Estonian Math Competitions 2010/2011

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Estonian Mathematical Olympiad

Mathematics Contests in Estonia

The Estonian Mathematical Olympiad is held annually in three rounds: at the school, town/regional and national levels. The best students of each round (except the final) are invited to participate in the next round. Every year, about 110 students altogether reach the final round.

In each round of the Olympiad, separate problem sets are given to the students of each grade. Students of grade 9 to 12 compete in all rounds, students of grade 7 to 8 participate at school and regional levels only. Some towns, regions and schools also organize olympiads for even younger students. The school round usually takes place in December, the regional round in January or February and the final round in March or April in Tartu. The problems for every grade are usually in compliance with the school curriculum of that grade but, in the final round, also problems requiring additional knowledge may be given.

The first problem solving contest in Estonia took place already in 1950. The next one, which was held in 1954, is considered as the first Estonian Mathematical Olympiad.

Apart from the Olympiad, open contests are held twice a year, usually in October and in December. In these contests, anybody who has never been enrolled in a university or other higher education institution is allowed to participate. The contestants compete in two separate categories: the Juniors and the Seniors. In the first category, students up to the 10th grade can participate; the other category has no restriction. Being successful in the open contests generally assumes knowledge outside the school curriculum.

Based on the results of all competitions during the year, about 20 IMO team candidates are selected. IMO team selection contest for them is held in April or May, lasting two days; each day, the contestants have 4.5 hours to solve 3 problems, similarly to the IMO. All participants are given the same problems. Some problems in our selection contest are at the level of difficulty of the IMO but somewhat easier problems are usually also included.

The problems of previous olympiads are available at the Estonian Mathematical Olympiad's website `http://www.math.olympiaadid.ut.ee/eng`.

Besides the above-mentioned contests and the quiz "Kangaroo" other regional and international competitions and matches between schools are held as well.

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This booklet presents the problems of the open contests, the final round of national olympiad and the team selection contest. For the open contests and the final round, selection has been made to include only problems that have not been taken from other competitions or problem sources and seem interesting enough. The team selection contest is presented entirely.

Selected Problems from Open Contests

OC-1. *(Juniors.)* Find all pairs (a,b) of real numbers with $a+b=1$, which satisfy $(a^2+b^2)(a^3+b^3)=a^4+b^4$.

Answer: $(0,1)$, $(1,0)$, and $(\frac{1}{2},\frac{1}{2})$.

Solution 1. As $a^3+b^3=(a+b)(a^2-ab+b^2)$, the given equation can be expressed as $(a^2+b^2)(a^2-ab+b^2)=a^4+b^4$. Expanding brackets gives $-a^3b+2a^2b^2-ab^3=0$, which factorizes to $-ab(a-b)^2=0$. Hence $a=0$ or $b=0$ or $a-b=0$. Together with the condition $a+b=1$, we get the following solutions: $(0,1)$, $(1,0)$, and $(\frac{1}{2},\frac{1}{2})$.

Solution 2. Denote $ab=c$. Then $a^2+b^2=(a+b)^2-2ab=1-2c$, $a^3+b^3=(a+b)(a^2-ab+b^2)=1\cdot(1-2c-c)=1-3c$, and $a^4+b^4=(a^2+b^2)^2-2a^2b^2=(1-2c)^2-2c^2=2c^2-4c+1$. The given equation $(a^2+b^2)(a^3+b^3)=a^4+b^4$ can now be expressed as $(1-2c)(1-3c)=2c^2-4c+1$, or equivalently, $c(4c-1)=0$. Hence, $c=0$ or $c=\frac{1}{4}$. Now the two simultaneous equations $a+b=1$ and $ab=c$ give the solutions $a=0$, $b=1$ and $a=1$, $b=0$ for $c=0$, and $a=\frac{1}{2}$, $b=\frac{1}{2}$ for $c=\frac{1}{4}$.

OC-2. *(Juniors.)* Consider a parallelogram $ABCD$.

a) Prove that if the incenter of the triangle ABC is located on the diagonal BD, then the parallelogram $ABCD$ is a rhombus.

b) Is the parallelogram $ABCD$ a rhombus whenever the circumcenter of the triangle ABC is located on the diagonal BD?

Answer: b) no.

Solution. a) As the incenter of the triangle ABC is located on diagonal BD (Fig. 1), we can conclude that BD is the bisector of $\angle ABC$. Therefore $\angle ABD=\angle CBD$. However, since $ABCD$ is a parallelogram, $\angle ABD=\angle CDB$. Hence the triangle BCD is isosceles, i.e. $|BC|=|CD|$. Thus, $ABCD$ is a rhombus.

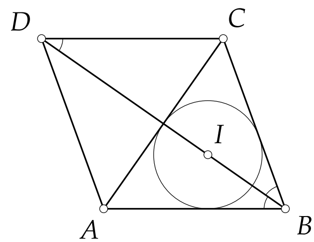


Fig. 1

b) Let $ABCD$ be a rectangle with different side lengths. The circumcenter of triangle ABC is located on the intersection of the diagonals of the rectangle. We see that all the required conditions are satisfied, however $ABCD$ is not a rhombus.

OC-3. *(Juniors.)* The numbers 0, 1, and 2 are written in the vertices of a triangle. One step involves increasing two of the three numbers by m or decreasing one of the three numbers by n. Is it possible that after some steps there are numbers 1, 2, and 3 (in an arbitrary order) written in the vertices if

a) $m=3$, $n=6$;

b) $m=4\frac{1}{2}$, $n=6$?

Answer: a) no; b) yes.

Solution 1. a) Both the step that involves increasing two of the numbers by 3 and the step that involves decreasing one of the numbers by 6 result in the sum of all three numbers being changed by 6. Thus the remainder when the sum of the three numbers is divided by 6 will always be the same regardless of the number of steps taken. But as the sums $0+1+2$ and $1+2+3$ give different remainders when divided by 6, it is impossible to reach the required end situation from the given initial situation.

b) First increase the second and the third numbers three times by $4\frac{1}{2}$; we end up with 0, $14\frac{1}{2}$, $15\frac{1}{2}$ in the vertices. Now increase the first and the second numbers by $4\frac{1}{2}$ and also increase the first and the third numbers by $4\frac{1}{2}$; so we end up with 9, 19 and 20 written in the three vertices, respectively. Finally decrease the first number once by 6 and the other two three times by 6, achieving the situation in question.

Solution 2. a) Consider one of the numbers. The remainder when this number is divided by 3 is the same regardless of the number of steps taken. Therefore, if we want to achieve the situation where 1, 2, 3 are located in the three vertices, the numbers 1 and 2 should stay in the same vertices where they were at the beginning and 3 has to be in the vertex where 0 was. Notice that two increasings are exactly cancelled out by one decreasing. Thus, the vertices where the numbers remain the same should have undergone an even number of increasings and the vertex where 0 is replaced by 3 should have been exposed to an odd number of increasings. Hence there should have been an odd number of increasings in total which is impossible since each increasing step influences the numbers in two vertices.

b) As in Solution 1.

OC-4. *(Juniors.)* Find all pairs (n, k) of positive integers that satisfy the equality $n! + (n+1)! = k! + 120$.

Answer: $(4, 4)$, $(5, 6)$.

Solution 1. Note that for every n, $n! + (n+1)! = n! + n! \cdot (n+1) = n! \cdot (n+2) \leqslant (n+2)!$. Thus if $n! + (n+1)! = k! + 120$, then due to $120 = 5!$ we have $k! + 5! \leqslant (n+2)!$. This inequality in turn implies $k < n+2$ and $5 < n+2$. Hence $5 \leqslant n+1$, leading to $0 \leqslant (n+1)! - 5! = k! - n!$. Consequently, $k \geqslant n$, i.e., the cases to be considered are $k = n$ and $k = n+1$. If $k = n$, then the initial equation leads to $(n+1)! = 120$, giving $n = 4$, $k = 4$. If $k = n+1$, then analogously $n = 5$, $k = 6$.

Solution 2. If $k < n$, then $n! > k!$. If, additionally, $n \geqslant 4$, then $(n+1)! \geqslant 120$, giving $n! + (n+1)! > k! + 120$. But if $n < 4$, then $n! + (n+1)! \leqslant 30 < k! + 120$. The desired equality can hold in neither of the cases.

If $k > n+1$, then $k! - (n+1)!$ is positive and is also divisible by $(n+1)!$, hence $k! - (n+1)! \geqslant (n+1)!$. On the other hand, $n! - 120 < (n+1)!$, giving $k! - (n+1)! > n! - 120$. Thus, there is no solution in this case either.

Hence $n = k$ or $n+1 = k$, leading to two solutions $(n, k) = (4, 4), (5, 6)$.

Solution 3. The equation implies that 120 is divisible by the minimum of $n!$ and $k!$. As $120 = 5!$, either $n \leqslant 5$ or $k \leqslant 5$. Consider both cases.

If $n \in \{1, 2, 3\}$, then $n! + (n+1)! - k! < 120$, whence the equation has no solution. If $n = 4$, then $120 = 144 - k!$, whence $k! = 24$ and $k = 4$. If $n = 5$, then $120 = 840 - k!$, whence $k! = 720$ and $k = 6$.

If $k \in \{1, 2, 3, 4, 5\}$, then the initial equation implies that $n! + (n+1)!$ lies between 121 and 240. This is possible only if $n = 4$, since if $n = 3$, then $n! + (n+1)! = 30$, and if $n = 5$, then $n! + (n+1)! = 840$. If $n = 4$, then $n! + (n+1)! = 144$, which corresponds to $k = 4$.

OC-5. *(Juniors.)* Consider the diagonals A_1A_3, A_2A_4, A_3A_5, A_4A_6, A_5A_1 and A_6A_2 of a convex hexagon $A_1A_2A_3A_4A_5A_6$. The hexagon whose vertices are the points of intersection of the diagonals is regular. Can we conclude that the hexagon $A_1A_2A_3A_4A_5A_6$ is also regular?

Answer: yes.

Solution 1. We show that the hexagon $A_1A_2A_3A_4A_5A_6$ has all its side lengths equal and all its angles equal. As the internal hexagon is regular, the grey triangles in Fig. 2 all have two angles of equal size and so they are isosceles. Additionally, all these six isosceles triangles have their bases of equal lengths, thus they are all congruent. The black triangles on Fig. 2 are isosceles because the grey triangles are isosceles. Additionally, their vertex angles are equal, as they all are equal to the angles of a regular hexagon. Therefore the black triangles are all congruent and thus their bases are of equal length. Now the angles of the external hexagon are all formed of the angles of two grey triangles and one black triangle. As both of the latter are congruent, the external hexagon has its angles of equal size.

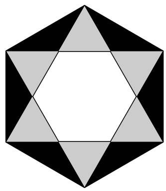


Fig. 2

Solution 2. Lengthen the sides of the internal regular hexagon until intersection. The points of intersection are exactly the vertices of the initial external hexagon. Because of the symmetry of the internal hexagon the points of intersection are symmetrically located about the midpoint of the internal hexagon. Thus, the external (initial) hexagon is also regular.

OC-6. *(Juniors.)* A cashier has a stack of n notes lying on top of each other. He has to turn all notes front side up, however the order of the notes is not important. Every step consists of taking a block of consecutive notes and turning them around in the stack. Find the smallest number of steps that will suffice him to turn all notes in the stack front side up, irrespective of the initial position of notes.

Answer: $\frac{n}{2}$ for even n, and $\frac{n+1}{2}$ for odd n.

Solution 1. Let the cashier always choose the part of the stack which starts from the top most note facing the wrong way and ends with the bottom most note facing the wrong way. Then after k steps there are at least

k notes facing the right way on top of the stack and so are k notes at the bottom of the stack. Therefore for even n, after at least $\frac{n}{2}$ steps all the notes are facing upwards. For odd n it is guaranteed that after $\frac{n-1}{2}$ steps there are $n-1$ notes facing the right way, and one more step may be needed to turn the middle note the right way. So, overall at least $\frac{n+1}{2}$ steps are sufficient.

To show the necessity, assume that the stack itself is located in between a bigger stack where all the notes are already facing the right way, and call the number of pairs of neighboring notes where one of the notes is the right way around and the other is the wrong way around *degree of disarrangement*. If the degree of disarrangement of the stack is 0, we have achieved the required situation (we introduce the concept of the bigger stack to ensure that if all the notes in our stack are the wrong way around the degree of disarrangement of the stack is non-zero). At every step, the degree of disarrangement of the stack cannot decrease by more than 2, since the relative order of consecutive notes can only change at the boundaries of the block to be turned around. Assume the initial stack consists of notes facing alternately the right way and the wrong way, whereby for odd n assume the top and the bottom notes are the wrong way around. Then for odd n the degree of disarrangement to start with is $n+1$ and so we need at least $\frac{n+1}{2}$ steps. For even n the initial degree of disarrangement is n because one of the endmost notes is the right way around and thus at least $\frac{n}{2}$ steps are needed.

Remark. In the first part of the solution we cannot claim that at every step the *number of notes* facing the wrong way decreases by 2. In addition to the two endmost notes facing the wrong way, all the notes between them are also turned around and as a result the number of notes facing the wrong way may even increase. For that reason it is necessary to monitor the progression of success from the stack's ends onwards.

Solution 2. Let us show that every possible stack can be put in order as required by no more than $\frac{n}{2}$ (for even n) or $\frac{n+1}{2}$ (for odd n) steps as follows.

For $n=2$ the statement holds. Indeed, if both notes are facing the wrong way, we simply turn them around together, and if only one of the notes is facing the wrong way we only turn this one around. Now partition the stack into smaller stacks, each containing two consecutive notes and solve the problem for each stack separately. For even n no more than $\frac{n}{2}$ steps are required. For odd n the last note has no counterpart, however it still may require one step, so overall no more than $\frac{n+1}{2}$ steps are needed.

The necessity is shown as in Solution 1.

Solution 3. Let us present one more strategy to achieve the required situation by $\frac{n}{2}$ (for even n) or $\frac{n+1}{2}$ (for odd n) steps for any stack of notes. If no more than half of the notes are facing the wrong way we simply turn each of them around individually. This requires as many steps as is the number of such notes, i.e. for even n no more than $\frac{n}{2}$ steps and for odd n no more than $\frac{n-1}{2}$ steps. If more than half of the notes are facing the wrong way, first turn the whole stack around and then individually turn around all those notes facing the wrong way. For even n no more than $1 + (\frac{n}{2} - 1) = \frac{n}{2}$ steps are required. For odd n no more than $1 + \frac{n-1}{2} = \frac{n+1}{2}$ steps are required.

The necessity is again shown as in Solution 1.

OC-7. *(Juniors.)* Consider a positive integer N with exactly 6 positive divisors $d_1, \ldots, d_6$ such that $1 = d_1 < d_2 < d_3 < d_4 < d_5 < d_6 = N$. Call such an integer N *good* if the sum $d_4 + d_5$ is divisible by the sum $d_2 + d_3$.

a) Find the smallest positive integer N which has exactly 6 positive divisors and which is not good.

b) Prove that there are infinitely many positive integers N all with exactly 6 positive divisors and all not good.

Answer: a) 20.

Solution 1. a) Considering the numbers from 1 to 20 we see that exactly three of them have 6 divisors: 12 (the divisors are $1, 2, 3, 4, 6, 12$), 18 ($1, 2, 3, 6, 9, 18$), and 20 ($1, 2, 4, 5, 10, 20$). For 12 the sum $d_4 + d_5 = 4 + 6$ is divisible by the sum $d_2 + d_3 = 2 + 3$ and similarly, for 18 the sum $d_4 + d_5 = 6 + 9$ is divisible by $d_2 + d_3 = 2 + 3$. However, for 20 the sum $d_4 + d_5 = 5 + 10$ is not divisible by the sum $d_2 + d_3 = 2 + 4$. Thus, the smallest non-good number with exactly 6 factors is 20.

b) Take $N = 4p$, where p is an arbitrary prime number larger than 4. Then N has exactly 6 different divisors: $1, 2, 4, p, 2p, 4p$, in increasing order. Indeed, as $N = 2^2 p^1$, where 2 and p are two different prime numbers, all of its divisors can be expressed as $2^i p^j$ where $i \leqslant 2$ and $j \leqslant 1$. From here we obtain exactly $3 \cdot 2 = 6$ choices: i can be either 0, 1 or 2 and for every i we have two choices for j: 0 or 1.

Here, $d_4 + d_5 = p + 2p = 3p$ is odd because $p > 2$ and thus is not divisible by an even number $d_2 + d_3 = 2 + 4 = 6$. Therefore, none of the numbers expressed as $N = 4p$ where $p > 4$ is a prime is good. As there are infinitely many prime numbers, there must also be infinitely many such numbers N.

Solution 2. Let us find all $N > 1$ which have exactly 6 divisors.

1) If $N = p^k$, where p is a prime, then it has the divisors $1, p, \ldots, p^k$, i.e. $k + 1$ divisors overall. Thus, all $N = p^5$ satisfy this requirement.

2) Let N have two different prime divisors, i.e. $N = p^k q^l$. For $k \geqslant 2$ and $l \geqslant 2$ we see that N has at least 9 different divisors: $1, p, p^2, q, q^2, pq, p^2q, pq^2$, and p^2q^2. For $l = 1$, N has the divisors $1, p, \ldots, p^k$, and $q, pq, \ldots, p^kq$, i.e. $2(k+1)$ divisors in total. Thus, all $N = p^2q$ satisfy the requirement.

3) Let N have at least three prime divisors p, q, r. Then N has at least 8 different divisors: $1, p, q, r, pq, pr, qr$, and pqr, and we get no more numbers.

Let us now consider $N = p^5$ and $N = p^2q$ in more detail.

i) If $N = p^5$, then $d_i = p^{i-1}$ and $d_4 + d_5 = p^3 + p^4 = p^3(1 + p)$ is divisible by $d_2 + d_3 = p + p^2 = p(1 + p)$. Thus they are all good.

ii) If $N = p^2q$, where $q < p$, then N has the divisors 1, q, p, pq, p^2, and p^2q, in increasing order, and $d_4 + d_5 = pq + p^2 = p(q + p)$ is divisible by $d_2 + d_3 = q + p$ and thus, they are all good, too.

iii) If $N = p^2q$, where $p < q < p^2$, then N has the divisors 1, p, q, p^2, pq, and p^2q, in increasing order, and $d_4 + d_5 = p^2 + pq = p(p + q)$ is divisible by $d_2 + d_3 = p + q$. Thus, they are all good, too.

iv) Finally, let $N = p^2q$, where $q > p^2$. Then N has the divisors 1, p, p^2, q, pq, and p^2q, in increasing order and $d_4 + d_5 = q + pq = q(1 + p)$ is not divisible by $d_2 + d_3 = p + p^2 = p(1 + p)$ because the prime number q cannot be divisible by another prime number p. Thus all these numbers have exactly 6 different divisors and they all are non-good. To get the smallest of these numbers, we have to take p and q as small as possible, i.e. $p = 2$ and $q = 5$ (to achieve $q > p^2 = 4$). Then $N = 2^2 \cdot 5 = 20$. Finally, there are infinitely many of these numbers N because we have infinitely many choices for prime numbers p and q such that $q > p^2$. For example, we can take $p = 2$ and q an arbitrary prime number bigger than 5. As there are infinitely many prime numbers, we have proven the statement.

OC-8. *(Seniors.)* Prove that none of the integers that contain one 2, one 1, and all the rest zeros, can be expressed as a sum of two perfect squares or as a sum of two perfect cubes.

Solution. All the numbers described in the problem are divisible by 3 (as their sum of digits is divisible by 3). Note that all perfect squares leave the remainder 0 or 1 when divided by 3, and therefore, for the sum of the two perfect squares to be divisible by 3, they both have to be divisible by 3. Now, as the numbers are both divisible by 3, their squares are divisible by 9 and thus the sum of the squares is divisible 9. However, the number described in the problem is not divisible by 9, a contradiction.

Note that when a cube number is divided by 3, it will leave the remainder of either 0, 1 or -1. Indeed, $(3k)^3 = 9(3k^3)$ and $(3k \pm 1)^3 = 27k^3 \pm 27k^2 + 9k \pm 1 = 9(3k^3 \pm 3k^2 + k) \pm 1$. The numbers described in the problem give the remainder 3 when divided by 9 (as their sum of digits gives the remainder 3 when divided by 9), therefore we conclude that it is impossible to express them as sums of two cubes.

OC-9. *(Seniors.)* Consider an acute-angled triangle ABC and its circumcircle. Let D be a point on the arc AB which does not include point C and let A_1 and B_1 be points on the lines DA and DB, respectively, such that $CA_1 \perp DA$ and $CB_1 \perp DB$. Prove that $|AB| \geqslant |A_1B_1|$.

Solution 1. If CD is the diameter of the circumcircle of triangle ABC, then $A_1 = A$ and $B_1 = B$ and the statement holds. Assume that CD is not the diameter (Fig. 3). Then $A_1 \neq A$ and $B_1 \neq B$. The point A_1 lies on the ray AD if and only if the point B_1 does not lie on the ray BD (depending on which side of the diameter through point C point D is located). Thus, $\angle CAA_1 =$

$\angle CBB_1$ (because the sum of opposite angles of a cyclic quadrilateral $ADBC$ is $180°$). Thus, the right-angled triangles AA_1C and BB_1C are similar. From $\angle ACA_1 = \angle BCB_1$ we see that $\angle ACB = \angle A_1CB_1$. This together with $\frac{|AC|}{|BC|} = \frac{|A_1C|}{|B_1C|}$ gives that ACB and A_1CB_1 are similar. As $|AC| > |A_1C|$, we conclude that $|AB| > |A_1B_1|$.

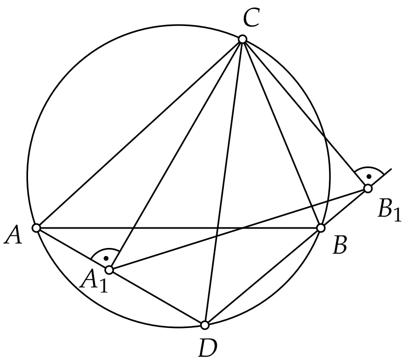


Fig. 3

Solution 2. Since the angles CA_1D and CB_1D are right angles, the points C, A_1, D, and B_1 form a cyclic quadrilateral and thus $\angle CAB = \angle CDB = \angle CA_1B_1$. Similarly, $\angle CBA = \angle CB_1A_1$. Therefore the triangles ABC and A_1B_1C are similar. As $|CA| \geqslant |CA_1|$, we deduce that $|AB| \geqslant |A_1B_1|$.

Solution 3. The radius R of the circumcircle of the quadrilateral $CADB$ is at least as large as the radius R_1 of the circumcircle of the quadrilateral CA_1DB_1 because CD is a chord in the first one and a diameter in the second one. The sine law in triangles ADB and A_1DB_1 gives $|AB| = 2R \sin \angle D$ and $|A_1B_1| = 2R_1 \sin \angle D$. As $R \geqslant R_1$, we deduce $|AB| \geqslant |A_1B_1|$.

Remark. The statement holds for all triangles ABC and all points D on the circumcircle, given D is not one of the vertices of the triangle.

OC-10. *(Seniors.)* Find all pairs (m, n) of positive integers for which the $m \times n$ grid contains exactly 225 rectangles whose side lengths are odd and whose edges lie on the lines of the grid.

Answer: $(1, 29)$, $(5, 9)$, $(9, 5)$, and $(29, 1)$.

Solution. The $m \times n$ grid is formed by $m + 1$ horizontal and $n + 1$ vertical lines. Number the horizontal lines with numbers from 1 to $m + 1$ and the vertical lines with numbers from 1 to $n + 1$. Rectangles with odd side lengths arise if and only if two horizontal lines with different parity and two vertical lines with different parity intersect.

Assume that at least one of the numbers m and n is even. We can assume without loss of generality that $m = 2k$. Then there are exactly $k + 1$ odd-numbered and k even-numbered horizontal lines and thus there are $k(k+1)$ pairs of lines of different parity. But this means that overall the number of rectangles with odd side lengths is even and cannot be 225. Therefore m and n are both odd numbers. Let now $m = 2k - 1$ and $n = 2l - 1$. Then we have exactly k even-numbered and k odd-numbered horizontal lines and l even-numbered and l odd-numbered vertical lines. Overall it is possible to form $k \cdot k \cdot l \cdot l = (kl)^2$ rectangles with odd side lengths. From $(kl)^2 = 225$ we get $kl = 15$. The solutions are $k = 1$, $l = 15$ or $k = 3$, $l = 5$ (or vice versa). So $m = 1$, $n = 29$ or $m = 5$, $n = 9$ (or vice versa).

OC-11. *(Seniors.)* Given a triangle ABC where $|BC| = a$, $|CA| = b$ and $|AB| = c$, prove that the equality

$$\frac{1}{a+b} + \frac{1}{b+c} = \frac{3}{a+b+c}$$

holds if and only if $\angle ABC = 60°$.

Solution. By finding the common denominator on the left hand side, transform the equation to $(a + 2b + c)(a + b + c) = 3(a + b)(b + c)$. Expanding the brackets and simplifying gives $b^2 = a^2 + c^2 - ac$. Comparing the latter with the cosine law $b^2 = a^2 + c^2 - 2ac\cos\beta$, we see that the equality holds if and only if $\cos\beta = \frac{1}{2}$, i.e., $\beta = 60°$.

OC-12. *(Seniors.)* A square $ABCD$ lies in the coordinate plane with its vertices A and C lying on different coordinate axes. Prove that one of the vertices B or D lies on the line $y = x$ and the other one on $y = -x$.

Solution 1. Assume without loss of generality that A is located on the x-axis and C is located on the y-axis, let these points have coordinates of $A\,(a, 0)$ and $C\,(0, c)$. As the diagonals of a square bisect each other, we know that the intersection point P of diagonal is also the mid-point of AC, i.e. $P\,(\frac{a}{2}, \frac{c}{2})$ and $\overrightarrow{PC} = (-\frac{a}{2}, \frac{c}{2})$.

As the diagonals of a square are perpendicular to each other and of the same length, the vectors $\overrightarrow{PB}$ and $\overrightarrow{PD}$ have the same length as the vector $\overrightarrow{PC}$ and are perpendicular to it. But for a given vector $\vec{u} = (s, t)$, there are exactly two vectors perpendicular to and having the same length as it: $\vec{v} = (-t, s)$ and $-\vec{v} = (t, -s)$. For the vector $\vec{u} = \overrightarrow{PC} = (-\frac{a}{2}, \frac{c}{2})$ we get $\vec{v} = (\frac{c}{2}, \frac{a}{2})$ and w.l.o.g. we can assume that $\overrightarrow{PB} = \vec{v}$ and $\overrightarrow{PD} = -\vec{v}$. Now from here $B(\frac{a+c}{2}, \frac{a+c}{2})$ and $D(\frac{a-c}{2}, \frac{c-a}{2})$. Thus, we see that the point B is located on the line $y = x$ and point D is located on the line $y = -x$.

Solution 2. W.l.o.g., assume that the vertices of the square are labelled counter-clockwise with $A\,(a, 0)$, $C\,(0, c)$, where a, $c \geqslant 0$ (other cases are similar). Let O be the origin, then $\angle AOC = 90°$, i.e. the circumcircle (with

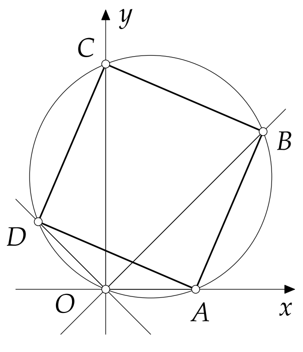


Fig. 4

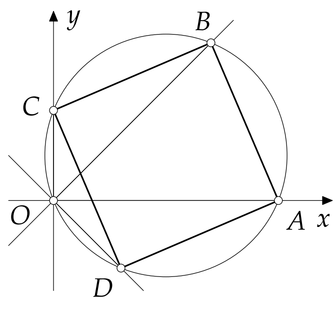


Fig. 5

diameter AC) of the square $ABCD$ passes through the origin O. Based on the assumptions made, B definitely lies in the first quadrant and D has to lie in the second quadrant (Fig. 4) or in the fourth quadrant (Fig. 5), otherwise the circle with the diameter BD cannot pass the origin. Now note that the vertices of the square divide its circumcircle into four equal arcs of $90°$, each having an inscribed angle of $45°$ subtending on it. Thus, $\angle AOB = \angle BOC = 45°$, i.e., B lies on the line with equation $y = x$ (if $A = O$ or $C = O$, then one of those angles will lose its meaning, however, the other one is still $45°$ and that is sufficient). Similarly, $\angle COD = 45°$, if D lies in the second quadrant, or $\angle AOD = 45°$, if D lies in the fourth quadrant. In both cases, D lies on the line $y = -x$; this condition is also met in the special case $D = O$.

OC-13. *(Seniors.)* Let a, b, c be fixed real numbers, where $0 \leqslant a, b, c \leqslant 4$. Prove that the system of equations

$$\begin{cases} p^2 - aq = -3 \\ q^2 - br = -4 \\ r^2 - cp = -5 \end{cases}$$

has no real solutions (p, q, r).

Solution. Adding up all equations gives $p^2 - cp + q^2 - aq + r^2 - br = -12$. From the inequality $(p - \frac{c}{2})^2 \geqslant 0$ we have $p^2 - cp \geqslant -\frac{c^2}{4} \geqslant -4$ and similarly, $q^2 - aq \geqslant -4$ and $r^2 - br \geqslant -4$. Adding up these inequalities, we see that to avoid a contradiction with the equality derived first, all three inequalities must actually be equalities, i.e. $a = b = c = 4$ and $p = q = r = 2$. But this does not satisfy the initial equations.

OC-14. *(Seniors.)* Let ABC be a triangle with integral side lengths. The angle bisector drawn from B and the altitude drawn from C meet at point P inside the triangle. Prove that the ratio of areas of triangles APB and APC is a rational number.

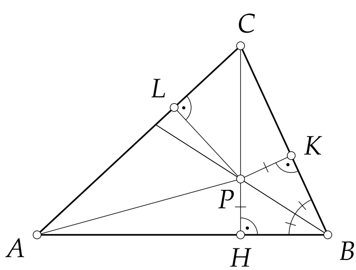


Fig. 6

Solution 1. Let H be the foot of the altitude drawn from C. First prove that $|AH|$ and $|BH|$ are rational numbers. For that, use the Pythagorean theorem for triangles ACH and BCH to obtain $|AH|^2 + |CH|^2 = |AC|^2$ and $|BH|^2 + |CH|^2 = |BC|^2$. Therefore $|AC|^2 - |BC|^2 = |AH|^2 - |BH|^2 = (|AH| - |BH|) \cdot (|AH| + |BH|) = (|AH| - |BH|) \cdot |AB|$. We see that $|AH| - |BH| = \frac{|AC|^2 - |BC|^2}{|AB|}$ is rational and so are $|AH| = \frac{|AB| + (|AH| - |BH|)}{2}$ and $|BH| = |AH| - (|AH| - |BH|)$. Let now K be the projection of P to BC (see Fig. 6). As P lies on the angle bisector of B, it is equidistant from both AB and BC, i.e., $|PH| = |PK|$. Consequently,

$\dfrac{S_{APB}}{S_{BPC}} = \dfrac{|AB| \cdot |PH|}{|BC| \cdot |PK|} = \dfrac{|AB|}{|BC|}$. As $CH \perp AB$, also $\dfrac{S_{BPC}}{S_{APC}} = \dfrac{|CP| \cdot |BH|}{|CP| \cdot |AH|} = \dfrac{|BH|}{|AH|}$. Thus, $\dfrac{S_{APB}}{S_{APC}} = \dfrac{|AB|}{|BC|} \cdot \dfrac{|BH|}{|AH|}$ is rational as a product of two rational numbers.

Solution 2. Let H be the foot of the altitude drawn from C and let L be the projection of P to AC (see Fig. 6). Now $\angle CPL = 90^\circ - \angle ACH = 90^\circ - (90^\circ - \angle CAB) = \angle CAB$, giving $\frac{|PH|}{|PL|} = \frac{|PH|}{|PC| \cdot \cos \angle CAB}$. The angle bisector theorem gives $\frac{|PH|}{|PC|} = \frac{|BH|}{|BC|} = \cos \angle ABC$. Consequently,

$$\frac{S_{APB}}{S_{APC}} = \frac{|AB| \cdot |PH|}{|AC| \cdot |PL|} = \frac{|AB|}{|AC|} \cdot \frac{\cos \angle ABC}{\cos \angle CAB}.$$

As the side lengths of the triangle ABC are integers, $\dfrac{|AB|}{|AC|}$ is rational. By the cosine law, the cosines of the angles of triangle ABC are rational, whence $\dfrac{\cos \angle ABC}{\cos \angle CAB}$ is rational. Altogether, $\dfrac{S_{APB}}{S_{APC}}$ is rational.

OC-15. *(Seniors.)* Prove that the set of integers $\{0, 1, 2, \ldots, 2^n - 1\}$ can be partitioned into $n+1$ disjoint subsets $A_0, A_1, \ldots, A_n$ such that both of the following hold:

a) If $k + l = n$, then the subsets A_k and A_l have the same number of elements.

b) If s and t are non-negative integers and $s + t \leqslant n$, then for an arbitrary element z in the set A_{s+t}, there exist elements x and y from the sets A_s and A_t, respectively, such that $x + y = z$.

Solution 1. Divide the set A into subsets such that the subset A_k consists of only those numbers which have exactly k ones in their binary representation. Then $A_0 = \{0\}$, $A_1 = \{1, 2, 4, \ldots, 2^{n-1}\}, \ldots, A_n = \{2^n - 1\}$. Let us show that both conditions are met. The first condition is met because the numbers with k ones are in one-to-one correspondence with the numbers with $n-k$ ones: given a number, simply replace all ones in its binary representation by zeros and vice versa. To show that the second condition is met, choose an arbitrary number z from the set A_{s+t}. Its binary representation contains exactly $s+t$ ones. Construct a binary number x by choosing s ones from the binary representation of z and filling all other binary places by zeros, analogously construct a second number y based on remaining ones in z. Then $x \in A_s$, $y \in A_t$ and $x + y = z$.

Solution 2. Let us prove the statement by induction. If $n = 1$, then $A = \{0, 1\}$, and taking $A_0 = \{0\}$ and $A_1 = \{1\}$ we get a partition that satisfies both of the requirements.

Assume now that we have a partition $C_0, C_1, \ldots, C_n$ for the set $C = \{0, 1, \ldots, 2^n - 1\}$. Construct a partition of the set $A = \{0, 1, \ldots, 2^{n+1} - 1\}$ based on that. First, generate the sets $B_0, B_1, \ldots, B_n$ as follows: the elements of the subset B_i are derived from the elements of the subset C_i by adding 2^n to them. The subsets $B_0, B_1, \ldots, B_n$ form a partition of the set $A \setminus C = \{2^n, 2^n+1, \ldots, 2^{n+1}-1\}$ and from the construction for all $i = 0, 1, \ldots, n$ the corresponding subsets B_i and C_i have the same number of elements.

Now, let $A_i = C_i \cup B_{i-1}$ for all $i = 1, 2, \ldots, n$ and in addition to that, $A_0 = C_0$ and $A_{n+1} = B_n$. Then, $|A_0| = |A_{n+1}| = 1$, and if $k+l = n+1$, then also $k, l \neq 0$, $|A_k| = |B_{k-1}| + |C_k| = |C_{k-1}| + |C_k|$ and $|A_l| = |B_{l-1}| + |C_l| = |C_{l-1}| + |C_l|$. As $(k-1)+l = k+(l-1) = n$, we see that $|C_{k-1}| = |C_l|$ and $|C_k| = |C_{l-1}|$; thus $|A_k| = |A_l|$.

To verify that the second condition is met, let z be an arbitrary element of A_{s+t}. If $t = 0$, then $A_s = A_{s+t}$ and $A_t = A_0 = \{0\}$, so we can take $x = z$ and $y = 0$. Now assume $t \geqslant 1$. If $z < 2^n$, then z is an element of C_{s+t} and thus there exist elements x and y in the sets $C_s \subset A_s$ and $C_t \subset A_t$, respectively, such that $x + y = z$. If $z \geqslant 2^n$, then z is an element of B_{s+t-1}, i.e. $z - 2^n$ is an element of the set C_{s+t-1} and thus the sets C_s and C_{t-1} contain elements x and y, respectively, such that $x + y = z - 2^n$. But now $x + (y + 2^n) = z$, where y and $z + 2^n$ are elements of the sets $C_s \subset A_s$ and $B_{t-1} \subset A_t$, respectively. So the statement holds for all positive n.

Remark. The subsets $A_0, A_1, \ldots, A_n$ formed in Solution 2 are actually the same as those formed in Solution 1.

OC-16. *(Seniors.)* How many positive integers are there that are divisible by 2010 and that have exactly 2010 divisors (1 and the integer itself included)?

Answer: 24.

Solution. Let N be a positive integer that is divisible by 2010 and that has exactly 2010 positive divisors. Since $2010 = 2 \cdot 3 \cdot 5 \cdot 67$, also N should be divisible by these four primes. Thus, $N = 2^a \cdot 3^b \cdot 5^c \cdot 67^d \cdot s$, where a, b, c, $d > 0$ and s is not divisible by any of the primes 2, 3, 5, 67. All the factors of N can be expressed as $2^i \cdot 3^j \cdot 5^k \cdot 67^l \cdot t$, where $0 \leqslant i \leqslant a$, $0 \leqslant j \leqslant b$, $0 \leqslant k \leqslant c$, $0 \leqslant l \leqslant d$, and t is a factor of s. There are $a+1$ choices for i (from 0 to a) and similarly, there are $b+1$, $c+1$ and $d+1$ choices for j, k and l, respectively. Therefore, N has $\delta(N) = (a+1)(b+1)(c+1)(d+1)\delta(s)$ different factors, where $\delta(x)$ stands for the number of factors of x. We require $\delta(N) = 2010$. As $a+1 > 1$, $b+1 > 1$, $c+1 > 1$, and $d+1 > 1$, we see that each of these numbers is divisible by some prime numbers and the number $\delta(N) = (a+1)(b+1)(c+1)(d+1)\delta(s)$ can thus be expressed as a product of at least four prime numbers. But as 2010 itself is a product of exactly four prime numbers, we conclude that $a+1$, $b+1$, $c+1$, and $d+1$ are exactly those primes 2, 3, 5, and 67, in some order, and $\delta(s) = 1$. From the latter condition we see that $s = 1$ because any numbers bigger than 1 has more than one factor. So for N to satisfy the conditions, N must be expressible as $2^a \cdot 3^b \cdot 5^c \cdot 67^d$, where a, b, c, d are the numbers 1, 2, 4, and 66 in some order. Thus there are $4! = 24$ numbers satisfying the conditions.

Selected Problems from the Final Round of National Olympiad

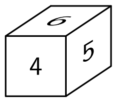
FR-1. *(Grade 9.)* Juku discovered that of the things in his satchel, 60 percent were ugly and 76 percent were useless. He scrapped all things that were both ugly and useless, and added things that were both beautiful and useful. After this, of the things in Juku's satchel, 25 percent are ugly and 45 percent are useless. How many percent of the things in Juku's satchel were both beautiful and useful initially?

Answer: 4 percent.

Solution. Observe that the amount of things that are beautiful but useless and things that are useful but ugly remained unchanged. The difference between the percentages of these things was $76\% - 60\% = 16\%$ before displacement but is $45\% - 25\% = 20\%$ after that. Hence the overall number of things in the satchel decreased $20 : 16 = 1.25$ times. Things that are beautiful but useless form 45% of all things after the displacement, hence they formed $45\% : 1.25 = 36\%$ before it. As there were $100\% - 60\% = 40\%$ of beautiful things in total, the things that were both beautiful and useful constituted $40\% - 36\% = 4\%$ of the content of the satchel.

Remark. This problem can of course be solved in completely standard ways via linear equations.

FR-2. *(Grade 9.)* There are 8 identical dice. The numbers 4, 5, 6 are written on three faces of the dice, as shown in the figure, and the remaining faces carry the numbers 1, 2, 3 so that the sum of the numbers written on each pair of opposite faces is 7.



a) Show that using these dice, it is possible to form a $2 \times 2 \times 2$ cube so that every two faces that touch each other carry the same number.

b) Is it possible to do this in such a way that only numbers 4, 5, 6 occur on the outer surface of the resulting cube?

Answer: b) no.

Solution. a) Put together four dice as depicted in Fig. 7. These dice form the lower layer of the cube. On top of this, place another similar layer turned upside down. By the construction of the layer, the numbers on the faces touching each other within one layer coincide everywhere. As the second layer is turned upside down, the numbers on faces of cubes of different layers that touch each other also coincide.

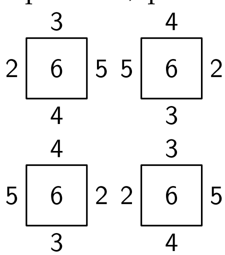


Fig. 7

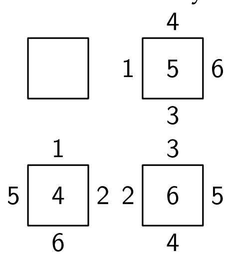


Fig. 8

b) Suppose it is possible to form a $2 \times 2 \times 2$ cube so that its surface contains only numbers $4, 5, 6$. As exactly 3 faces of each unit cube are visible, all three numbers must occur on those. Place the cube in such a way that the upper layer has 6 in its southeastern corner (see Fig. 8). Then, as the only possibility, the upper layer must have 4 in its southwestern corner and 5 in its northeastern corner. Now it is impossible to place a dice in the northwestern corner since it should touch both of its neighbors with number 1.

FR-3. *(Grade 10.)* Prove that

$$\frac{a^2+bc}{b+c}+\frac{b^2+ca}{c+a}+\frac{c^2+ab}{a+b} \geqslant a+b+c$$

for all positive real numbers a, b, c.

Solution 1. W.l.o.g., assume $a \geqslant b \geqslant c$. Then

$$\frac{a^2+bc}{b+c}+\frac{b^2+ca}{c+a}+\frac{c^2+ab}{a+b} =$$

$$= \frac{a^2+(b+c)c-c^2}{b+c}+\frac{b^2+(c+a)a-a^2}{c+a}+\frac{c^2+(a+b)b-b^2}{a+b} =$$

$$= \frac{a^2-c^2}{b+c}+c+\frac{b^2-a^2}{c+a}+a+\frac{c^2-b^2}{a+b}+b \geqslant a+b+c,$$

since

$$\frac{a^2-c^2}{b+c}+\frac{b^2-a^2}{c+a}+\frac{c^2-b^2}{a+b} = \frac{a^2-b^2+b^2-c^2}{b+c}+\frac{b^2-a^2}{c+a}+\frac{c^2-b^2}{a+b} =$$

$$= (a^2-b^2)\left(\frac{1}{b+c}-\frac{1}{c+a}\right)+(b^2-c^2)\left(\frac{1}{b+c}-\frac{1}{a+b}\right) =$$

$$= \frac{(a^2-b^2)(a-b)}{(b+c)(c+a)}+\frac{(b^2-c^2)(a-c)}{(b+c)(a+b)} \geqslant 0.$$

Solution 2. Rearranging and transforming the expression gives

$$\frac{a^2+bc}{b+c}-a+\frac{b^2+ca}{c+a}-b+\frac{c^2+ab}{a+b}-c =$$

$$= \frac{a^2+bc-ab-ac}{b+c}+\frac{b^2+ca-bc-ba}{c+a}+\frac{c^2+ab-ca-cb}{a+b} =$$

$$= \frac{(a-b)(a-c)}{b+c}+\frac{(b-c)(b-a)}{c+a}+\frac{(c-a)(c-b)}{a+b} =$$

$$= \frac{(a^2-b^2)(a^2-c^2)+(b^2-c^2)(b^2-a^2)+(c^2-a^2)(c^2-b^2)}{(b+c)(c+a)(a+b)} =$$

$$= \frac{a^4+b^4+c^4-a^2b^2-a^2c^2-b^2c^2}{(b+c)(c+a)(a+b)} =$$

$$= \frac{(a^2-b^2)^2+(c^2-a^2)^2+(b^2-c^2)^2}{2(b+c)(c+a)(a+b)} \geqslant 0.$$

FR-4. *(Grade 10.)* Find all quadruples (w, x, y, z) of positive integers such that $w^x + w^y = w^z$.

Answer: $(w, x, y, z) = (2, n, n, n+1)$, where n is any positive integer.

Solution 1. Consider the following cases.

If $w = 1$, then no solution can exist, since the l.h.s. of the equality equals 2 while the r.h.s. equals 1.

If $w \geqslant 2$, then $x < z$ and $y < z$, i.e., $x \leqslant z-1$ and $y \leqslant z-1$. Thus $w^x + w^y \leqslant w^{z-1} + w^{z-1} = 2 \cdot w^{z-1} \leqslant w \cdot w^{z-1} = w^z$. To satisfy the equation, equalities must hold in both inequalities and thus $x = y = z-1$ and $w = 2$. This gives the solutions $(w, x, y, z) = (2, n, n, n+1)$, where n is an arbitrary positive integer.

Solution 2. In the case $w = 1$ there are no solutions because $1 + 1 = 2$ is not a power of 1. Assume in the rest that $w > 1$. W.l.o.g., assume $x \leqslant y < z$. Then the equation takes the form $w^x(1 + w^{y-x}) = w^z$, whence $1 + w^{y-x} = w^{z-x}$. Consequently, $1 + w^{y-x}$ is a positive power of w and is divisible by w. If $y - x$ were positive, then w^{y-x} would also be divisible by w, whence 1 should be divisible by w, which is impossible. The remaining case $y - x = 0$ leads to $2 = w^{z-x}$ that gives $w = 2$ and $z - x = 1$ as the only possibility. Hence the solutions of the equation are of the form $(w, x, y, z) = (2, n, n, n+1)$, where n is any positive integer. Checking shows that all these quadruples satisfy the equation.

FR-5. *(Grade 10.)* Each side of a convex quadrangle $ABCD$ is a diameter of a circle. All four circles pass through the same point O, different from the vertices of the quadrangle, and no two circles have common points other than those mentioned. Prove that $ABCD$ is a rhombus.

Solution. Since AB, BC, CD, and DA are diameters (Fig. 9), AOB, BOC, COD, and DOA are right angles. Hence AOC and BOD are straight angles, i.e., the diagonals of the quadrangle meet at O. The circles drawn on the opposite sides AB and CD cannot have common points besides O, since otherwise one circle would pass through three collinear points (two vertices of the quadrangle and the point O). Consequently, the circles drawn on AB and CD must touch at O. Let P and Q be the centers of these circles, respectively; then O lies on the segment PQ. As $|AP| = |PO|$ and $|CQ| = |QO|$, isosceles triangles give $\angle BAC = \angle PAO = \angle POA = \angle QOC = \angle QCO = \angle DCA$. Thus AB and CD are parallel. Similarly, the remaining sides are parallel. Thus $ABCD$ is a rectangle; the diagonals of a rectangle are perpendicular only if the rectangle is a rhombus.

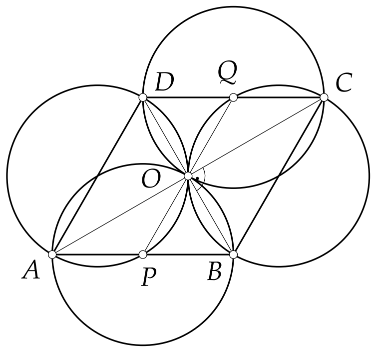


Fig. 9

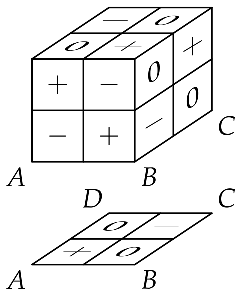


Fig. 10

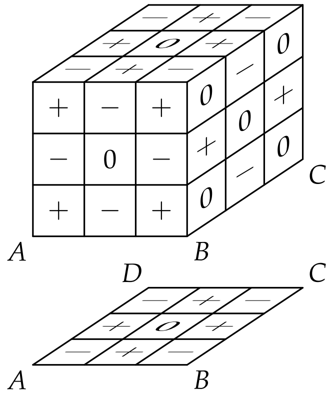


Fig. 11

FR-6. *(Grade 10.)* Every face of a unit cube has one of numbers -1, 0, 1 written on it in such a way that every two faces with a common edge contain different numbers. Is it possible to form

a) a cube of size $2 \times 2 \times 2$;

b) a cube of size $3 \times 3 \times 3$

so that in the grids that come up on the faces, every two squares with a common side contain different numbers and the sum of all numbers on each face equals 0?

Answer: a) yes; b) yes.

Solution. First note that the placement of the numbers on the faces of the unit cube is unique. Indeed, let a number x be written on some face; then the neighboring faces contain alternately the other numbers y and z, while the opposite face again contains x. This means that each of the numbers -1, 0, 1 occurs in one pair of opposite faces. Figures 10 and 11 show suitable constructions (where $-$ and $+$ denote -1 and 1, respectively).

Remark. These constructions can be easily generalized to arbitrary (even or odd, respectively) integral size of the cube.

FR-7. *(Grade 10.)* The size of the angle ABC, expressed in degrees, in a right triangle ABC is an integer. It is known that for some positive integer n, one can choose points $K_0 = A, K_2, \ldots, K_{2n}$ on the hypotenuse AB and points $K_1 = C, K_3, \ldots, K_{2n+1} = B$ on the leg CB in such a way that each triangle $K_{i-1}K_iK_{i+1}$ with $i = 1, \ldots, 2n$ is isosceles with base $K_{i-1}K_{i+1}$. Find all possible values of the size of angle ABC.

Answer: $2^\circ, 6^\circ, 10^\circ, 18^\circ$, and 30°.

Solution. Let $\angle ABC = \alpha$ (Fig. 12). Then the base angle of the last isosceles triangle $K_{2n-1}K_{2n}K_{2n+1}$ is α. The base angle of the second last isosceles triangle $K_{2n-2}K_{2n-1}K_{2n}$ has the size $180^\circ - (180^\circ - 2\alpha) = 2\alpha$. The base angle of the next triangle before it, $K_{2n-3}K_{2n-2}K_{2n-1}$, has the size

$180° - (180° - 4\alpha) - \alpha = 3\alpha$. Generally, the size of the base angle of triangle $K_{2n-i}K_{2n-i+1}K_{2n-i+2}$ is $180° - (180° - 2 \cdot (i-1)\alpha) - (i-2)\alpha = i\alpha$ ($i = 3, \ldots, 2n$). Thus the base angle of triangle $ACK_2 = K_0K_1K_2$ has the size $2n\alpha$. Now in the triangle ABC we get $90° = \angle BAC + \angle ABC = 2n\alpha + \alpha$, whence $\alpha = \frac{90°}{2n+1}$. By the conditions of the problem, α must be an integer, hence $2n+1$ is an odd divisor of 90 and is greater than 1 (as a triangle cannot have two angles of the size 90°). Such divisors are 3, 5, 9, 15, and 45 that give the solutions 30°, 18°, 10°, 6°, and 2°, respectively.

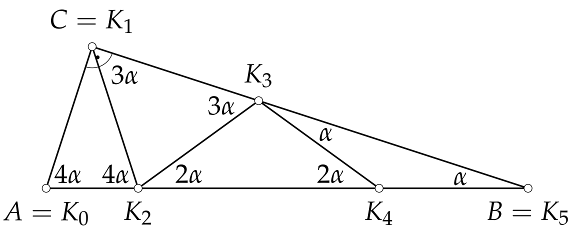


Fig. 12

FR-8. *(Grade 11.)* Find all integers that cannot be expressed as a sum of at least three consecutive terms of some non-constant arithmetic sequence of integers.

Answer: 1 and -1.

Solution 1. First prove that 1 and -1 are not expressible as the sum of at least three consecutive terms of an arithmetic sequence of integers. Let a_1, a_2, $\ldots$, a_k be k consecutive terms of an arithmetic sequence, where $k \geqslant 3$. They sum up to $s = \frac{a_1+a_k}{2} \cdot k$. If k is odd, then s is divisible by k. If k is even, then s is divisible by $\frac{k}{2} > 1$. In both cases, s differs from 1 and -1.

Now prove that every integer s other than 1 or -1 is expressible as the sum of at least three consecutive terms of an arithmetic sequence of integers. If $s = 0$, then $s = -1 + 0 + 1$. If s is different from zero and is even, i.e., $s = 2t$, where $t \neq 0$, then $-t$, 0, t, $2t$ sum up to $2t = s$. If s is odd, i.e., $s = 2t + 1$, then $-t+1, \ldots, 0, 1, \ldots, t-1, t, t+1$ are consecutive terms of an arithmetic sequence; they sum up to $t + (t+1) = 2t+1 = s$, since the terms $-t+1$ through $t-1$ mutually cancel.

Solution 2. Let a_1 be the first of the consecutive terms and d be the common difference of consecutive terms. The sum of n consecutive terms is $s = \dfrac{2a_1 + d(n-1)}{2} \cdot n$. Thus $2s = (2a_1 + d(n-1))n$. If $s = 1$ or $s = -1$, then this equality cannot hold because $n \geqslant 3$ divides neither 2 nor -2. If $s = 0$, then choose the portion of the arithmetic progression to be -1, 0, 1. If s differs from these numbers, then let $n = 2|s|$, d be an arbitrary odd number, and $a_1 = \dfrac{1-(n-1)d}{2}$ if $s > 0$, and $a_1 = \dfrac{-1-(n-1)d}{2}$ if $s < 0$.

FR-9. *(Grade 11.)* Find all integral solutions of the equation $x^3 - y^3 = 3xy + 1$.

Answer: $(x, y) = (n+1, n)$, where n is any integer, or $(x, y) = (-1, 1)$.

Solution. First assume $x > y$. Then $3xy + 1 = x^3 - y^3 = (x-y)(x^2 + xy + y^2) \geqslant (x-y) \cdot 3xy$. Thus $3xy = x^3 - y^3 - 1 \geqslant 1 - 1 = 0$ because $x > y$. If $3xy > 0$, then $3xy \geqslant 3$, hence the inequality $3xy + 1 \geqslant (x-y) \cdot 3xy$ derived above implies $x - y = 1$. If $3xy = 0$, then either $x = 0$ or $y = 0$ and in both cases the only possibility is $x - y = 1$ again. An elementary check shows that all pairs $(x, y) = (n+1, n)$, where n is an integer, satisfy the initial equation.

Now assume $x = y$. Then the equation has no solutions, since the l.h.s. is 0 while the r.h.s. is positive.

Finally assume $x < y$. Then the l.h.s. of the equation is negative, showing that xy is negative. Hence $x < 0$ and $y > 0$. Denoting $-x = z$ and multiplying the equation by (-1) leads to new equation $z^3 + y^3 = 3zy - 1$. Then $3zy - 1 = z^3 + y^3 = (z+y)(z^2 - zy + y^2) \geqslant (z+y)zy$. Hence $z + y < 3$, giving $z = y = 1$, i.e., $x = -1$, $y = 1$, as the only possibility. It is easy to check that this satisfies the equation.

FR-10. *(Grade 11.)* Let CM be the median of a triangle ABC. Prove that the product of the circumradius of ACM and the altitude drawn from M in ACM equals the product of the circumradius of BCM and the altitude drawn from M in BCM.

Solution 1. Let $\angle CAB = \alpha$ and $\angle CBA = \beta$ (see Fig. 13), and let r and s be the circumradii of the triangles ACM and BCM, respectively. By the sine law in the triangle ACM we obtain $\frac{|CM|}{\sin\alpha} = 2r$, reducing to $r = \frac{|CM|}{2\sin\alpha}$. Analogously, $s = \frac{|CM|}{2\sin\beta}$. Let U and V be the feet of altitudes drawn from the point M in triangles ACM and BCM, respectively. Then $|MU| = |AM|\sin\alpha = \frac{|AB|}{2}\sin\alpha$. Analogously, $|MV| = \frac{|AB|}{2}\sin\beta$. Thus $r \cdot |MU| = \frac{|CM|\cdot|AB|}{4} = s \cdot |MV|$.

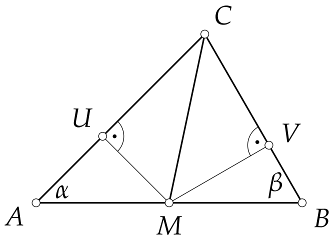


Fig. 13

Solution 2. Let x and y be the altitudes drawn from M in the triangles ACM and BCM, respectively. Let r and s be the circumradii of these triangles, respectively. The areas of triangles ACM and BCM are equal because of $|AM| = |BM|$ and the common altitude drawn from C. Therefore $|AC| \cdot x = |BC| \cdot y$. Denote $\angle AMC = \gamma$. The sine law gives $|AC| = 2r\sin\gamma$ and $|BC| = 2s\sin(180^\circ - \gamma) = 2s\sin\gamma$. Hence $2r\sin\gamma \cdot x = 2s\sin\gamma \cdot y$. As $\gamma \neq 0$, this implies $rx = sy$.

Solution 3. The formulas $S = \frac{abc}{4R}$ and $S = \frac{ah}{2}$, where a, b, c are the side lengths, R is the circumradius and h is the altitude corresponding to a, together give $Rh = \frac{bc}{2}$. The product of the circumradius of ACM and the altitude drawn from M is thus $\dfrac{|AM| \cdot |CM|}{2}$. Analogously, $\dfrac{|BM| \cdot |CM|}{2}$ for triangle BCM. These two products are equal, since $|AM| = |BM|$.

FR-11. *(Grade 11.)* The inhabitants of a city of naturals are natural numbers. Every two different inhabitants may either be or not be friends. Call a city *neighborly* if every two inhabitants share a common friend if and only if one of the numbers is divisible by the other. Can a city whose inhabitants are precisely $1, 2, \ldots, 2011$ be neighborly?

Answer: no.

Solution 1. Let p be a prime inhabitant of a neighborly city (see Fig. 14). As p is divisible by 1 and $p \neq 1$, the inhabitants 1 and p share a common friend k. As k is divisible by 1 and $k \neq 1$, the inhabitants 1 and k share a common friend i.

If $i \neq p$, then k is a common friend of i and p. This means that i is divisible by p because the primality of p does not permit the divisibility the other way round. If $i = p$, then 1 is a common friend of k and p. This analogously means that k is divisible by p. In both cases, the city contains a multiple of p that is greater than p.

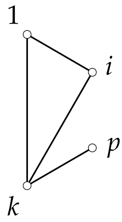


Fig. 14

In the city with inhabitants $1, \ldots, 2011$, the prime inhabitant 2011 has no larger multiples. Hence this city cannot be neighborly.

Solution 2. Suppose that this city is neighborly. Choose arbitrary 12 prime inhabitants (for instance, the first 12 primes). Each of them shares a common friend with 1; let these friends be $a_1, a_2, \ldots, a_{12}$. These numbers are all different since otherwise one of them would be a common friend to two prime numbers. W.l.o.g., assume $a_1 < a_2 < \ldots < a_{12}$. As 1 is a common friend of all them, each of $a_2, \ldots, a_{12}$ must be divisible by the previous term. This implies that $a_{12} \geqslant 2^{12} = 2048$, a contradiction.

Remark. The use of primality of 2011 in Solution 1 can be replaced with the application of Chebyshev's theorem, choosing p arbitrarily in such a way that $1005 < p < 2010$.

FR-12. *(Grade 12.)* Find the last digit of the number $1^1 + 2^2 + 3^3 + \ldots + 2011^{2011}$.

Answer: 8.

Solution. Consider the sum modulo 2 and modulo 5. As powers of odd numbers are odd and powers of even numbers are even, the number of odd summands equals the number of odd elements in set $\{1, \ldots, 2011\}$. As there are an even number of odd elements in this set, the sum given in the problem is even. Concerning modulo 5, note that 0 to every power is congruent to 0 and a^4 is congruent to 1, whenever $1 \leqslant a \leqslant 4$. Thus for all a such that $1 \leqslant a \leqslant 20$, $(a+20)^{a+20} \equiv a^{a+20} = a^a \cdot a^{20} = a^a \cdot (a^4)^5 \equiv a^a \cdot 1 = a^a \pmod 5$. Hence the remainders modulo 5 repeat periodically with the period 20. As there are 100 full periods and 100 is divisible by 5, the sum of the last 2000 summands is congruent to 0 modulo 5. It remains to compute the remainders of the first 11 summands: $1^1 = 1$, $2^2 = 4$, $3^3 = 27 \equiv 2$, $4^4 \equiv 4^0 = 1$, $5^5 \equiv 0^1 = 0$, $6^6 \equiv 1^2 = 1$, $7^7 \equiv 2^3 \equiv 3$, $8^8 \equiv 3^0 = 1$,

$9^9 \equiv 4^1 = 4$, $10^{10} \equiv 0^2 = 0$, $11^{11} \equiv 1^3 = 1$. Hence the overall sum is congruent to 3 modulo 5. Consequently, the last digit of this sum is 8.

FR-13. *(Grade 12.)* Does there exist a positive real number C such that the inequality

$$x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4 \leqslant C(x_1x_2 + x_2x_3 + x_3x_4 + x_4x_1)$$

holds for arbitrary positive real numbers x_1, x_2, x_3, x_4?

Answer: no.

Solution. For simplicity, denote $A = x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4$ and $B = x_1x_2 + x_2x_3 + x_3x_4 + x_4x_1$. Choose $x_1 = x_3 = u$ and $x_2 = x_4 = 1$. Then $A \geqslant x_1x_3 = u^2$ and $B = 4u$. Hence $\frac{A}{B} \geqslant \frac{u}{4}$. As u can be arbitrarily large, no constant C such that $A \leqslant CB$ exists.

FR-14. *(Grade 12.)* In a rectangle $ABCD$ we have $|AB| = a$ and $|BC| = b$, where $a \geqslant b$. Let E be a point in the interior of side AB such that there is exactly one possibility to choose points F, G, H on the sides BC, CD, DA, respectively, in such a way that $EFGH$ is a rectangle, too. Find the ratio of the areas of rectangles $EFGH$ and $ABCD$.

Answer: $\frac{1}{2}$.

Solution. The rectangles $ABCD$ and $EFGH$ have a common center O (see Fig. 15). As rectangles are cyclic quadrangles, the point F lies on the circle with center O and radius $|OE|$. This circle intersects the side BC at two points symmetric w.r.t. the midpoint of the side. To have exactly one point common to the circle and the side, the side must be tangent to the circle and F must be the midpoint of BC. Analogously, H must be the midpoint of DA. In triangle EFH, the side HF has length a and the corresponding altitude is $\frac{b}{2}$, giving $\frac{1}{2} \cdot a \cdot \frac{b}{2} = \frac{ab}{4}$ as the area of the triangle. The triangle GFH has the same area. Hence the area of rectangle $EFGH$ is $2 \cdot \frac{ab}{4} = \frac{ab}{2}$ that makes up a half of the area ab of the rectangle $ABCD$.

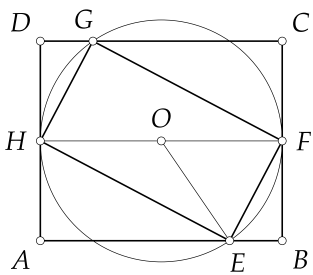


Fig. 15

FR-15. *(Grade 12.)* Ants has three pencils, each of a different color. In how many ways can he paint the faces of a regular octahedron in such a way that faces with a common edge always have different colors? Colorings that can be obtained from each other via rotations of the octahedron are considered the same.

Answer: 15.

Solution. One color can occur at most 4 times (at most twice among the faces adjacent to either one of some two opposite vertices). Thus the possible numbers of colors are 4, 4, 0 or 4, 3, 1 or 4, 2, 2 or 3, 3, 2.

i) Case 4, 4, 0. There are 3 possibilities to choose two colors from the three. After that, there is only one possibility to paint the octahedron. Thus there are 3 possibilities to paint.

ii) Case 4, 3, 1. Ordering the 3 colors can be done in 6 ways. After that, there is only one possibility to paint the octahedron, since the color used 4 times must occur twice among the faces adjacent to one vertex and twice among the faces adjacent to the opposite vertex. The remaining two colors can be deployed in principle in only one way. Thus there are 6 possibilities to paint.

iii) Case 4, 2, 2. Choosing the color that is used 4 times can be done in 3 different ways. After that, the octahedron can be painted in only 1 way, since after 4 faces have been painted with the same color, faces with either of the other colors must meet at the same vertex. Thus there are 3 possibilities to paint.

iv) Case 3, 3, 2. Choosing the color that is used only twice can be done in 3 ways. If the faces painted with this color met at a common vertex V, the faces adjacent to the opposite vertex would be painted alternately with the other two colors. But then the remaining two faces adjacent to V would have to be painted with the same color, that contradicts the case assumption. Hence the color that occurs twice is used on a pair of opposite sides. The other colors occur alternately on the surface formed by the remaining six faces. Thus there are 3 possibilities to paint.

Consequently, the number of all colorings is $3 + 6 + 3 + 3 = 15$.

FR-16. *(Grade 12.)* Inside a regular $2n$-gon, an arbitrary point is chosen and connected to every vertex of the $2n$-gon. The triangles obtained are colored alternately black and white so that triangles with a common side are of different color. Prove that the sum of the areas of all white triangles equals the sum of the areas of all black triangles.

Solution 1. If $n = 2$, i.e., the $2n$-gon is a square, then the claim holds because the base sides of white triangles are the opposite sides of the square and the altitudes lie on the same line, so the total area of white triangles is a half of the area of the square. Assume in the following that $n > 2$. Consider the regular n-gon whose sides are obtained by prolonging all sides of the $2n$-gon that belong to white triangles (see Fig. 16). Join the point chosen inside the initial $2n$-gon with all vertices of the n-gon. The altitude drawn in any white triangle coincides with the altitude drawn in the corresponding triangle in the n-gon, while the ratio of the corresponding base sides equals the ratio of the side length of the $2n$-gon and the side length of the n-gon, denote it by c. Thus the total area of white triangles is cS_n, where S_n is the area of the n-gon. Analogously, the total area of black triangles is cS_n, too.

Solution 2. The claim of the problem is equivalent to the statement that the sum of the altitudes of black triangles drawn to the sides that coincide to the sides of the $2n$-gon is equal to that of white triangles. Let O be

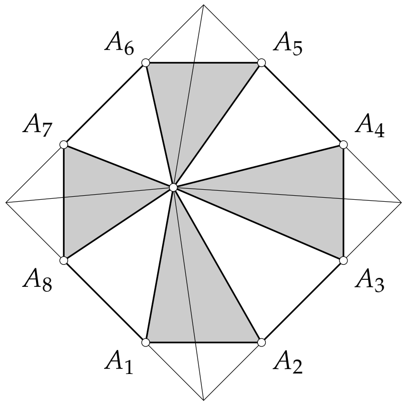


Fig. 16

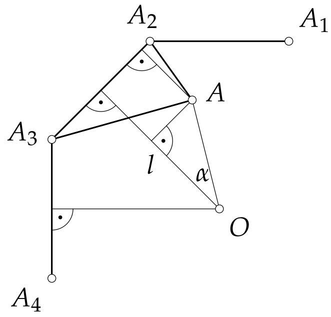


Fig. 17

the center of the $2n$-gon, A be the point chosen inside, α be the angle between line OA and the line perpendicular to a side, and l be the distance between O and any side of the $2n$-gon (see Fig. 17). Then the altitude of the corresponding triangle is $l - |OA|\cos\alpha$. The altitude of the next triangle of the same color can be expressed similarly but α is replaced with $\alpha + \frac{360^\circ}{n}$. Thus the sum of all altitudes of the triangles of the same color is $nl - |OA| \cdot (\cos\alpha + \cos(\alpha + \frac{360^\circ}{n}) + \ldots + \cos(\alpha + \frac{(n-1)360^\circ}{n}))$. To show that the sum inside parentheses equals 0, multiply the sum by $\sin\frac{360^\circ}{2n}$. Since $\cos(\alpha + \frac{k\cdot 360^\circ}{n})\sin\frac{360^\circ}{2n} = \frac{1}{2}(\sin(\alpha + \frac{(k+\frac{1}{2})\cdot 360^\circ}{n}) - \sin(\alpha + \frac{(k-\frac{1}{2})\cdot 360^\circ}{n}))$, a telescoping sum emerges and after reduction one obtains $-\sin(\alpha - \frac{360^\circ}{2n}) + \sin\left(\alpha + \frac{(n-\frac{1}{2})\cdot 360^\circ}{n}\right) = 0$. Hence for both colors, the sum of the altitudes of all triangles of this color is nl.

Remark. The sum $\cos\alpha + \cos(\alpha + \frac{360^\circ}{n}) + \ldots + \cos(\alpha + \frac{(n-1)\cdot 360^\circ}{n})$ in Solution 2 can also be computed as follows. Denote $z = \cos\alpha + i\sin\alpha$, where i is the imaginary unit and let $z_k = \cos\frac{k\cdot 360^\circ}{n} + i\sin\frac{k\cdot 360^\circ}{n}$, $k = 0, 1, \ldots, n-1$. Then the sum under consideration is the real part of the complex number $z\cdot z_0 + z\cdot z_1 + \ldots + z\cdot z_{n-1}$. Thus $z\cdot z_0 + z\cdot z_1 + \ldots + z\cdot z_{n-1} = z\cdot(z_0 + z_1 + \ldots + z_{n-1}) = z\cdot(z_1^0 + z_1^1 + z_1^2 + \ldots + z_1^{n-1}) = z\cdot\frac{z_1^n - 1}{z_1 - 1} = z\cdot 0 = 0$, whence the sum under consideration is equal to 0.

IMO Team Selection Contest

First day

TS-1. Two circles lie completely outside each other. Let A be the point of intersection of internal common tangents of the circles and let K be the projection of this point onto their external common tangent. The tangents, different from the common tangent, to the circles through point K meet the circles at M_1 and M_2. Prove that the line AK bisects the angle M_1KM_2.

Solution 1. Let L_1 and L_2 be the points of tangency of the external common tangent of the circles, N_1 and N_2 be the points of tangency of an internal common tangent, and O_1 and O_2 be the centers of the two circles (see Fig. 18). As all the lines O_1L_1, AK, and O_2L_2 are perpendicular to the line L_1L_2, they are parallel to each other and thus $\frac{|L_1K|}{|L_2K|} = \frac{|O_1A|}{|O_2A|}$. The triangles O_1AN_1 and O_2AN_2 are similar because they are both right-angled and have the same vertical angles. Thus, $\frac{|O_1A|}{|O_2A|} = \frac{|O_1N_1|}{|O_2N_2|} = \frac{|O_1L_1|}{|O_2L_2|}$. Therefore, the right-angled triangles O_1L_1K and O_2L_2K are similar due to proportionality of their legs. Hence, $\angle L_1KO_1 = \angle L_2KO_2$. As $\angle L_1KM_1 = 2\angle L_1KO_1$ and $\angle L_2KM_2 = 2\angle L_2KO_2$, we also get that $\angle L_1KM_1 = \angle L_2KM_2$. Together with the equality $\angle L_1KA = \angle L_2KA = 90°$ this implies $\angle M_1KA = \angle M_2KA$.

Solution 2. Both of the circles appear at the same angle, when viewed from the point A. To solve the problem, it is enough to show that both of the circles also appear at the same angle, when viewed from the point K. Let the centers of the circles have the coordinates $O_1(a_1, b_1)$ and $O_2(a_2, b_2)$ and let r_1 and r_2 be the radii of the circles. The two circles appear at the same angle from the point $P(x, y)$ if and only if $\frac{r_1}{|O_1P|} = \frac{r_2}{|O_2P|}$, i.e., $\frac{r_1}{\sqrt{(x-a_1)^2+(y-b_1)^2}} = \frac{r_2}{\sqrt{(x-a_2)^2+(y-b_2)^2}}$. Simple algebra shows that this equation is equivalent to $(r_1^2 - r_2^2)x^2 + (r_1^2 - r_2^2)y^2 + c_1x + c_2y + c_3 = 0$, where c_1, c_2, and c_3 are some

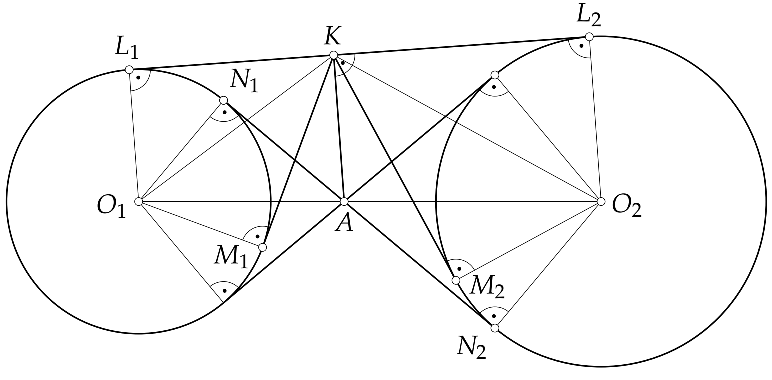


Fig. 18

constants. If $r_1 = r_2$, then the statement clearly holds. If $r_1 \neq r_2$, then the last equation is that of a circle. Point A as well as the point D of intersection of the external common tangents both lie on that circle, and from symmetry, the diameter of that circle is AD. As AK is perpendicular to the external common tangent of the circles, the point K also lies on that circle.

Remark. The statement would hold even if we swapped the internal and external tangents of the circles and considered angle M_1KM_2 as the angle between the lines KM_1 and KM_2 instead.

TS-2. Let n be a positive integer. Prove that for each factor m of the number $1 + 2 + \ldots + n$ such that $m \geqslant n$, the set $\{1, 2, \ldots, n\}$ can be partitioned into disjoint subsets, the sum of the elements of each being equal to m.

Solution. For every positive integer k, denote $S_k = \{1, 2, \ldots, k\}$ and $s_k = 1 + 2 + \ldots + k = \frac{k\cdot(k+1)}{2}$. Prove the claim by induction: assume there exists the required partitions of $S_1, \ldots, S_{n-1}$ and prove the same for S_n. Fix an arbitrary m such that $m \mid s_n$, $m \geqslant n$.

First assume $m \geqslant 2n$. Let $d = \frac{s_n}{m}$. To construct d disjoint subsets of S_n with equal sum, partition the set $\{n, n-1, \ldots, n-2d+1\}$ into subsets $M_i = \{n+1-i, n-2d+i\}$, where $i = 1, \ldots, d$. As $m \geqslant 2n$ implies $n+1 \geqslant 4d$, one gets $\frac{s_{n-2d}}{d} = \frac{(n-2d)\cdot(n+1-2d)}{2d} \geqslant \frac{(n-2d)\cdot 2d}{2d} = n-2d$. Note also that $d \mid \frac{(n-2d)\cdot(n+1-2d)}{2}$, since $(n-2d)\cdot(n+1-2d) \equiv n(n+1) \equiv 0 \pmod{2d}$. Hence by the induction hypothesis, there exist disjoint subsets $L_1, \ldots, L_d$ of S_{n-2d} with equal sum. Taking $M_i \cup L_i$ for each $i = 1, \ldots, d$ forms the desired partition of S_n.

Now assume $n \leqslant m < 2n$. If $m = n$, then the task is trivial (n must be odd to be a divisor of s_n, so take sets $\{i, n-i\}$, $i = 1, 2, \ldots, \frac{n-1}{2}$, and $\{n\}$). If $m > n$, then form the subsets $M_i = \{m-1-n+i, n+1-i\}$ of the set $\{n, n-1, \ldots, m-n\}$ for $i = 1, \ldots, n - \lceil \frac{m-1}{2} \rceil$, the sum of the elements of each being m. The solution is complete if the remaining numbers in S_n can be divided into sets, the sum of the elements of each being also m.

If m is odd, then the set of remaining numbers is $\{1, 2, \ldots, m-n-1\}$. For $m = n+1$, this set is empty and the partition is trivial, so assume $m > n+1$. As $m > m-n-1$ and $m \mid \frac{(m-n-1)\cdot(m-n)}{2}$ (the latter following from $(m-n-1)\cdot(m-n) \equiv (n+1)n \equiv 0 \pmod{m}$ and the parity of m), the desired partition exists by the induction hypothesis.

If m is even, then the set of remaining numbers of S_n also includes $\frac{m}{2}$. But $\frac{m}{2} \mid s_{m-n-1}$ again by $m \mid (m-n-1)\cdot(m-n)$. The inequality $m < 2n$ implies $\frac{m}{2} < n$, so also $\frac{m}{2} > m-n > m-n-1$. Here, $m = n+1$ is impossible, since $n+1 \mid s_n$ implies $2 \mid n$ and $2 \nmid n+1$. Hence the induction hypothesis gives the existence of a partition of S_{m-n-1} into subsets, the sum of the elements of each being $\frac{m}{2}$. The number of these subsets is $\frac{s_{m-n-1}}{m/2} = \frac{(m-n-1)\cdot(m-n)}{m} = m-2n-1+2\cdot\frac{(n+1)n}{2m}$, which is odd. Together with the number $\frac{m}{2}$, the subsets can be grouped by two to form the desired partition.

TS-3. Does there exist an operation $*$ on the set of all integers such that the following conditions hold simultaneously:

(1) for any integers x, y, z, $(x * y) * z = x * (y * z)$;

(2) for any integers x and y, $x * x * y = y * x * x = y$?

Answer: yes.

Solution. Define an operation $\oplus$ on the set of all non-negative integers, which maps two non-negative integers a and b to a non-negative integer $a \oplus b$, such that for all $i = 0, 1, \ldots$, $(a \oplus b)_i = (a_i + b_i) \bmod 2$, where n_i stands for the binary digit corresponding to 2^i in the binary representation of n. This operation satisfies condition (1) for all non-negative integers because addition modulo 2 satisfies it. The operation also satisfies condition (2) because if $x, y \in \{0, 1\}$, then $(x + x + y) \bmod 2 = y \bmod 2 = (y + x + x) \bmod 2$. As the set of non-negative integers as well as the set of all integers are countable, there exists one-to-one correspondence f between these sets (e.g. mapping a non-negative integer x to the integer $(-1)^x \lfloor \frac{x+1}{2} \rfloor$). Every integer can therefore be uniquely expressed in the form $f(n)$, where n is a non-negative integer. Therefore we can define the operation $*$ by the formula $f(x) * f(y) = f(x \oplus y)$. Following from the construction, both conditions (1) and (2) still hold.

Remark. This problem was inspired by problem 5 on Baltic Way 2006.

Second day

TS-4. Let a, b, c be positive real numbers such that $2a^2 + b^2 = 9c^2$. Prove that

$$\frac{2c}{a} + \frac{c}{b} \geqslant \sqrt{3}.$$

Solution 1. Using the AM-GM inequality for three terms twice, one gets

$$\frac{2c}{a} + \frac{c}{b} = \frac{(2b + a)c}{ab} = \frac{(2b + a)\sqrt{2a^2 + b^2}}{3ab} = \frac{(b + b + a)\sqrt{a^2 + a^2 + b^2}}{3ab} \geqslant$$
$$\geqslant \frac{3\sqrt[3]{b^2 a}\sqrt{3\sqrt[3]{a^4 b^2}}}{3ab} = \frac{3\sqrt{3}\sqrt[3]{b^2 a \cdot a^2 b}}{3ab} = \frac{3\sqrt{3}ab}{3ab} = \sqrt{3}.$$

Solution 2. Using HM-QM inequality for a, a, b gives

$$\frac{3}{\frac{2}{a} + \frac{1}{b}} = \frac{3}{\frac{1}{a} + \frac{1}{a} + \frac{1}{b}} \leqslant \sqrt{\frac{a^2 + a^2 + b^2}{3}} = \sqrt{\frac{2a^2 + b^2}{3}} = \sqrt{\frac{9c^2}{3}} = \sqrt{3}c.$$

Thus

$$\left(\frac{2}{a} + \frac{1}{b}\right) \cdot c \geqslant \frac{3}{\sqrt{3}} = \sqrt{3},$$

which implies the desired inequality.

Solution 3. By AM-GM for two terms,

$$\Big(\frac{2c}{a}+\frac{c}{b}\Big)^2=\frac{(2b+a)^2c^2}{a^2b^2}=\frac{(2b+a)^2(2a^2+b^2)}{9a^2b^2}=$$
$$=\frac{((a^2+b^2)+4ab+3b^2)(a^2+(a^2+b^2))}{9a^2b^2}\geqslant\frac{(6ab+3b^2)(a^2+2ab)}{9a^2b^2}=$$
$$=\frac{(2a+b)(a+2b)}{3ab}=\frac{2a^2+5ab+2b^2}{3ab}\geqslant\frac{4ab+5ab}{3ab}=3.$$

Solution 4. The square of the l.h.s. of the desired inequality is

$$\Big(\frac{2c}{a}+\frac{c}{b}\Big)^2=\frac{(2b+a)^2(2a^2+b^2)}{9a^2b^2}=\frac{1}{9}\Big(2+\frac{a}{b}\Big)^2\Big(2+\frac{b^2}{a^2}\Big).$$

Denoting $\frac{a}{b}=x$, the desired inequality reduces to $\frac{1}{9}(2+x)^2(2+\frac{1}{x^2})\geqslant 3$, which is equivalent to $(2+x)^2(2x^2+1)\geqslant 27x^2$. This in turn is equivalent to $2x^4+8x^3-18x^2+4x+4\geqslant 0$, that is $2(x-1)^2(x^2+6x+2)\geqslant 0$ after factorization. This inequality holds, since on positive arguments the quadratic polynomial x^2+6x+2 is positive.

Remark. There are also plenty of solutions via derivative.

TS-5. Prove that if n and k are positive integers such that $1<k<n-1$, then the binomial coefficient $\binom{n}{k}$ is divisible by at least two different primes.

Solution 1. Assume w.l.o.g. that $n\geqslant 2k$ (if $n<2k$, then interchange the roles of k and $n-k$). Let p be an arbitrary prime number. Consider the numbers that remain into the numerator of the expression

$$\binom{n}{k}=\frac{n\cdot(n-1)\cdot\ldots\cdot(n-k+1)}{k\cdot(k-1)\cdot\ldots\cdot 1}$$

after reducing all factors by the highest power of p by which they are divisible. Suppose that some two of the k factors resulting after this step are equal. Then the corresponding initial factors are of the form $s\cdot p^i$ and $s\cdot p^j$, where $i>j$. But then $n\geqslant s\cdot p^i\geqslant p\cdot s\cdot p^j>p\cdot(n-k)\geqslant 2\cdot(n-k)$, which contradicts the assumption $n\geqslant 2k$. Hence the k new factors are pairwise different. As $1<k<n-1$, the numerator initially contains at least two consecutive numbers, at least one of which is not divisible by p. This number does not change in the process described above. By the assumption $n\geqslant 2k$, this number is greater than k. Consequently, the product remaining in the numerator after elimination of powers of p is greater than the denominator $k\cdot(k-1)\cdot\ldots\cdot 1$. This means that the powers of p in the original numerator cannot be completely cancelled out with the denominator. So the canonical representation of $\binom{n}{k}$ cannot consist of a power of p only.

Solution 2. Suppose that for some n and k,

$$\binom{n}{k}=\frac{n\cdot(n-1)\cdot\ldots\cdot(n-k+1)}{k\cdot(k-1)\cdot\ldots\cdot 1}=p^t,$$

where p is a prime number and t is some positive integer. Let m be a number in $\{n, n-1, \ldots, n-k+1\}$, in the canonical representation of which the exponent of p is the largest. Then the exponent of p in the canonical representation of n, $n-1$, ..., $m+1$ coincides with that in the canonical representation of $n-m$, $n-m-1$, ..., 1, respectively. Similarly, the exponent of p in the canonical representation of $m-1$, ..., $n-k+1$ coincides with that in the canonical representation of 1, ..., $m-1-n+k$, respectively. Consequently, the exponent of p in the canonical representation of the product $n(n-1)\ldots(m+1)(m-1)\ldots(n-k+1)$ equals to that in the canonical representation of the product $(n-m)!\,(m-1-n+k)!$. Since

$$\frac{k!}{(n-m)!\,(m-1-n+k)!} = k \cdot \frac{(k-1)!}{(n-m)!\,(k-1-n+m)!} = k \cdot \binom{k-1}{n-m}$$

is clearly an integer, the exponent of p in the canonical representation of $(n-m)!\,(m-1-n+k)!$ does not exceed that in the canonical representation of $k!$. Hence, the exponent of p in the canonical representation of $\binom{n}{k}$ does not exceed that in the canonical representation of m. As the assumptions of the problem imply $\binom{n}{k} \geqslant \binom{n}{2} > n \geqslant m$, this leads to a contradiction.

Remark. More straightforward solutions can be presented using either Legendre's formula or Kummer's theorem.

TS-6. On a square board with m rows and n columns, where $m \leqslant n$, some squares are colored black in such a way that no two rows are alike. Find the biggest integer k such that for every possible coloring to start with one can always color k columns entirely red in such a way that no two rows are still alike.

Answer: $n-m+1$.

Solution 1. Prove that if $m \leqslant n$, one of the columns can always be colored red. Then, when excluding this column, we can continue the process until the number of columns is smaller than the number of rows, i.e. $n-m+1$ times. Suppose we cannot color a single column red such that no two rows still appear alike. Then, for every column there are at least two rows that differ by only one square in that column. Consider those two rows for every column. Now consider a graph with its vertices being the rows and its edges being the pairs of rows of interest. As this graph has at least as many edges as vertices, the graph contains a cycle. Consider an arbitrary row (i.e. vertex) x of the cycle. Then the row it is followed by differs from row x exactly by one square, assume this square is located at the column y. Every next row differs from the previous one in exactly one column. As all these columns differ from column y, all the other edges of this cycle correspond to passing from one row to another such that the square in column y remains the same. Therefore, the square in column y remains the same in all the rows after x and it has to remain the same at the last passage which takes us back to the row x. This, however, means that all squares in row x have to be of the same color as the square at the row after x, a contradiction.

In general, no more rows can be colored red. Assume that all the squares are colored black to start with, apart from the diagonal of an $m \times m$-subsquareboard. If we color $n - m + 2$ columns red, there are always at least 2 columns which lie in that subsquareboard. But this means that we will over-color the only two white squares in two rows and thus we end up with two equally colored rows.

Solution 2. Consider the columns of the squareboard one by one. The first column divides the set of all the rows into two subsets: one of them consists of the rows which have the square at the first column white and the other consists of those rows that have their first square black. If the first column has all its squares white or all its squares black, then we can color it red. Similarly, for every next column divide the subsets even further depending on whether there is a black or a white square in that row in that column. If no subsets are divided further at a particular step, we can color that column red. Therefore, for all columns, either the number of subsets increases by at least one or this column is colored red. As we started off with one set and ended up with m subsets each containing one row only, there has to be no more than $m - 1$ columns that were not colored red and thus at least $k = n - m + 1$ that were. All these rows are still different from each other because we colored only those columns which had no new information about the differences between rows compared with previous columns.

Similarly to Solution 1 we can show that there always exists a coloring for which no more columns can be colored red.

Solution 3. Prove by induction on the number of rows that we can always color at least $k = n - m + 1$ columns red. If $m = 1$ then we can color all the columns red, i.e. $k = n = n - m + 1$. Assume that for $m = l$ we can color at least $n - l + 1$ columns red such that no two rows appear alike. Assume now that $m = l + 1$. Following from the induction hypothesis we can color at least $n - l + 1$ columns red such that the first l rows remain different. If after that coloring process the last row is different from the rest, all the conditions required are satisfied and we can color at least $n - m + 2$ columns red. If the last row however is similar to any of the rows above it (there can be only one of those) then as these rows were all different to start with, there should exist a column at which the last row and the row that appears similar after coloring actually differ. If we do not color this column, all the rows will appear different and we can color $n - m + 1$ columns red.

Similarly to Solution 1 we can show that there always exists a coloring for which no more columns can be colored red.

Problems Listed by Topic

Number theory: OC-4, OC-7, OC-8, OC-16, FR-7, FR-8, FR-11, FR-12, TS-5
Algebra: OC-1, OC-11, OC-13, FR-1, FR-3, FR-4, FR-9, FR-13, TS-3, TS-4
Geometry: OC-2, OC-5, OC-9, OC-12, OC-14, FR-5, FR-10, FR-14, FR-16, TS-1
Discrete mathematics: OC-3, OC-6, OC-10, OC-15, FR-2, FR-6, FR-15, TS-2, TS-6