part of the strip is covered with one unit L-shape. Now we have

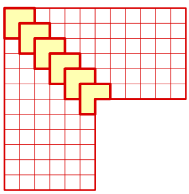


Fig. 12

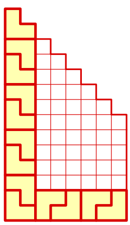


Fig. 13

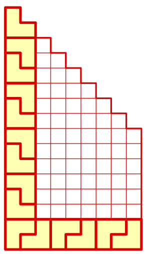


Fig. 14

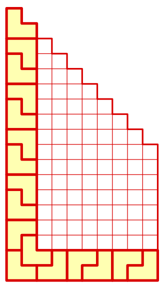


Fig. 15

to cover the rest of the strip whose lower and left sides have correspondingly the lengths $k+2$ and $2k$.

- If k is divisible by 3, then cut the figure into two strips of sizes $2 \times 2k$ and $k \times 2$ and cover both of them with 2×3 rectangles consisting of two unit L-shapes (Fig. 13) and we are done.
- If $k \equiv 1 \pmod 3$, then cut the figure into two strips of sizes $2 \times (2k-2)$ and $(k+2) \times 2$ and cover both of them with 2×3 rectangles (Fig. 14). This is possible because $2k-2$ and $k+2$ are divisible by 3.
- If $k \equiv 2 \pmod 3$, then cut the figure into two strips of sizes $2 \times (2k-4)$ and $(k-2) \times 2$, and a corner part, which is a L-shape with $k=2$. Both strips can be covered by 2×3 rectangles since $2k-4$ and $k-2$ are divisible by 3; the corner part can be covered by induction basis (Fig. 15).

FR-11. A ball bearing consists of two cylinders with the same axis and n equal balls between them. The centers of all the balls are on the same plane perpendicular to the axis of the cylinders and each ball touches both cylinders and two adjacent balls. Let r be the radius of the balls and let R be the radius of the outer cylinder. Prove that $\frac{r}{R} < \frac{\pi}{n+\pi}$. *(Grade 11.)*

Solution 1. Consider the regular n-gon with vertices at the centers of the balls (Fig. 16). Its edges are of length $2r$ and its perimeter is $n \cdot 2r$. The radius of the circumcircle of the n-gon is $R-r$ and the length of the circumcircle is $2\pi(R-r)$. Since a chord of a circle is always shorter than the corresponding arc of the circle, we have $n \cdot 2r < 2\pi(R-r)$ or $nr + \pi r < \pi R$, which implies $\frac{r}{R} < \frac{\pi}{n+\pi}$.

Solution 2. Consider the isosceles triangle with vertices at the centers of two adjacent balls and at the closest point to them on the common axis of the cylinders (Fig. 17). The two equal sides of the triangle are of length $R-r$, the base is of length $2r$ and the vertex angle is $\frac{2\pi}{n}$. The altitude drawn onto the base divides the triangle into two equal right triangles with the hypotenuse $R-r$, one of the legs r and the opposite angle $\frac{\pi}{n}$. Hence $\frac{r}{R-r} = \sin\frac{\pi}{n} < \frac{\pi}{n}$, whence $nr < \pi R - \pi r$, which implies $\frac{r}{R} < \frac{\pi}{n+\pi}$.

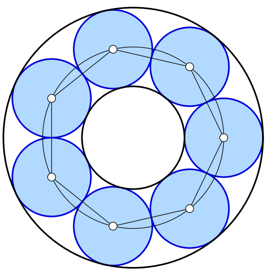


Fig. 16

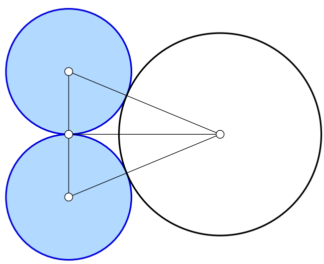


Fig. 17

FR-12. The sequence (a_n) is defined by $a_1 = 1$ and $a_n = n \cdot (a_1 + \ldots + a_{n-1})$ for all $n > 1$. Find all indices n for which a_n is divisible by $1 \cdot 2 \cdot \ldots \cdot n$. *(Grade 12.)*

Answer: 1 and all positive even numbers.

Solution. For each $n \geqslant 2$ denote $S_n = a_1 + \ldots + a_{n-1}$. Then $a_n = S_n \cdot n$ and for all $n > 2$ we have $S_n = S_{n-1} + a_{n-1} = S_{n-1} + S_{n-1} \cdot (n-1) = S_{n-1} \cdot n$. Hence $S_n = S_{n-1} \cdot n = S_{n-2} \cdot (n-1)n = \ldots = S_2 \cdot 3 \cdot \ldots \cdot n = \frac{n!}{2}$ because $S_2 = 1 = \frac{1 \cdot 2}{2}$. Consequently $a_n = S_n \cdot n = n! \cdot \frac{n}{2}$ for all $n \geqslant 2$. Therefore, for $n \geqslant 2$, a_n is divisible by $n!$ iff n is even, and $n = 1$ also satisfies the condition.

FR-13. The lengths of the sides of a quadrilateral are a, b, c, d and its area is S. Prove that $a^2 + b^2 + c^2 + d^2 \geqslant 4S$. For which quadrilaterals does the equality hold? *(Grade 12.)*

Answer. The equality holds only for squares.

Solution. Without loss of generality we can assume that a, b, c and d are the lengths of consecutive sides of the quadrilateral. A diagonal divides the quadrilateral into two triangles. From one partition we get the inequality $\frac{ab}{2} + \frac{cd}{2} \geqslant S$, whence $ab + cd \geqslant 2S$, and from the other partition $\frac{bc}{2} + \frac{da}{2} \geqslant S$, whence $bc + da \geqslant 2S$. Therefore $ab + bc + cd + da \geqslant 4S$.

On the other hand, by adding the inequalities $a^2 + b^2 \geqslant 2ab$, $b^2 + c^2 \geqslant 2bc$, $c^2 + d^2 \geqslant 2cd$, and $d^2 + a^2 \geqslant 2da$, and dividing by 2 we get $a^2 + b^2 + c^2 + d^2 \geqslant ab + bc + cd + da$, which implies the required inequality.

The equality holds iff all the inequalities used are, in fact, equalities. In the first inequality the equality holds iff all the angles are right angles. In the second step the equalities hold iff all sides are of equal length.

FR-14. In a coordinate city there are $n \geqslant 3$ tramlines parallel to the x-axis such that each line begins from x-coordinate 0 and ends at x-coordinate n. Exactly one tram of length 1 is moving on each line: on the first line with speed 1, on the second line with speed 2 etc, until on the last line with the speed n. When a tram reaches the end of its line it instantly starts moving back without turning around. In the morning all trams start moving at the same time from the starting position where the x-coordinate of the back end of the tram is 0. Prove that the trams' projections onto the x-axis never cover the whole interval from 0 to n. *(Grade 12.)*

Solution. The projections of the trams can cover the whole interval only when one projection covers $[0, 1]$, another $[1, 2]$ etc. until $[n-1, n]$. Consider the moments when the projection of the slowest tram covers one of these intervals. When the slowest tram moves by 1 unit, then the fastest and the third fastest trams move correspondingly by n and $n-2$ units. Together these two trams move by $2n-2$ units which is exactly one to and fro cycle.

Denote the integer positions of the trams on the round trip by numbers 0 to $2n-3$, i.e the starting position is 0 and each next one until returning to the starting point is greater by one. Call these numbers the position characteristics. If the sum of the position characteristics of two trams is $2n-2$, then their projections cover the same interval because one of them has moved the same amount from the starting point as the other one still has to go to reach it. If the sum of the position characteristics is 0, then they both are in the starting positions, so they again cover the same interval. Hence, when the sum of the position characteristics is divisible by $2n-2$, the projections cover the same interval.

At the beginning the sum of the position characteristics of the fastest and the third fastest tram is 0 and each time they together move by $2n-2$ units the sum of their position characteristics stays divisible by $2n-2$. Consequently, when the projection of the slowest tram covers an interval with integer endpoints, the projections of these two trams cover the same interval, hence at least one of the intervals is not covered.

FR-15. Find the minimal distance between two points, one of which is on the graph of function $y = e^x$ and the other on the graph of function $y = \ln x$. *(Grade 12.)*

Answer: $\sqrt{2}$.

Solution. The graphs of functions $y = e^x$ and $y = \ln x$ are symmetrical w.r.t. line $y = x$ (Fig. 18). Hence the distance between points on these graphs is minimal iff both points are at minimal distance from line $y = x$. The distance between the graph of the function $y = e^x$ and the graph of the function $y = x$ is minimal at the point where the tangent is parallel to $y = x$. Then, $y' = 1$ for the function $y = e^x$, whence $e^x = 1$, giving $x = 0$ and $y = 1$. The minimal distance is therefore between points $(0, 1)$ and $(1, 0)$ and it is $\sqrt{2}$.

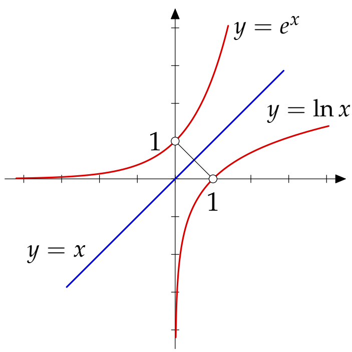


Fig. 18

IMO Team Selection Contest

First day

TS-1. For arbitrary positive integers a, b, denote $a \ominus b = \dfrac{a-b}{\gcd(a,b)}$.

Let n be a positive integer. Prove that the following conditions are equivalent:

(i) $\gcd(n, n \ominus m) = 1$ for every positive integer $m < n$;

(ii) $n = p^k$ where p is a prime number and k is a non-negative integer.

Solution. Note at first that $da \ominus db = a \ominus b$ for all positive integers a, b, and d. Indeed,

$$da \ominus db = \frac{da - db}{\gcd(da, db)} = \frac{d \cdot (a-b)}{d \cdot \gcd(a,b)} = \frac{a-b}{\gcd(a,b)} = a \ominus b.$$

Show now that if n is a prime power and $m < n$, then $n \ominus m$ is relatively prime to n. Indeed, let $n = p^k$ where p is a prime number, and let $m = p^i s$ where $\gcd(p, s) = 1$. Then $m < n$ implies $i < k$. Now

$$n \ominus m = p^{k-i} \ominus s = \frac{p^{k-i} - s}{\gcd(p^{k-i}, s)} = p^{k-i} - s$$

because $\gcd(p^{k-i}, s) = 1$ by the choice of s. Also, for the same reason, $\gcd(p, p^{k-i} - s) = 1$, hence $\gcd(n, n \ominus m) = \gcd(p^k, p^{k-i} - s) = 1$.

It remains to show that if n is not a prime power, then there exists a positive integer m such that $m < n$ and the integers $n \ominus m$ and n share a common prime factor. Since n is not a prime power, it has at least two different prime factors. Let p and q be some prime factors of n, whereby $p < q$. Let $n = p^k t$ where $\gcd(p, t) = 1$. Take $m = n - p^{k+1}$. As n is divisible by both p^k and q which are relatively prime, it is also divisible by their product $p^k q$. Consequently, $p^{k+1} < p^k q \leqslant n$, i.e., $0 < m < n$. Now

$$n \ominus m = n \ominus (n - p^{k+1}) = t \ominus (t - p) = \frac{t - (t-p)}{\gcd(t, t-p)} = \frac{p}{\gcd(t,p)} = p$$

since $\gcd(t, p) = 1$. We see that $n \ominus m$ and n have a common prime factor p.

TS-2. Let n be a positive integer. Find the largest integer N for which there exists a set of n weights such that it is possible to determine the mass of all bodies with masses of 1, 2, ..., N using a balance scale (i.e. to determine whether a body with unknown mass has a mass 1, 2, ..., N, and which namely).

Answer: $N = \frac{3^n - 1}{2}$.

Solution. The possibility to determine mass m means the possibility to place the weights on the two scalepans so that the difference of total masses on the two scalepans is exactly m.

Every weight can be placed on either of the two pans or on neither of the pans. For n weights this makes 3^n different placements. Note that the placement where none of the weights is on the scales does not determine any mass. Also, for each placement there is a symmetric placement with all the weights on the two pans swapped, which determines the same mass. Therefore with n weights it is possible to determine at most $\frac{3^n-1}{2}$ different masses.

We show by induction that it is possible to determine all masses from 1 to $\frac{3^n-1}{2}$ using n weights with masses $1, 3, \ldots, 3^{n-1}$. For $n = 1$ it is obvious. Assume that the claim holds for $n = k$. By the induction assumption we can determine all masses from 1 to $\frac{3^k-1}{2}$ by weights $1, 3, \ldots, 3^{k-1}$. Using the weight with mass 3^k, we can determine the mass 3^k, and using it together with the other weights also the masses $3^k + 1, \ldots, 3^k + \frac{3^k-1}{2}$ and $3^k - 1, \ldots, 3^k - \frac{3^k-1}{2}$. Since $3^k - \frac{3^k-1}{2} = \frac{3^k-1}{2} + 1$ and $3^k + \frac{3^k-1}{2} = \frac{3^{k+1}-1}{2}$, the claim is also true for $n = k + 1$.

TS-3. Let the angles of a triangle be α, β, and γ, the perimeter $2p$ and the radius of the circumcircle R. Prove the inequality

$$\cot^2\alpha + \cot^2\beta + \cot^2\gamma \geqslant 3\left(\frac{9R^2}{p^2} - 1\right).$$

When is the equality achieved?

Answer: the equality holds for equilateral triangles.

Solution. Let the opposite sides of the angles α, β, and γ be correspondingly a, b, and c. Since $\cot^2\alpha = 1/\sin^2\alpha - 1$ and from the law of sines $1/\sin\alpha = 2R/a$, we have $\cot^2\alpha = 4R^2/a^2 - 1$; similarly $\cot^2\beta = 4R^2/b^2 - 1$ and $\cot^2\gamma = 4R^2/c^2 - 1$. The inequality can therefore be written as

$$4R^2 \cdot \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) - 3 \geqslant 3 \cdot \left(\frac{4 \cdot 9R^2}{(a+b+c)^2} - 1\right),$$

or

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \geqslant \frac{27}{(a+b+c)^2}.$$

Dividing both sides by 3 and taking the square root gives

$$\sqrt{\frac{1}{3} \cdot \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right)} \geqslant \frac{3}{a+b+c}.$$

The left side is the quadratic mean of $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ and the right side is the harmonic mean of the same numbers, hence the inequality holds.

The equality holds iff $a = b = c$.

Second day

TS-4. In an acute triangle ABC the angle C is greater than the angle A. Let AE be a diameter of the circumcircle of the triangle. Let the intersection point of the ray AC and the tangent of the circumcircle through the vertex B be K. The perpendicular to AE through K intersects the circumcircle of the triangle BCK for the second time at point D. Prove that CE bisects the angle BCD.

Solution. Since AE is a diameter of the circumcircle of the triangle ABC, $\angle ACE = \angle ECK = 90°$. So it suffices to show that $\angle ACB = \angle DCK$ (Fig. 19). Let L be the point of intersection of lines AE and DK. Then $\angle BAC = \angle CBK = \angle CDK$ by the inscribed angles theorem. Also $\angle ABC = \angle AEC = \angle CKD$ where the latter equality follows from the similarity of the right triangles ACE and ALK. Hence the two triangles ABC and DKC are similar, and therefore $\angle ACB = \angle DCK$.

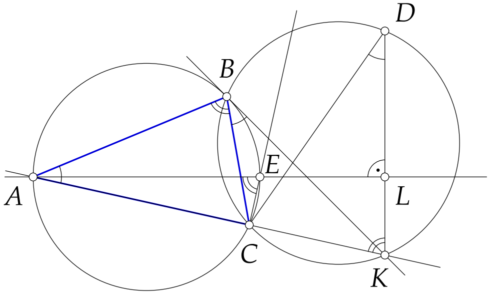


Fig. 19

Remark. This problem was proposed to the Baltic Way competition in 2008 (not by Estonia) but was not selected.

TS-5. Let $P(x,y)$ be a non-constant homogeneous polynomial with real coefficients such that $P(\sin t, \cos t) = 1$ for every real number t. Prove that there exists a positive integer k such that $P(x,y) \equiv (x^2+y^2)^k$.

Solution 1. Let n be the degree of the polynomial P, i.e.,

$$P(x,y) = a_n x^n + a_{n-1}x^{n-1}y + \ldots + a_1 xy^{n-1} + a_0 y^n,$$

where $n > 0$. Note that n must be even because otherwise the condition $P(\sin t, \cos t) = 1$ for $t = 0$ would imply $a_0 = 1$ while the same condition for $t = \pi$ would imply $a_0 = -1$.

Since P has no constant term, $P(0,0) = 0$. Now assume that $x \neq 0$ or $y \neq 0$ and take $c = \sqrt{x^2+y^2}$. Since

$$\left(\frac{x}{\sqrt{x^2+y^2}}\right)^2 + \left(\frac{y}{\sqrt{x^2+y^2}}\right)^2 = 1,$$

there exists some real number t such that $\sin t = x/\sqrt{x^2+y^2}$ and $\cos t = y/\sqrt{x^2+y^2}$ and therefore $P(\sin t, \cos t) = 1$. By homogenicity, $P(x,y) = c^n \cdot P\left(\frac{x}{c}, \frac{y}{c}\right)$, hence

$$P(x,y) = \left(\sqrt{x^2+y^2}\right)^n \cdot P\left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right) = \left(\sqrt{x^2+y^2}\right)^n.$$

for all x, y. The case $n = 2k$ implies $P(x,y) = (x^2 + y^2)^k$ which satisfies also the condition $P(0,0) = 0$.

Solution 2. Like in Solution 1, express the polynomial as a sum of $n+1$ monomials with coefficients $a_0, \ldots, a_n$ and show that $n = 2k$.

We prove the claim of the problem by induction on k. In case $k = 0$ (omitting the extra assumption that P is non-constant) the claim holds obviously. Assume now that $k > 0$ and the claim holds for $k-1$. Substituting $t = 0$ and $t = \frac{\pi}{2}$ into $P(\sin t, \cos t) = 1$ gives $a_0 = 1$ and $a_n = 1$, respectively. Hence the polynomial $P(x,y) - (x^2+y^2)^k$ does not have terms with x^n and y^n. Let $Q(x,y)$ be such that $P(x,y) - (x^2+y^2)^k \equiv xy \cdot Q(x,y)$. Then $\sin t \cos t \cdot Q(\sin t, \cos t) = 0$ for every real number t, hence $Q(\sin t, \cos t) = 0$ for every t such that $\sin 2t \neq 0$. By continuity of $Q(\sin t, \cos t)$ as a function of t, it follows that $Q(\sin t, \cos t) \equiv 0$. Now define $R(x,y) = Q(x,y) + (x^2 + y^2)^{k-1}$. As both Q and R are homogeneous polynomials of degree $2(k-1)$, the assumptions of the problem hold for polynomial R. By the induction hypothesis, $R(x,y) \equiv (x^2+y^2)^{k-1}$. Hence $Q(x,y) \equiv 0$ and $P(x,y) \equiv (x^2+y^2)^k$.

TS-6. Every unit square of a $n \times n$ board is colored either red or blue so that among all 2×2 squares on this board all possible colorings of 2×2 squares with these two colors are represented (colorings obtained from each other by rotation and reflection are considered different).

a) Find the least possible value of n.

b) For the least possible value of n find the least possible number of red unit squares.

Answer: a) 5; b) 10.

Solution. a) Since there are $2^4 = 16 = 4^2$ possibilities to color a 2×2 square in two colors and a $n \times n$ square contains $(n-1)^2$ such subsquares, we must have $n - 1 \geqslant 4$, or $n \geqslant 5$. For $n = 5$ a suitable coloring is given in Fig. 20.

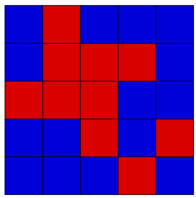


Fig. 20

b) Fig. 20 presents a coloring with 10 red squares. We will show that this is the least possible.

Note that in the 5×5 square there are 4 unit squares in the corners, 12 squares on the sides (not in the corners), and 9 inner squares. Each corner square is contained in exactly one, side square in two and inner square in four 2×2 squares. All 16 colourings of 2×2 squares contain a total of 64 unit squares of which 32 are red by symmetry. Therefore, if the 5×5 square contains k red squares, among them a corner squares, b side squares and c inner squares, then $a + b + c = k$ and $a + 2b + 4c = 32$. The equation $a + 2b + 4c = 32$ implies $c \leqslant 8$. If $c = 8$, then $a = b = 0$. If $c = 7$, then the only possibility to have $k < 10$ is $b = 2$ and $a = 0$. If $c \leqslant 6$, then always $k = a + b + c \geqslant 10$.

Thus it is enough to show that there are no colorings with required properties with $a = 0$ and $b \leqslant 2$. Indeed, in this case the 5×5 square has at least two sides not containing any red squares. Without loss of generality, let one of them be the upper side. We saw in part a) that for $n = 5$ each coloring of 2×2 squares must occur exactly once. Since among all 16 colorings of 2×2 squares there are 4 such where both upper unit squares are blue, and two upper rows of the 5×5 square contain exactly 4 2×2 squares, all four such colorings must be located in the two upper rows, among these the completely blue coloring. Since the same is true for the other side which does not contain any red squares, the two sides must meet and a completely blue 2×2 square must be in the corner where the two sides meet. Without loss of generality, let it be the left side. Then the two squares on Fig. 21 must be red, because otherwise there would be more than one completely blue 2×2 square. But now there are two 2×2 squares with red square in the lower right corner and the rest of them blue. Therefore there is no coloring satisfying the conditions with $a = 0$ and $b \leqslant 2$ and the least number of red squares is $k = 10$.

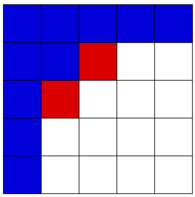


Fig. 21

Problems Listed by Topic

Number theory: OC-1, OC-6, OC-7, OC-12, FR-1, FR-7, FR-12, TS-1
Algebra: OC-4, OC-5, OC-13, FR-2, FR-8, FR-15, TS-5
Geometry: OC-2, OC-9, OC-11, FR-4, FR-6, FR-9, FR-11, FR-13, TS-3, TS-4
Discrete mathematics: OC-3, OC-8, OC-10, FR-3, FR-5, FR-10, FR-14, TS-2, TS-6