Estonian Math Competitions 2011/2012

The Gifted and Talented Development Centre
Tartu 2012

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WE THANK:

Estonian Ministry of Education and Research

University of Tartu

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Estonian Mathematical Olympiad

Mathematics Contests in Estonia

The Estonian Mathematical Olympiad is held annually in three rounds: at the school, town/regional and national levels. The best students of each round (except the final) are invited to participate in the next round. Every year, about 110 students altogether reach the final round.

In each round of the Olympiad, separate problem sets are given to the students of each grade. Students of grade 9 to 12 compete in all rounds, students of grade 7 to 8 participate at school and regional levels only. Some towns, regions and schools also organize olympiads for even younger students. The school round usually takes place in December, the regional round in January or February and the final round in March or April in Tartu. The problems for every grade are usually in compliance with the school curriculum of that grade but, in the final round, also problems requiring additional knowledge may be given.

The first problem solving contest in Estonia took place already in 1950. The next one, which was held in 1954, is considered as the first Estonian Mathematical Olympiad.

Apart from the Olympiad, open contests are held twice a year, usually in October and in December. In these contests, anybody who has never been enrolled in a university or other higher education institution is allowed to participate. The contestants compete in two separate categories: the Juniors and the Seniors. In the first category, students up to the 10th grade can participate; the other category has no restriction. Being successful in the open contests generally assumes knowledge outside the school curriculum.

Based on the results of all competitions during the year, about 20 IMO team candidates are selected. IMO team selection contest for them is held in April or May, lasting two days; each day, the contestants have 4.5 hours to solve 3 problems, similarly to the IMO. All participants are given the same problems. Some problems in our selection contest are at the level of difficulty of the IMO but somewhat easier problems are usually also included.

The problems of previous olympiads are available at the Estonian Mathematical Olympiad's website `http://www.math.olympiaadid.ut.ee/eng`.

Besides the above-mentioned contests and the quiz "Kangaroo" other regional and international competitions and matches between schools are held as well.

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This booklet presents the problems of the open contests, the final round of national olympiad and the team selection contest. For the open contests and the final round, selection has been made to include only problems that have not been taken from other competitions or problem sources and seem interesting enough. The team selection contest is presented entirely.

Selected Problems from Open Contests

O1. *(Juniors.)* Find all four-digit numbers, which after deleting any one digit turn into a three-digit number that is a divisor of the original number.

Answer: 1100, 1200, 1500, 2200, 2400, 3300, 3600, 4400, 4800, 5500, 6600, 7700, 8800, 9900.

Solution. Let $\overline{abcd}$ be a such number. Since $\overline{abcd}$ is divisible by $\overline{abc}$, we have $d = 0$. Since $\overline{abcd} = \overline{abc0}$ is divisible by $\overline{abd} = \overline{ab0}$, we have $c = 0$. Since $\overline{abcd} = \overline{ab00}$ is divisible by $\overline{acd} = \overline{a00}$ and by $\overline{bcd} = \overline{b00}$, the number $\overline{ab}$ is divisible by a and b. So $b = ax$ and $10a = by$ with integer x and y. Therefore $10a = axy$, whence $xy = 10$. If $x = 1$, $y = 10$, then $a = b$, which gives 9 possible numbers 1100, 2200, 3300, 4400, 5500, 6600, 7700, 8800, 9900. If $x = 2$, $y = 5$, then $2a = b$, which gives 4 possibilities 1200, 2400, 3600, 4800. If $x = 5$, $y = 2$, then $5a = b$, which gives 1 number 1500. The case $x = 10$, $y = 1$ is impossible, since a and b must be one-digit numbers.

O2. *(Juniors.)* Find the minimum number of colours required to paint all points with integer coordinates in the plane in such a way that no two points which are exactly five units apart have the same color.

Answer: 2.

Solution. Obviously at least 2 colors are necessary. Color all points (x, y) with even sum of coordinates with one color and all other points with another color. All points that are at distance 5 from (x, y) are $(x \pm 4, y \pm 3)$, $(x \pm 3; y \pm 4)$, $(x \pm 5, y)$, $(x, y \pm 5)$. In each case, the sum of coordinates has the parity different from that of (x, y). Therefore they are colored differently from (x, y).

O3. *(Juniors.)* A hiking club wants to hike around a lake along an exactly circular route. On the shoreline they determine two points, which are the most distant from each other, and start to walk along the circle, which has these two points as the endpoints of its diameter. Can they be sure that, independent of the shape of the lake, they do not have to swim across the lake on any part of their route?

Answer: No.

Solution. Suppose the shape of the lake is an equilateral triangle. Then the two points which are the most distant from each other are two vertices of the triangle. The circle, which has these two points as the endpoints of its diameter, does not cover the whole triangle, because the distance of the third vertex from the center of the circle is $\frac{\sqrt{3}}{2}$ of the length of the side of the triangle, but the radius of the circle is only $\frac{1}{2}$ of this length.

O4. *(Juniors.)* Two circles c and c' with centers O and O' lie completely outside each other. Points A, B, and C lie on the circle c and points A',

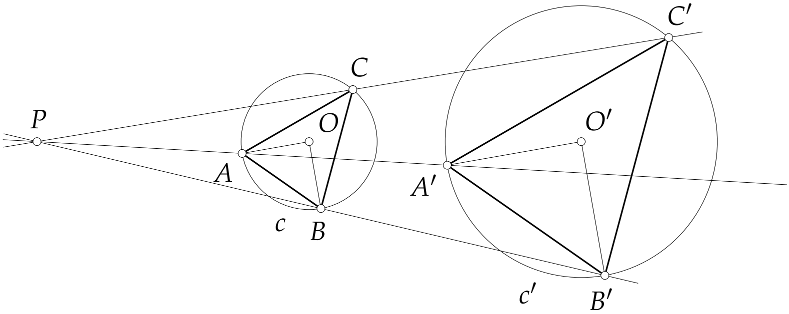


Fig. 1

B', and C' lie on the circle c' so that segment $AB \parallel A'B'$, $BC \parallel B'C'$, and $\angle ABC = \angle A'B'C'$. The lines AA', BB', and CC' are all different and intersect in one point P, which does not coincide with any of the vertices of the triangles ABC or $A'B'C'$. Prove that $\angle AOB = \angle A'O'B'$.

Solution. The triangles ABP and $A'B'P$ are similar, because their corresponding sides are parallel (Fig. 1). Hence $\frac{|AB|}{|A'B'|} = \frac{|BP|}{|B'P|}$. Likewise the triangles BCP and $B'C'P$ are similar, hence $\frac{|BC|}{|B'C'|} = \frac{|BP|}{|B'P|}$. Thus $\frac{|AB|}{|A'B'|} = \frac{|BC|}{|B'C|}$, and since $\angle ABC = \angle A'B'C'$, the triangles ABC and $A'B'C'$ are also similar. From the equality of the angles ACB and $A'C'B'$ the equality of the central angles AOB and $A'O'B'$ now follows.

Remark. Figure 1 corresponds to the case when the vectors $\overrightarrow{AB}$ and $\overrightarrow{A'B'}$ have the same direction. If they have the opposite directions, then the figure is different (the intersection point P lies on the segments AA', BB' and CC') but the argument is still correct.

O5. *(Juniors.)* Let n be a positive integer and $a_1, \ldots, a_{2n}$ be real numbers in $\left[-\frac{1}{2}, \frac{1}{2}\right]$. Leaving out any one of the numbers, the sum of the remaining $2n-1$ numbers is always an integer. Prove that $a_1 = \ldots = a_{2n}$.

Solution. Assume that there exist a_i and a_j which are not equal. Let $S = a_1 + \ldots + a_{2n}$. Since $S - a_i$ and $S - a_j$ are integers, their difference $(S - a_i) - (S - a_j) = a_j - a_i$ is also an integer. Since $a_j - a_i \neq 0$, and they belong to $\left[-\frac{1}{2}, \frac{1}{2}\right]$, their difference can be only ± 1, this happens when a_i and a_j are $\frac{1}{2}$ and $-\frac{1}{2}$ in any order. Let a_k be any of the given numbers. Since $(S - a_i) - (S - a_k) = a_k - a_i$ is an integer, a_k must also be either $\frac{1}{2}$ or $-\frac{1}{2}$. Hence all numbers a_i are either $\frac{1}{2}$ or $-\frac{1}{2}$. It follows that the sum of any two numbers a_i and a_j is an integer. As we have an even number of them, the sum of all the numbers S is also an integer. But then $S - a_i = S \pm \frac{1}{2}$ cannot be an integer, a contradiction. Therefore all numbers a_i must be equal.

O6. *(Juniors.)* Is it possible that the perimeter of a triangle whose side lengths are integers, is divisible by the double of the longest side length?

Answer: no.

Solution. Let the side lenghts of the triangle be integers a, b, c. Without loss of generality we may assume that $c \geq a$ and $c \geq b$. Suppose that the perimeter of the triangle $a+b+c$ is divisible by double of the longest side length $2c$. Since $0 < a+b+c \leq 3c < 2 \cdot 2c$, the perimeter $a+b+c$ can be divisible by $2c$ only in the case when $a+b+c = 2c$. But then $a+b = c$, which violates the triangle inequality $a+b > c$.

O7. *(Seniors.)* For any positive integer n let a_n be the largest power of 2 that divides n (e.g. $a_{2011} = 1$, $a_{2012} = 4$). Prove that for any positive integers i and j with $i < j$, the sum $\frac{1}{a_i} + \frac{1}{a_{i+1}} + \ldots + \frac{1}{a_j}$ is a fractional number.

Solution. First prove that the largest power of 2 among the numbers a_i, a_{i+1}, ..., a_j is unique. Let 2^s be the largest of the numbers a_i, a_{i+1}, ..., a_j. If there were k and l with $i \leq k < l \leq j$ such that $a_k = a_l = 2^s$, then they must be of the form $k = 2^s u$ and $l = 2^s v$, where u and v are odd numbers. Since $k < l$, we have $u < v$ and $u+1 < v$. Since $u+1$ is even, the number $m = 2^s(u+1)$ has a divisor 2^{s+1}, and $k < m < l$, which contradicts the choice of s. Thus the largest power of 2 appears only once among the numbers a_i, a_{i+1}, ..., a_j. Converting the fractions to the common denominator the fraction with the largest denominator gives 1 in the numerator, all others give a positive power of 2, i.e. an even number. Consequently the numerator is odd and cannot cancel with the denominator.

O8. *(Seniors.)* Let a be a real number, $0 \leq a \leq 1$. Prove that for any nonnegative integer n the inequality $(n+1)a \leq n + a^{n+1}$ holds.

Solution 1. The inequality is equivalent to the inequality $na - n \leq a^{n+1} - a$, or the inequality $n(a-1) \leq a(a-1)(a^{n-1} + a^{n-2} + \ldots + 1)$. If $a = 1$, then the inequality obviously holds. If $a < 1$ then $a - 1 < 0$, and dividing both sides of the inequality by $a-1$ we get an equivalent inequality $n \geq a(a^{n-1} + a^{n-2} + \ldots + 1)$, or $n \geq a^n + a^{n-1} + \ldots + a$. Since $a < 1$, in the last sum all terms are less than 1, hence the sum does not exceed n.

Solution 2. If $n = 0$ then the inequality is $a \leq a$, which obviously holds. Suppose that the inequality holds for $n = k$ and prove that it then holds for $n = k+1$ as well. From the inequality $(k+1)a \leq k + a^{k+1}$ we get the inequality $(k+2)a \leq k+1+a^{k+2}$ by adding the inequality $a \leq 1 + a^{k+2} - a^{k+1}$. The last inequality is equivalent to the inequality $(1-a)(1-a^{k+1}) \geq 0$, which obviously holds, because both factors are nonnegative.

Solution 3. If $a = 0$, then the inequality is $0 \leq n$, which obviously holds. If $a > 0$, then AM-GM gives

$$\frac{1 + \ldots + 1 + a^{n+1}}{n+1} \geq \sqrt[n+1]{1 \cdot \ldots \cdot 1 \cdot a^{n+1}},$$

which is obviously equivalent with the original inequality.

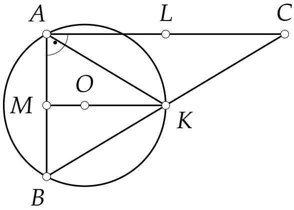


Fig. 2

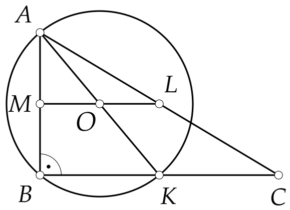


Fig. 3

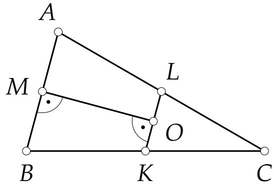


Fig. 4

Remark 1. There are also solutions via derivative.

Remark 2. Actually the inequality holds for all nonnegative real numbers a and n.

O9. *(Seniors.)* Let ABC be a triangle with median AK. Let O be the circumcenter of the triangle ABK.

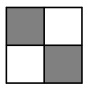
a) Prove that if O lies on a midline of the triangle ABC, but does not coincide with its endpoints, then ABC is a right triangle.

b) Is the statement still true if O can coincide with an endpoint of the midsegment?

Solution. a) Let L and M be the midpoints of the sides CA and AB, respectively. If O lies on the segment KM (Fig. 2), then the segment KM and the perpendicular bisector of AB have two different common points O and M, hence KM is the perpendicular bisector of AB. Since KM is parallel to AC and is perpendicular to AB, the angle at vertex A must be right. If O lies on the segment LM (Fig. 3), then we get similarly that the angle at vertex B must be right. If O lies on the segment KL (Fig. 4), then on one hand $\angle ABC$ is acute, because OK and MB are perpendicular to MO, the perpendicular bisector of AB, and $|OK| < |LK| = |MB|$. On the other hand, $\angle ABK$ must be obtuse, since the circumcenter O of the triangle ABK lies outside of the triangle, a contradiction. Thus this case is not possible.

b) If the triangle ABC is equilateral, then the median AK is also the altitude and ABK is a right triangle with the hypotenuse AB. The circumcenter O of the last triangle is the midpoint of AB, i.e. an endpoint of a midsegment of the triangle, but ABC is not a right triangle.

O10. *(Seniors.)* Determine the least number of the dark squares which remain visible if one covers the $n \times n$ squared paper with 2×2 squares shown on the figure (they can be turned) so that all of the squares on the paper are covered at least once?



Answer: n.

Solution. No matter how we cover the squared paper, there must be at least one dark square in each column, because on the last figure we put on this column one dark square always remains visible. Hence there must always be at least n dark squares visible. On the other hand, we can leave exactly n dark squares visible by covering the paper as follows. First put one figure on the lower left corner of the paper. Then put two figures so that they cover the dark squares of the first figure and their dark squares form a diagonal of length 3. Then add 3 figures so that they cover the dark squares of the previous figures and their dark squares form a diagonal of length 4. Repeat this until we have a diagonal of length n from one corner of the paper to the opposite corner. To cover all the squares repeat the same steps starting from the upper right corner of the paper.

O11. *(Seniors.)* The teacher drew a 3×3 table in Juku's exercise book and wrote a number in every position of the table. Then he gave Juku the following task.

1) Turn the next page and draw a similar table. Write in the first row the numbers obtained by subtracting the numbers in the third row of the corresponding column from the numbers in the second row of the corresponding column in the previous table. Similarly, the numbers in the second and third row are obtained as differences of the third and the first, and the first and the second row.
2) Turn the next page and draw a new table. Write in the first column the numbers obtained by subtracting the numbers in the third column from the numbers in the second column in the corresponding row in the previous table. Similarly, the numbers in the second and third column are obtained as differences of the third and the first, and the first and the second column.

Repeat in turns steps 1 and 2 until you reach a table where all the numbers are zeroes. Juku has reached the end of the third page and has not yet reached the table with all zeroes in it. Prove that his task never ends.

Solution. First note that after step 1 we get a table where the column sums of the table are 0, and after step 2 we get a table where the row sums of the table are 0. Suppose that after some steps we reach the table with all zeroes in it. By symmetry we can consider the case where we get this table after step 2. Then the table on the previous step was

$$\begin{matrix} a & a & a \\ b & b & b \\ c & c & c \end{matrix}$$

where at least one of the numbers a, b, c is not zero. This table was obtained after step 1, hence $a + b + c = 0$. The table on the previous step was

$$\begin{matrix} d & e & f \\ d - c & e - c & f - c \\ d + b & g + b & f + b \end{matrix}$$

This was also written by Juku, because he computed at least 2 tables. Since this table was obtained after step 2, we must have $d + e + f = d - c + e -$

$c+f-c=d+b+g+b+f+b=0$. From the first equality it follows that $c=0$, from the equality of the first and third expression $b=0$, and since $a+b+c=0$, we have $a=0$, which contradicts the assumption that at least one of the numbers a, b, c is not zero.

O12. *(Seniors.)* Prove that for any positive integer n the sum of the first n primes is greater than n^2.

Solution. First notice that the n-th prime p_n satisfies the inequality $p_n \geq 2n-1$. Indeed, the claim holds for the first prime $p_1=2$. Since all other primes are odd and there is exactly $n-1$ odd numbers between 2 and $2n$, there are at most n prime numbers less or equal to $2n-1$, hence $p_n \geq 2n-1$. Now consider the sum of the n first primes $P=p_1+p_2+\ldots+p_n$. Since $p_k \geq 2k-1$ for any k, and additionally $p_1=2>1$, the sum P is strictly greater than the sum of n first odd numbers $S=1+3+\ldots+2n-1=$ $=(1^2-0^2)+(2^2-1^2)+\ldots+(n^2-(n-1)^2)=n^2$. So $P>S=n^2$.

O13. *(Seniors.)* Find all triples (a,b,c) of positive integers such that

$$a^{bc}+b^{ca}+c^{ab}=3abc.$$

Answer: $(1,1,1)$, $(1,2,3)$, $(1,3,2)$, $(2,1,3)$, $(3,1,2)$, $(2,3,1)$, $(3,2,1)$.

Solution. First assume $a \geq 2$, $b \geq 2$, $c \geq 2$. W.l.o.g., let c be the greatest among the three numbers. Then $a^{bc}+b^{ca}+c^{ab} \geq a^4+b^4+c^4 > b^4+c^4 \geq 2b^2c^2=2b\cdot c\cdot bc > 3\cdot a\cdot bc$. Thus there are no solutions in this case.

It remains to study triples that contain 1. W.l.o.g., let $a=1$. The equation reduces to $1+b^c+c^b=3bc$. Assume $b\geq 3$, $c\geq 3$. W.l.o.g., $c\geq b$, leading to $1+b^c+c^b \geq 1+b^3+c^3 > c^3 \geq 3\cdot b\cdot c$. Thus there are no solutions in this case either. Now assume $b\geq 2$, $c\geq 2$ and one of the numbers is 2. W.l.o.g. let $b=2$. The equation reduces to $1+2^c+c^2=6c$ which can be interpreted as a quadratic equation w.r.t. c that leads to $c=3\pm\sqrt{9-(2^c+1)}$. Hence $8-2^c$ is a perfect square. The only candidates for this are 4 and 0 that give $c=2$ and $c=3$, respectively, but $c=2$ leads to contradiction (the above formula would give $c=1$ or $c=5$). The case $c=3$ gives the solution $(1,2,3)$ of the original equation. By symmetry, also $(1,3,2)$, $(2,1,3)$, $(2,3,1)$, $(3,1,2)$, $(3,2,1)$ are solutions. If one of the numbers b and c is 1 then, w.l.o.g., $b=1$. The equation reduces to $1+1+c=3c$, whence $c=1$. This gives the trivial solution $(1,1,1)$.

O14. *(Seniors.)* Let ABC be an acute triangle and D an interior point of its side AC. We call a side of the triangle ABD *friendly*, if the excircle of ABD tangent to that side has its center on the circumcircle of ABC. Prove that there are exactly two friendly sides of ABD if and only if $|BD|=|DC|$.

Solution. Let E, F and G be the centers of excircles touching BD, AD and AB respectively, and let ω be the circumcircle of ABC (see Fig. 5). To prove the assertion of the problem, we will show that F and G cannot both lie on ω and that $E\in\omega \iff |BD|=|DC| \iff F\in\omega$.

As AF and AG are bisectors of the two complementary angles of BAD, the point A lies on the segment FG. Thus, only one of the rays AF and AG can cut the circle ω again, and therefore only one of F and G can lie on ω.

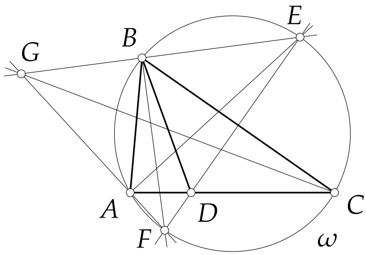


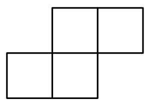
Fig. 5

To show that $E \in \omega$ iff $|BD| = |DC|$, we first note that E lies on ω iff $\angle CAE = \angle CBE$. Since $\angle CAE = \frac{1}{2}\angle BAD = \frac{1}{2}(\pi - \angle ADB - \angle ABD) = \frac{1}{2}(\pi - (\pi - 2\angle BDE) - (\pi - 2\angle DBE)) = \angle BDE + \angle DBE - \frac{\pi}{2} = \frac{\pi}{2} - \angle BED$, we see that $E \in \omega$ is equivalent to $\angle CBE = \frac{\pi}{2} - \angle BED$, i.e BC and DE being perpendicular. Since DE is the bisector of BDC, this occurs iff $|BD| = |DC|$.

It remains to show that $|BD| = |DC|$ iff $F \in \omega$. Point F lies on ω iff $\angle AFB = \angle BCD$. Using the fact that $\angle BAF = \angle BAD + \frac{1}{2}(\pi - \angle BAD) = \frac{1}{2}(\pi + \angle BAD)$, we get $\angle AFB = \pi - \angle ABF - \angle BAF = \frac{\pi - \angle ABD - \angle BAD}{2} = \frac{\angle ADB}{2} = \frac{\angle BCD + \angle CBD}{2}$. Thus $\angle AFB = \angle BCD$ is equivalent to $\angle BCD = \angle CBD$, i.e., $|BD| = |DC|$.

Remark. The claim holds in the case of right or obtuse triangle, too. The problem with the above proof is that if $\angle ABC > 90^\circ$ then point E may fall inside triangle ABC, whence equality $\angle CAE = \angle CBE$ is no more equivalent to $E \in \omega$. Nevertheless, one can show that if $|BD| = |DC|$, then E must lie outside triangle ABC, extending the validity of the claim to the obtuse case.

O15. *(Seniors.)* Let k be a positive integer. Determine the largest number of *snakes*, consisting of four squares (see figure), which can be placed on a $(2k+1) \times (2k+1)$ chessboard so that the snakes neither overlap nor stick out across the edges of the chessboard. The snakes can be turned and reflected.



Answer: k^2.

Solution. First show that k^2 snakes can be placed on a $(2k+1) \times (2k+1)$ chessboard. Divide the chessboard into strips of width 2 (one strip of width 1 remains). On any strip we can place k snakes, one after another; so on k strips, it is possible to place k^2 snakes.

It remains to prove that one can not place more than k^2 snakes on the chessboard. Write numbers $0, 1, 0, 1, \ldots, 0$ in the odd rows, and numbers $2, 3, 2, 3, \ldots, 2$ in the even rows. Notice that no matter how we place the snake on the board, it always covers numbers 0, 1, 2 and 3. Since all numbers 3 are in the squares with even row and column numbers, there is exactly k^2 of them, hence there can be at most k^2 snakes.

Selected Problems from the Final Round of National Olympiad

F1. *(Grade 9.)* Integers a, b, c are such that $a + b + c$ is divisible by 6, and $a^2 + b^2 + c^2$ is divisible by 36. Does it imply that $a^3 + b^3 + c^3$ is divisible by a) 8; b) 27?

Answer: a) yes; b) no.

Solution. a) As the sum of a, b, and c is divisible 6, and is therefore even, there must be either 0 or 2 odd numbers among the three. If we had 2 odd numbers, the sum of the squares $a^2 + b^2 + c^2$ would give a remainder of $0 + 1 + 1 = 2$ when dividing by 4. But this is not possible, since the sum is divisible by 36, and therefore also by 4. So, all the numbers a, b, c are even. Hence, all the numbers a^3, b^3, c^3 are divisible by 8, and so is their sum.

b) If $a = 8$, $b = c = 2$, then all of the premises are fulfilled: $8 + 2 + 2 = 12$ is divisible by 6 and $8^2 + 2^2 + 2^2 = 72$ is divisible by 36. But $8^3 + 2^3 + 2^3 = 528$ is not divisible by 9, and therefore, it is not divisible by 27.

F2. *(Grade 9.)* Let ABC be an isosceles triangle with $|AB| = |AC|$. The bisector of angle ABC meets the side AC at the point D.

a) Is the triangle ABD isosceles whenever the triangle BCD is isosceles?

b) Is the triangle BCD isosceles whenever the triangle ABD is isosceles?

Answer: a) yes; b) no.

Solution. a) Denote $\angle BAC = \alpha$ and $\angle ABC = \angle ACB = \beta$ (Fig. 6). Assume that the triangle BCD is isosceles. If $|CB| = |CD|$, then the angles $\angle CBD$, $\angle CDB$ and $\angle BCD$ would be $\frac{\beta}{2}$, $\frac{\beta}{2}$ and β respectively, which implies $2 \cdot \frac{\beta}{2} + \beta = 180°$ giving $\beta = 90°$. This is impossible, since the triangle ABC has two angles equal to β. If $|DB| = |DC|$, then we would get $\frac{\beta}{2} = \beta$, which is also impossible. This leaves the only option $|BC| = |BD|$. Then, the triangles ABC and BCD are similar, since all corresponding angles are the same. Therefore $\angle DBA = \angle DBC = \angle BAC = \angle BAD$, showing that the triangle ABD is isosceles with $|DA| = |DB|$.

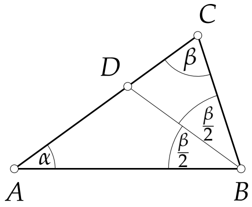


Fig. 6

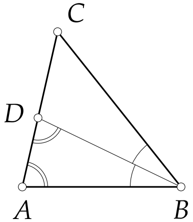
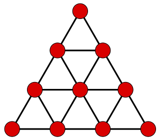


Fig. 7

b) If the angles of the triangle ABC are $\frac{3}{7} \cdot 180°$, $\frac{2}{7} \cdot 180°$, $\frac{2}{7} \cdot 180°$ (Fig. 7), then $\angle ADB = 180° - \angle BAD - \angle ABD = 180° - \frac{3}{7} \cdot 180° - \frac{1}{7} \cdot 180° = \frac{3}{7} \cdot 180° = \angle BAD$, which shows that the triangle ABD is isosceles with $|BA| = |BD|$. At the same time, the angles in the triangle BCD are $\frac{1}{7} \cdot 180°$, $\frac{2}{7} \cdot 180°$, $\frac{4}{7} \cdot 180°$, which are pairwise different, so the triangle BCD is not isosceles.

F3. *(Grade 9.)* An equilateral triangle with side length 3 is divided into 9 equilateral triangles with side length 1. An integer from 1 to 10 is written into every point that is a vertex of a small triangle (colored vertices on the figure), such that all numbers are written exactly once. For every small triangle, the sum of the numbers in its three vertices is written inside it. Prove that at least three of those sums are greater than 11.



Solution. In a triangle, which has 10 at one vertex, the sum is at least 13. If 10 is not at one of the vertices of the large triangle, the number of triangles with sum greater than 12 is at least 3 and the problem is solved. If 10 is at the vertex of the large triangle, then look, where is the number 9. If 9 does not lie at a vertex of the large triangle, then there are at least 3 small triangles, with a sum of at least 12, and the problem is solved. If 9 is at a vertex a the large triangle, then look, where is the number 8. If 8 does not lie at the vertex of a large triangle, then it is at a vertex of at least 3 small triangles. In at least two of them the sum is at least 12, and as at most one of these can overlap with one of the triangles found earlier, the problem is solved. If 8 is at a vertex of the large triangle, then either the sum in that triangle is at least 12, meaning the problem is solved, or the other vertices in the triangle have numbers 1 and 2. In the last case, the numbers 3, 4 and 5 are the smallest numbers whose positions are not set, but these give a sum of 12. So, we can simply choose any triangle who vertices we have not looked at yet as our third triangle (there are three of these triangles).

F4. *(Grade 9.)* Jüri wishes to draw n circles and any number of lines on the plane such that all the lines meet at one point, and for every two circles there exist two lines that touch both of these circles.

a) Is it possible for Jüri to solve this problem for any $n \geq 2$?

b) For which natural numbers n is it possible to solve this problem if in addition all the circles must have the same radius?

Answer: a) yes; b) $n \leq 4$.

Solution. a) Jüri can draw two lines and draw any number of circles such that they touch both of the lines (Fig. 8).

b) Assume that Jüri has solved the problem for some n, where $n > 1$.

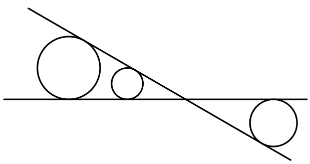


Fig. 8

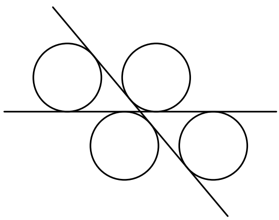


Fig. 9

Let O be the intersection point of all the lines. Look at any circle c. From the premises of the problem we see that the circle c touches two of the lines drawn by Jüri, which we call k and l. But any one circle can only touch up to two lines drawn from one point. So, the circle c does not have any more lines touching it. If c' is any other circle drawn by Jüri, then the common tangents of c and c' can only be k and l. Therefore, k and l are the common tangents of all the circles. Two lines divide the plane into four sectors, inside each can be only one circle with the previously set radius (Fig. 9). So, for $n > 4$ the problem has no solution but for $n \leq 4$, it obviously has.

F5. *(Grade 10.)* Find all pairs (n, m) of positive integers such that the arithmetic and geometric means of m and n are different two-digit numbers consisting of the same digits.

Answer: $(32, 98)$, $(98, 32)$.

Solution. Let $10a + b$ be the arithmetic mean of the given numbers, where a and b are decimal digits. Let $10a + b + x$ and $10a + b - x$ be the numbers we are searching for. Then, by the premises

$$\sqrt{(10a + b + x)(10a + b - x)} = 10b + a,$$

which after squaring and simplifying gives $x^2 = 99(a^2 - b^2)$. So, x^2 is divisible by 99, implying x^2 is divisible by 3 and 11. Since 3 and 11 are primes, x itself is divisible by 3 and 11, and therefore by 33. Denoting $x = 33z$, we get

$$11z^2 = \frac{99 \cdot 11z^2}{99} = \frac{x^2}{99} = a^2 - b^2 = (a + b)(a - b),$$

from which we see, that the product $(a + b)(a - b)$ is divisible by 11. Since 11 is a prime, either $a + b$ or $a - b$ is divisible by 11. Since $a - b \neq 0$ and a and b are single-digit numbers, we must have $a + b = 11$. Therefore $a - b = z^2$. Since $a - b$ and $a + b$ are either both odd or both even, z must be odd. So, $z = 1$, since $z \geq 3$ implies $x \geq 99$, but $10a + b - x$ must be positive. So, $a = 6$, $b = 5$, $x = 33$, and the corresponding pair is $(98, 32)$.

F6. *(Grade 10.)* We say that two real numbers r and s are *close* if $|r - s| = 10^u$ for some integer u. Let $y = ax + b$ be a linear function, for which there exist close numbers x_1 and x_2 so that the corresponding y_1 and y_2 are also close. Prove that for any close numbers x_1' and x_2', the corresponding y_1' and y_2' are also close.

Solution. From the premises we get $|x_1 - x_2| = 10^u$ and $|y_1 - y_2| = |(ax_1 + b) - (ax_2 + b)| = 10^v$ for some integers u, v. So,

$$10^v = |(ax_1 + b) - (ax_2 + b)| = |a(x_1 - x_2)| = |a| \cdot |x_1 - x_2| = |a| \cdot 10^u,$$

which gives $|a| = \frac{10^v}{10^u} = 10^{v-u}$. Let x_1', x_2' be any close real numbers, $|x_1' - x_2'| = 10^w$. Then $|y_1' - y_2'| = |(ax_1' + b) - (ax_2' + b)| = |a(x_1' - x_2')| =$

$|a| \cdot |x_1' - x_2'| = 10^{v-u} \cdot 10^{w} = 10^{w+v-u}$. Since u, v, w are integers, $w + v - u$ is also an integer, which shows that y_1', y_2' are close.

F7. *(Grade 10.)* Let ABC be a triangle on the plane. The angle bisector from the vertex A meets the side BC at P, and the median from the vertex B meets the side AC at M. The lines AB and MP meet at the point K. Prove that if $\frac{|PC|}{|BP|} = 2$, then AP and CK are perpendicular.

Solution 1. Let K' be a point on the ray AB, such that B is the midpoint of the line segment AK' (Fig. 10). Then CB is the median of the triangle ACK'. As P divides this line segment in the ratio $2:1$, P must be the centroid of the triangle ACK'. So, $K'M$, which is also a median of the triangle ACK', must pass through the point P. Therefore, $K = K'$. So, B is the midpoint of the line segment AK. As AP passes through the point P, AP is also a median of the triangle ACK. By the premises it is also an angle bisector. So, the triangle ACK is isosceles with $|AC| = |AK|$ and AP is its height. Therefore, $AP \perp CK$.

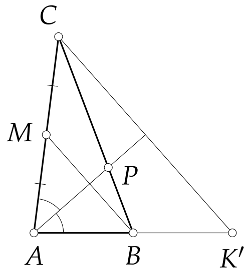
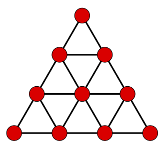


Fig. 10

Solution 2. As in solution 1, we show that B is the midpoint of AK. So BM joins the midpoints of sides in the triangle ACK and therefore $BM \parallel CK$. An the angle bisector divides the opposite side in the same ratio as the corresponding sides, so $\frac{|AC|}{|AB|} = \frac{|PC|}{|BP|} = 2$. As M is the midpoint of AC, we have $|AB| = |AM|$. An angle bisector, drawn from the vertex opposite the base in an isosceles triangle, is also the height in that triangle, giving $AP \perp BM$. It follows that $AP \perp CK$.

F8. *(Grade 10.)* An equilateral triangle with side length 3 is divided into 9 equilateral triangles with side length 1. An integer from 1 to 10 is written into every point that is a vertex of a small triangle (colored vertices on the figure), such that all numbers are written exactly once. For every small triangle, the sum of the numbers in the three vertices is written inside it. Prove that there exist three small triangles such that the sum of the numbers inside them is at least 48.



Solution 1. Any three small triangles, from which no two have common vertices, take up nine of the ten numbers written into the vertices of the small triangles. So, the sum of the numbers inside those small triangles is $55 - a$, where a is the number at the last vertex. Now it is sufficient to prove that we can choose the three small triangles, from which no two have common vertices, in four different ways, always leaving a different vertex out. So, in at least one case, the number at the last vertex is at most 7, and the sum of the numbers in the three chosen triangles is at least $55 - 7 = 48$.

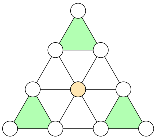


Fig. 11

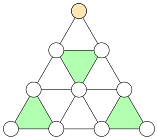


Fig. 12

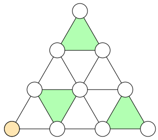


Fig. 13

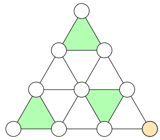


Fig. 14

Indeed, the three triangles can be chosen so that they leave uncovered the central number (Fig. 11) or one of the corner numbers (Fig. 12, 13, 14).

Solution 2. Let m be the number in the center of the large triangle. Then, when adding the sums of the three corner triangles we get the sum of all the numbers from 1 to 10, except m, so the sum of the corner triangles is $55 - m$. If $m \leq 7$, then the sum is at least 48.

In the rest of the cases, consider any three triangles around the center point, such that no two of them share a side. Adding the numbers in them, we get the sum of all the numbers from 1 to 10, except the three numbers in the corners, while we add the center number three times. So the sum of those triangles is at least $21 + 3m$. If $m \geq 9$, then the sum is at least 48.

This leaves the case $m = 8$. The sum of the numbers in the triangles in the corners is at least $55 - 8 = 47$, so at least one of them contains a sum that is at least 16. If none of the triangles contains a sum 17 or greater, the number 16 must occur in two different triangles. The sum of the numbers in any three triangles around the center point, chosen like above, is at least $21 + 3 \cdot 8 = 45$. So, in both triples at least one of the triangles contains a sum of at least 15, and since triangles sharing an edge cannot contain the same sum as the corresponding sums differ by exactly one term, at least one of the six triangles around the center point contains a sum of at least 16. Thus we can pick the three desired triangles from among either one corner triangle and two central triangles or two corner triangle and one central triangle that contain the largest numbers.

F9. *(Grade 10.)* Find all triples of positive integers (x, y, z), for which $x \cdot y! + 2y \cdot x! = z!$.

Answer: $(2, 1, 3)$ and $(n, n+1, n+2)$ for every positive integer n.

Solution. Since the left-hand side is greater than both $x!$ and $y!$, obviously $z > x$ and $z > y$. So, both sides of the equation are divisible by both $x!$ and $y!$. Therefore, $x \cdot y!$ is divisible by $x!$, which means that $y!$ is divisible by $(x-1)!$, giving $y \geq x - 1$. Analogously, $2y \cdot x!$ is divisible by $y!$, meaning $2 \cdot x!$ is divisible by $(y-1)!$. The case $x = 1, y = 3$ is not a solution, the case $x > 1$ gives $2 \cdot x! < (x+1)!$, which implies $x \geq y - 1$. This leaves us to look through the cases $-1 \leq y - x \leq 1$.

- If $y = x - 1$, then the equation simplifies to $(2x-1) \cdot x! = z!$. As $(x+1)(x+2) > 2x - 1$, we have $2x - 1 = x + 1$ and $z = x + 1$. This gives the solution $x = 2, y = 1, z = 3$.

- If $y = x$, the equation simplifies to $3x \cdot x! = z!$. As $(x+1)(x+2) > 3x$, we have $3x = x+1$, but this does not give integer solutions.
- If $y = x+1$, the equation simplifies to $(x^2+3x+2) \cdot x! = z!$ or $(x+2)! = z!$. From here we get a family of solutions $x = n$, $y = n+1$, $z = n+2$.

F10. *(Grade 11.)* In his last research, professor P was concentrating on natural numbers with a certain property. It is known that whenever a natural number x has this property, all multiples of x also have this property. Let $a_1, \ldots, a_n$ be positive integers such that all their divisors that are greater than one have the property professor P studied. Is it true that all divisors greater than one of the product $a_1 \ldots a_n$ definitely have this property?

Answer: yes.

Solution 1. Let $k > 1$ be any divisor of the product $a_1 \ldots a_n$. Then k has a prime divisor p, which is also a divisor of the product $a_1 \ldots a_n$. As p is a prime, there exists i, such that p is a divisor of a_i. As all the divisors of a_i greater than 1 have the property, p also has this property. By the premise, all the multiples of p have the property, so k has the property.

Solution 2. Let $k > 1$ be any divisor of the product $a_1 \ldots a_n$. If k were relatively prime to all a_i, it would be relatively prime to the product $a_1 \ldots a_n$, but $\gcd(k, a_1 \ldots a_n) = k > 1$. Hence $\gcd(k, a_i) > 1$ for some a_i. As a divisor of a_i, the number $\gcd(k, a_i)$ has the property studied by professor P. As a multiple of $\gcd(k, a_i)$, also k has the same property.

F11. *(Grade 11.)* a) Find all positive integers n, such that the sum of all integers from 1 to $n+1$ can be represented as the sum of n consecutive integers.

b) Find all positive integers n, for which there exists an integer a, such that the sum of the integers from a to $a+n$ is equal to the sum of the integers from $a+n+1$ to $a+2n$.

Answer: a) 1; b) all positive integers.

Solution 1. a) Clearly the sum of the first two positive integers can be represented as the sum of one positive integer. Now, let $n \geq 2$ and let us show that the sum of the $n+1$ first positive integers cannot be represented as a sum of n consecutive integers. Indeed, on one hand $1+2+\ldots+n+(n+1) > 2+3+\ldots+n+(n+1)$, on the other hand $1+2+\ldots+n+(n+1) < 1+2+3+\ldots+n+(n+1)+1 = 3+\ldots+n+(n+1)+2+2 \leq 3+\ldots+n+(n+1)+(n+2)$. So the sum of $n+1$ first positive integers $1+\ldots+(n+1)$ lies between $2+\ldots+(n+1)$ and $3+\ldots+(n+2)$, which both are consecutive sums of n consecutive integers. So, the number $1+\ldots+(n+1)$ is not a sum of n consecutive integers.

b) Let n be any positive integer. To solve the problem, it suffices to see that $n^2+(n^2+1)+\ldots+(n^2+n) = n^2 \cdot (n+1)+(1+\ldots+n) = n \cdot (n^2+n)+(1+\ldots+n) = (n^2+n+1)+\ldots+(n^2+n+n)$.

Solution 2. We use the formula for the sum of arithmetic progression.

a) If n is odd, then the sum of n consecutive integers is divisible by n. So, if the number $1+2+\ldots+(n+1) = \frac{n+1}{2}\cdot(n+2)$ was the sum of n consecutive integers, it would be divisible by $\frac{n+1}{2}$ and by n. As $n+1$ and n are relatively prime, the same clearly holds for $\frac{n+1}{2}$ and n. Therefore, $\frac{n+1}{2}\cdot(n+2)$ should be divisible by $\frac{n+1}{2}\cdot n$, meaning that $n+2$ should be divisible by n.

If n is even, the sum of n consecutive integers is divisible by $\frac{n}{2}$. So, if $1+2+\ldots+(n+1) = (n+1)\cdot\frac{n+2}{2}$ was the sum of n consecutive integers, it would be divisible by both $n+1$ and $\frac{n}{2}$. As $n+1$ and n are relatively prime, also $n+1$ and $\frac{n}{2}$ are relatively prime. Thus, $(n+1)\cdot\frac{n+2}{2}$ should be divisible by $(n+1)\cdot\frac{n}{2}$, implying that $n+2$ is divisible by n.

So, in all cases $n+2$ is divisible by n, which is equivalent to saying 2 is divisible by n. So, $n=1$ or $n=2$. Clearly $1+2$ is the sum of one integer, but $1+2+3=6$, being an even number, cannot be represented as the sum of two consecutive integers.

b) If a is the first of the two consecutive integers, then the problem can be represented as the equation $(a+a+n)(n+1)/2 = (a+n+1+a+2n)n/2$. By simplifying we see that it is equivalent to $a=n^2$. This means that the sum of $n+1$ consecutive integers, first of which is n^2, is the sum of the next n consecutive integers. So, the desired numbers exist for every n.

F12. *(Grade 11.)* The sides AB and AC of the triangle ABC touch the circle c respectively at points B' and C'. The center L of the circle c lies on the side BC. The circumcenter O of triangle ABC lies on the shorter arc $B'C'$ of the circle c. Prove that the circumcircle of ABC and the circle c meet at two points.

Solution. Let r be the circumradius of ABC, let s be the radius of c and $\alpha = \angle BAC$ (Fig. 15). By tangency, $|AB'| = |AC'|$. Thus $\angle C'B'A = \angle B'C'A = \frac{\pi}{2} - \frac{\alpha}{2}$ whence, by property of inscribed angle, $\angle B'OC' = \pi - \left(\frac{\pi}{2} - \frac{\alpha}{2}\right) = \frac{\pi}{2} + \frac{\alpha}{2}$. Clearly $\angle B'OC' > \angle BOC = 2\alpha$, leading to $\frac{\pi}{2} + \frac{\alpha}{2} > 2\alpha$. Hence $\alpha < \frac{\pi}{3}$. Now let K be the midpoint of side BC. From the right triangle KOC, one gets $|KO| = |OC|\cos\angle KOC = r\cos\alpha$. By the inequality obtained above, $\cos\alpha > \cos\frac{\pi}{3} = \frac{1}{2}$. On the other hand, $|KO| \le |LO| = s$, leading to $\frac{1}{2}r < r\cos\alpha = |KO| \le s$ or $r < 2s$. As c passes through the circumcenter of ABC, this inequality shows that these circles must intersect.

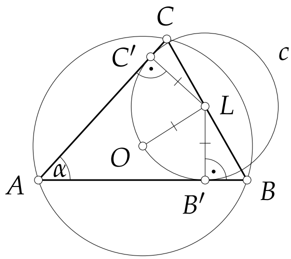


Fig. 15

Remark. This problem, proposed by Estonia, appeared in the IMO 2011 shortlist as G1.

F13. *(Grade 11.)* A finite grid is covered with 1×2 cards in such a way that the edges of the cards match with the lines of the grid, no card lies over the edge of the grid, and every square is covered by exactly two cards. Prove that one can remove some of the cards in such a way that every square will be covered by exactly one card.

Solution. Choose any square covered by two cards, and choose one of these cards. Move that card to a neighbouring square, and choose the other card that is covering that square. From there we move to the next square, etc., until we return to the first square. We cannot return to any other square visited previously, since in all squares except the first one, both cards have been chosen already. If we color the rectangular grid like a chessboard, then after an odd number of moves, we reach a square with the opposite color, and after an even number of moves, we reach a square with the same color. Therefore, the number of chosen cards is even. So, we can remove every second chosen card. All of the remaining squares we passed through will be covered by exactly one card. If after this, there are still squares that are covered by two cards, we repeat the process with a new randomly chosen square which is covered by two cards. We can never move from a square covered by two cards to a square covered by exactly one card, since all the squares covered by exactly one card were previously connected to squares now covered by exactly one card. So, after a finite number of steps we can find a new cycle, from which we can remove every second card. We repeat, until all squares are covered by exactly one card.

F14. *(Grade 11.)* There are 2012 points marked in a square with side length 11. Prove that one can choose an equilateral triangle with side length 12 which covers at least 671 points.

Solution. Place two equilateral triangles with side lengths 12 on the square in such a way that both have one vertex lie on the side of the square and the opposite sides of these vertices partially coincide with the other side of the square and with each other (Fig. 16). The area common to both triangles forms an equilateral triangle of side length 1. Position the third equilateral triangle with side length 12 between the two triangles, turned $180°$, such that the lowermost vertex of that triangle coincides with the uppermost vertex of the small triangle. To show that the square is fully covered by these triangles, we must show that the sum of the heights of the large and the small triangle is at least 11, which is equaivalent to showing $\frac{\sqrt{3}}{2} \cdot (12+1) > 11$. As we can simplify this equation to $13\sqrt{3} > 22$ and $3 \cdot 169 > 484$, we see that it holds. Therefore at least a third of the 2012 points or at least 671 points lie in one of the three chosen triangles.

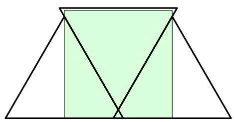


Fig. 16

F15. *(Grade 12.)* Find all pairs (x, y) of positive integers such that

$$\frac{1}{x^2} + \frac{249}{xy} + \frac{1}{y^2} = \frac{1}{2012}.$$

Answer: $(503, 1006), (1006, 503)$.

Solution 1. Let $\gcd(x, y) = d$ and $x = ad$, $y = bd$. Then the equation can be written as $\frac{a^2+249ab+b^2}{a^2b^2d^2} = \frac{1}{2012}$ or

$$a^2b^2d^2 = 2012(a^2 + 249ab + b^2).$$

As a and b are relatively prime, a^2 and b^2 are both relatively prime to $a^2 + 249ab + b^2$ and therefore both they must be divisors of 2012. As $2012 = 2^2 \cdot 503$ and 503 is a prime, the possible cases are $(a, b) = (1, 1)$, $(a, b) = (1, 2)$, $(a, b) = (2, 1)$. If we substitute $(a, b) = (1, 1)$ into the last equation, we get $d^2 = 2012 \cdot 251$, which is not solvable in integers. The other two cases give $4d^2 = 2012 \cdot 503$, from which $d = 503$. This leads to the solutions $(x, y) = (503, 1006)$ and $(x, y) = (1006, 503)$.

Solution 2. Multiplying both of the sides by $2012x^2y^2$, we get

$$2012x^2 + 249 \cdot 2012xy + 2012y^2 = x^2y^2.$$

From the left-hand side we see that both sides of the equation must be divisible by 503. As 503 is a prime, one of the numbers x and y must be divisible by 503. So x^2 or y^2 is divisible by 503^2, giving that both sides of the equation are divisible by 503^2. If x is divisible by 503, the summands $2012x^2$ and $249 \cdot 2012xy$ on the left-hand side are divisible by 503^2, meaning that $2012y^2$ is divisible by 503^2. Therefore y is divisible by 503. Analogously, we get that if y is divisible by 503, then x is also divisible by 503. Consequently, both x and y are divisible by 503. Denote $x = 503a$, $y = 503b$. Then the equation, after dividing both sides by 503^3, simplyfies to $4a^2 + 996ab + 4b^2 = 503a^2b^2$. Assume $a \geq b$. If $b \geq 2$, then $503a^2b^2 \geq 503a^2 \cdot 2b = 1006a^2b = 4a^2b + 4a^2b + 998a^2b > 4a^2 + 4b^2 + 996ab$, so the last equation cannot hold. Therefore $b = 1$. Now we get a quadratic equation $499a^2 - 996a - 4 = 0$ with respect to a, whose only positive solution is $a = 2$. From here we obtain the solution $(1006, 503)$ to our original equation. The case $b \geq a$ is symmetrical and gives the solution $(503, 1006)$.

F16. *(Grade 12.)* a) Prove that for every real number x the arithmetic mean of $\sqrt{1 + \sin x}$ and $\sqrt{1 - \sin x}$ is equal to one of the following: $\sin \frac{x}{2}$, $\cos \frac{x}{2}$, $-\sin \frac{x}{2}$, $-\cos \frac{x}{2}$.

b) Can one leave out one of the four numbers listed in part a) in such a way that the claim still holds?

Answer: b) no.

Solution 1. a) Denote the arithmetic mean given in the problem by $A(x)$. As

$$1 + \sin x = \sin^2 \frac{x}{2} + \cos^2 \frac{x}{2} + 2 \sin \frac{x}{2} \cos \frac{x}{2} = \left(\sin \frac{x}{2} + \cos \frac{x}{2}\right)^2,$$

$$1 - \sin x = \sin^2 \frac{x}{2} + \cos^2 \frac{x}{2} - 2 \sin \frac{x}{2} \cos \frac{x}{2} = \left(\sin \frac{x}{2} - \cos \frac{x}{2}\right)^2,$$

we get

$$A(x) = \frac{\sqrt{1 + \sin x} + \sqrt{1 - \sin x}}{2} = \frac{|\sin \frac{x}{2} + \cos \frac{x}{2}| + |\sin \frac{x}{2} - \cos \frac{x}{2}|}{2}.$$

Depending on the signs of the numbers $\sin \frac{x}{2} + \cos \frac{x}{2}$ and $\sin \frac{x}{2} - \cos \frac{x}{2}$, one of the trigonometric functions in the numerator cancels out and the other one is doubled, with either a positive or a negative sign. Therefore, $A(x)$ is equal to one of the numbers $\sin \frac{x}{2}$, $\cos \frac{x}{2}$, $-\sin \frac{x}{2}$, $-\cos \frac{x}{2}$.

b) Clearly $A(x) = 1$, whenever x is one of the numbers 0, π, 2π, 3π. Nevertheless, each of these four values makes a unique expression among $\sin \frac{x}{2}$, $\cos \frac{x}{2}$, $-\sin \frac{x}{2}$, $-\cos \frac{x}{2}$ evaluate to 1. Therefore, none of these four can be left out.

Solution 2. Part a) can also be proven as follows. Let $A(x)$ be the same as in the first solution. Then

$$\left(\frac{\sqrt{1 + \sin x} + \sqrt{1 - \sin x}}{2}\right)^2 = \frac{2 + 2\sqrt{1 - \sin^2 x}}{4} = \frac{1 + |\cos x|}{2},$$

so that $A(x) = \sqrt{\frac{1+|\cos x|}{2}}$. Therefore, if $\cos x \geq 0$, then $A(x) = \sqrt{\frac{1+\cos x}{2}} = \pm \cos \frac{x}{2}$; if $\cos x < 0$, then $A(x) = \sqrt{\frac{1-\cos x}{2}} = \pm \sin \frac{x}{2}$.

F17. *(Grade 12.)* In an acute triangle ABC, a point P is chosen such that all points symmetrical to P with respect to the sides of ABC lie on the circumcircle of ABC. Prove that P is the orthocenter of ABC.

Solution. Let A', B', C' be points symmetric to the point P with respect to the sides BC, CA, AB (Fig. 17). Then $|C'A| = |PA| = |B'A|$, giving that the arcs AC' and AB' of the circumcircle of the triangle ABC are equal. Since A and C' are on the same half-plane from the line BB', and C on the other one, we have $\angle C'CA = \angle B'CA = \angle PCA$. Since P and C' are on the same side from the line AC, the points P, C, C' are collinear. Since $PC' \perp AB$, we must also have $PC \perp AB$, that is, the point P lies on the height drawn from the vertex C in the triangle ABC. Analogously we see that P is on the other two heights.

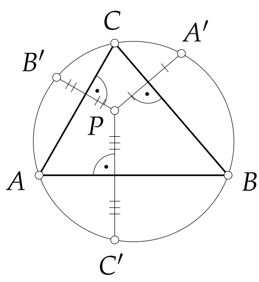


Fig. 17

Remark 1. The converse—the points symmetric to the orthocenter with respect to the sides of the triangle lie on the circumcircle of the triangle—is a known result in elementary geometry that can also be used to solve this problem. Namely, the point P lies on the reflections of arcs AB, BC, CA from the corresponding lines AB, BC, CA of the circumcircle. By the theorem mentioned, the intersection point of the height lies on the same arcs. But the circles, whose arcs are the reflections, already meet twice pairwise at the points A, B, C. Therefore, they cannot have two common intersection points.

Remark 2. The claim of the problem, as well as the theorem given in Remark 1, hold for all triangles, not just acute ones.

F18. *(Grade 12.)* There are 2^n soldiers standing in a line, where n is a positive integer. The soldiers can rearrange themselves into a new line only in the following way: the soldiers standing at odd numbered positions move to the front of the row, keeping their positions with respect to each other, and the soldiers previously standing at even numbered positions move to the end of the row, keeping their positions with respect to each other. Prove that after n rearrangements the soldiers stand in the same ordering as in the beginning.

Solution 1. The last soldier does not change its position. The rest of the soldiers regroup just as in the case, when the last soldier was not there, and the number of the soldiers was $2^n - 1$. So, it suffices to prove the claim for $2^n - 1$ soldiers. We show that after n rearrangements the soldiers are in positions, which can be found in the original line by counting cyclically every 2^i-th soldier (after the last soldier we go to the first one). Indeed, after 0 rearrangements, the claim clearly holds, and every rearrangement makes us cyclically count every second soldier in the previous line (after the last soldier we go to the second one), the first soldier will still be counted first. After n rearrangements the soldiers in the new line can be found by counting every 2^n-th soldier in the old line with $2^n - 1$ soldiers. Since the remainder of 2^n when divided by $2^n - 1$ is 1, this is equivalent to simply counting the soldiers. This means that we get back the original line.

Solution 2. Enumerate the soldiers starting from 0, and write the numbers in binary form (adding leading zeros to make the lengths of the binary codes equal; for example for $n = 3$ we have the numbers 000, 001, 010, 011, 100, 101, 110, 111). After a rearrangement the soldiers stand in such a way that when reinterpreting the last digit as the first one (but leaving the order of the rest of the digits the same), the soldiers are again enumerated by consecutive numbers. After n rearrangements the binary code of the soldiers has returned to the original, so every soldier's position corresponds to their original position in the line.

F19. *(Grade 12.)* a) Does there exist a function from real numbers to real numbers, which is not constantly zero and whose derivative's graph can be obtained by reflecting the graph of the original function with respect to the y-axis?

b) Does there exist a function from real numbers to real numbers, which is not constantly zero and whose derivative's graph can be obtained by shifting the graph of the original function towards the positive side of the x-axis by one unit?

Answer: a) yes; b) yes.

Solution. a) What the problem asks is equivalent to finding a function f such that $f'(x) = f(-x)$ for all real numbers x. This is so, for example, for the function $f(x) = \sin x + \cos x$, since $f'(x) = \cos x - \sin x = \cos(-x) + \sin(-x) = f(-x)$.

b) The premise about the graphs is equivalent to saying that for every real number x, we have $f'(x) = f(x-1)$. Assume that we have $a > 1$ such that $\ln a = a^{-1}$. Then, defining $f(x) = a^x$, we get $f'(x) = a^x \ln a = a^x \cdot a^{-1} = a^{x-1} = f(x-1)$. It remains to make sure that such a number $a > 1$ exists. Since $\ln 1 = 0 < 1 = 1^{-1}$ and $\ln \mathrm{e} = 1 > \mathrm{e}^{-1}$, the graphs of the continuous functions $g(x) = \ln x$ and $h(x) = x^{-1}$ intersect at some point $a > 1$. This is the number we were looking for.

Remark. One can prove that in part a) precisely all functions of the form $f(x) = c \cdot (\sin x + \cos x)$, where $c \neq 0$, satisfy the premises. The answers can be written in a different form, for example $\frac{\sqrt{2}}{2} \cdot (\sin x + \cos x) = \sin\left(x + \frac{\pi}{4}\right)$.

IMO Team Selection Contest

First day

S1. Prove that for any positive integer k there exist k pairwise distinct integers for which the sum of their squares equals the sum of their cubes.

Solution. For any integer $m > 1$ the numbers $2m^2+1$, $m(2m^2+1)$, $-m(2m^2+1)$ satisfy the conditions of the problem, because they are pairwise different and

$$\begin{aligned}&(2m^2+1)^2 + (m(2m^2+1))^2 + (-m(2m^2+1))^2\\ &= (1+m^2+m^2)\cdot(2m^2+1)^2 = (2m^2+1)^3 = (1+m^3-m^3)\cdot(2m^2+1)^3\\ &= (2m^2+1)^3 + (m(2m^2+1))^3 + (-m(2m^2+1))^3.\end{aligned}$$

With m growing, the numbers in these triples get arbitrarily large, hence for any set of these triples one can find a new triple, where all numbers are larger than the ones already used.

Any positive integer k can be written as $k = 3q + r$ with $0 \leq r < 3$. Choose q triples as above so that the numbers in them do not coincide. If $r = 1$, then add 0, and if $r = 2$, then add 0 and 1. Since for each group the sum of the squares of the numbers equals the sum of the cubes of the numbers, the same property holds for the whole set.

Remark 1. One can find these triples by looking for three numbers where two of them are opposites of each other. This gives the equation $x^2 + 2y^2 =$

x^3, or $2y^2 = x^2(x-1)$. Let $d = \gcd(x,y)$ and $x = dn$, $y = dm$. Then $2m^2 = n^2(dn-1)$. If n had a nontrivial prime divisor, then it must also divide m, a contradiction. Hence $n = 1$ and the equation is $2m^2 = d - 1$, or $d = 2m^2 + 1$. By choosing m freely we get the triples above.

Remark 2. One can also solve the problem by first showing that there exist infinitely many quadruples $(-m, m, -n, n+1)$ with $2m^2 = n^2 + n$ that satisfy the conditions of the problem.

S2. For a given positive integer n one has to choose positive integers a_0, $a_1, \ldots$ so that the following conditions hold:

(1) $a_i = a_{i+n}$ for any i;

(2) a_i is not divisible by n for any i;

(3) a_{i+a_i} is divisible by a_i for any i.

For which positive integers $n > 1$ is this possible only if the numbers a_0, a_1, $\ldots$ are all equal?

Answer: for all primes.

Solution. Let n be a prime. By condition (1) the sequence $a_0, a_1, \ldots$ contains only finitely many different numbers. If a_m is maximal of them, then by condition (3) a_{m+a_m} must also be maximal. Let us prove that if a_m is maximal of the numbers, then $a_{m+k\cdot a_m}$ is also maximal for any $k \geq 0$. This holds for $k = 0$. If the claim holds for k, then $a_{m+(k+1)\cdot a_m} = a_{m+k\cdot a_m + a_m} = a_{m+k\cdot a_m + a_{m+k\cdot a_m}} = a_{m+k\cdot a_m} = a_m$. This proves the claim. By condition (2) a_m is not divisible by n. Since n is prime, the numbers a_m and n are relatively prime. Hence among the numbers $m + k \cdot a_m$, where $0 \leq k < n$, there is one in each congruence class modulo n. Hence all members of the sequence are maximal, i.e. they are equal.

Suppose n is a composite number; let m be its divisor with $1 < m < n$. For any $k < m$ choose $a_k = m + k \cdot n$ and continue the sequence with period m. Condition (1) holds, since n is a multiple of m. Condition (2) holds, since all members of the sequence are congruent to m modulo n. For the condition (3) notice that all members of the sequence are divisible by m. Hence i and $i + a_i$ are always congruent modulo m, therefore $a_i = a_{i+a_i}$. At the same time not all the numbers are equal.

S3. In a cyclic quadrilateral $ABCD$ we have $|AD| > |BC|$ and the vertices C and D lie on the shorter arc AB of the circumcircle. Rays AD and BC intersect at point K, diagonals AC and BD intersect at point P. Line KP intersects the side AB at point L. Prove that $\angle ALK$ is acute.

Solution 1. From the properties of cyclic quadrilaterals we get $\angle KAB = \angle KCD$ and $\angle KBA = \angle KDC$. Let A', B', K' be the foots of the altitudes of the triangle ABK drawn from the vertices A, B, K, respectively, and let H be the orthocenter of the triangle ABK (Fig. 18). The points A, B, A', B' lie on a common circle, hence $\angle KA'B' = \angle KAB$ if $A' \neq B'$. Therefore A' and

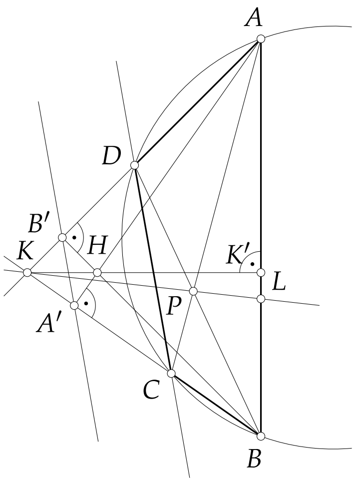


Fig. 18

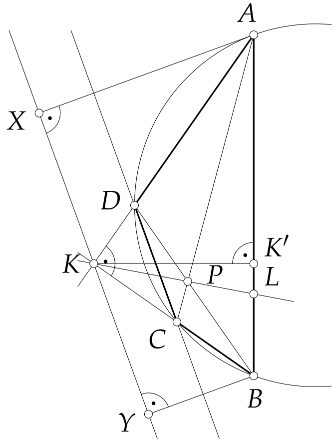


Fig. 19

B' lie on a line parallel to CD. Denote this line by $A'B'$ (even in the case $A' = B' = K$).

Let $d(X, l)$ be the distance of point X from line l, and let S_Δ be the area of triangle Δ. By two angles, $\triangle ACK \sim \triangle BDK$ and $\triangle PAD \sim \triangle PBC$, whence $\frac{|AK|}{|BK|} = \frac{|AC|}{|BD|}$ and $\frac{|AP|}{|BP|} = \frac{|AD|}{|BC|}$. At the same time

$$\frac{d(A, CD)}{d(B, CD)} = \frac{S_{\triangle ACD}}{S_{\triangle BCD}} = \frac{|AC| \cdot |AD| \cdot \sin \angle CAD}{|BD| \cdot |BC| \cdot \sin \angle CBD} = \frac{|AC|}{|BD|} \cdot \frac{|AD|}{|BC|},$$

$$\frac{d(A, KP)}{d(B, KP)} = \frac{S_{\triangle AKP}}{S_{\triangle BKP}} = \frac{|AK| \cdot |AP| \cdot \sin \angle KAP}{|BK| \cdot |BP| \cdot \sin \angle KBP} = \frac{|AK|}{|BK|} \cdot \frac{|AP|}{|BP|}.$$

Therefore

$$\frac{|AL|}{|LB|} = \frac{d(A, KP)}{d(B, KP)} = \frac{d(A, CD)}{d(B, CD)}.$$

Considering instead of the cyclic quadrilateral $ABCD$ the quadrilateral determined by points A, B, A', B', and instead of P and L the points H and K'correspondingly, we get similarly that

$$\frac{|AK'|}{|K'B|} = \frac{d(A, KH)}{d(B, KH)} = \frac{d(A, A'B')}{d(B, A'B')}.$$

This equality holds also in the special case $A' = B' = K$. Indeed, let the projections of points A and B to the line $A'B'$ be X and Y correspondingly (Fig. 19), then $\angle AKX = \angle KDC = \angle KBA = \angle AKK'$, $\angle BKY = \angle KCD = \angle KAB = \angle BKK'$, whence $\triangle AKX \cong \triangle AKK'$ and $\triangle BKY \cong \triangle BKK'$. It follows that $|AK'| = |AX|$, $|BK'| = |BY|$ and $\frac{|AK'|}{|K'B|} = \frac{d(A, A'B')}{d(B, A'B')}$.

Since C and D lie on the shorter arc AB, we have $\angle BCA = \angle BDA > \frac{\pi}{2}$. Thus the line $A'B'$ is farther from the points A and B than the line CD. Since $|AD| > |BC|$, we have $\angle ABD > \angle CAB$ and also $\angle KBA > \angle KAB$, which implies $|KA| > |KB|$. Hence $d(A, CD) + d(K, CD) > d(B, CD) + d(K, CD)$, or $d(A, CD) > d(B, CD)$. All together

$$\frac{|AL|}{|LB|} = \frac{d(A, CD)}{d(B, CD)} > \frac{d(A, A'B')}{d(B, A'B')} = \frac{|AK'|}{|K'B|}.$$

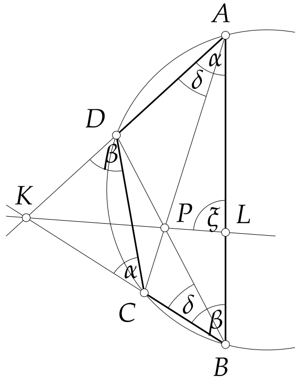


Fig. 20

Hence L lies farther from A than K' on the segment AB, therefore $\angle ALK < \angle AK'K = \frac{\pi}{2}$, i.e. $\angle ALK$ is acute.

Solution 2. Denote $\angle KAB = \angle KCD = \alpha$, $\angle KBA = \angle KDC = \beta$, $\angle KAC = \angle KBD = \delta$ and $\angle ALK = \xi$ (Fig. 20). Then $\angle KDB = \alpha + \beta - \delta = \angle KCA$. The condition $|AD| > |BC|$ is equivalent to $\beta > \alpha$, and points C and D being located in the shorter arc AB is equivalent to the inequality $\alpha + \beta - \delta < \frac{\pi}{2}$. In triangle KCD we get $\frac{|KD|}{|KC|} = \frac{\sin\alpha}{\sin\beta}$. From triangles KDP and KCP we obtain $\frac{|KP|}{\sin(\alpha+\beta-\delta)} = \frac{|KD|}{\sin(\xi-(\beta-\delta))}$, $\frac{|KP|}{\sin(\alpha+\beta-\delta)} = \frac{|KC|}{\sin(\xi+(\alpha-\delta))}$, respectively. Consequently, $\frac{|KD|}{|KC|} = \frac{\sin(\xi-(\beta-\delta))}{\sin(\xi+(\alpha-\delta))}$. Expressions of $\frac{|KD|}{|KC|}$ together yield $\frac{\sin\alpha}{\sin\beta} = \frac{\sin(\xi-(\beta-\delta))}{\sin(\xi+(\alpha-\delta))} = \frac{\sin\xi\cos(\beta-\delta)-\cos\xi\sin(\beta-\delta)}{\sin\xi\cos(\alpha-\delta)+\cos\xi\sin(\alpha-\delta)}$, that in turn implies

$$\begin{aligned}\sin\xi\,(\sin\beta\cos(\beta-\delta) - \sin\alpha\cos(\alpha-\delta))\\ = \cos\xi\,(\sin\beta\sin(\beta-\delta) + \sin\alpha\sin(\alpha-\delta)). \quad (1)\end{aligned}$$

Clearly $\sin\xi > 0$ and $\sin\beta\sin(\beta-\delta) + \sin\alpha\sin(\alpha-\delta) > 0$ because $\xi < \pi$ and $\beta > \delta$, $\alpha > \delta$. By the formula $\sin x\cos y = \frac{1}{2}(\sin(x+y) + \sin(x-y))$, we get $\sin\beta\cos(\beta-\delta) - \sin\alpha\cos(\alpha-\delta) = \frac{1}{2}(\sin(2\beta-\delta) - \sin(2\alpha-\delta))$. Now $\alpha + \beta - \delta < \frac{\pi}{2}$ implies $(2\beta-\delta) + (2\alpha-\delta) < \pi$, i.e., there exists a triangle whose two angles are $2\beta - \delta$ and $2\alpha - \delta$. But $2\beta - \delta > 2\alpha - \delta$ since $\beta > \alpha$, therefore the law of sines in that triangle implies $\sin(2\beta-\delta) > \sin(2\alpha-\delta)$ (the larger the angle, the large its opposite side in a triangle). Hence the second factor in the l.h.s. of equation (1) is positive. Altogether, we obtain $\cos\xi > 0$, whence $\xi < \frac{\pi}{2}$.

Remark. This problem can also be solved by coordinates.

Second day

S4. Let ABC be a triangle where $|AB| = |AC|$. Points P and Q are different from the vertices of the triangle and lie on the sides AB and AC, respectively. Prove that the circumcircle of the triangle APQ passes through the circumcenter of ABC if and only if $|AP| = |CQ|$.

Solution. Without loss of generality, we can assume that $|AP| \leq |AQ|$. Let O be the circumcenter of ABC. Let R be the intersection point of the bisector of $\angle BAC$ with the circumcircle of the triangle PAQ — we then have $|RB| = |RC|$ (Fig. 21). Also, $\angle APR = 180^\circ - \angle AQR = \angle CQR$ and $|RP| = |RQ|$ (since $\angle RAP = \angle RAQ$). So, $|AP| = |CQ| \iff \triangle APR \cong \triangle CQR \iff |RA| = |RC| \iff R = O$ (where $|RA| = |RC| \Rightarrow \triangle APR \cong \triangle CQR$ by two sides and obtuse angle).

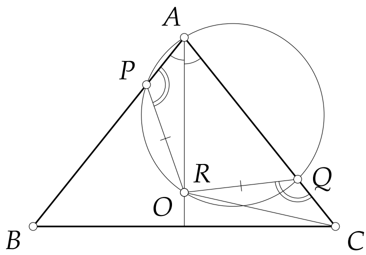


Fig. 21

Remark. This problem has been taken from the booklet "The Coins of Harpland and 20+10 other maths problems from Ireland" (edited by Bernd Kreussler), its author is Jim Leahy. The solution here is new.

S5. Let x, y, z be positive real numbers whose sum is 2012. Find the maximum value of

$$\frac{(x^2+y^2+z^2)(x^3+y^3+z^3)}{(x^4+y^4+z^4)}.$$

Answer: 2012.

Solution. If $x = y = z = \frac{2012}{3}$, then $\frac{(x^2+y^2+z^2)(x^3+y^3+z^3)}{(x^4+y^4+z^4)} = 2012$. Now we prove that for all x, y, z satisfying the premises we have

$$\frac{(x^2+y^2+z^2)(x^3+y^3+z^3)}{(x^4+y^4+z^4)} \leq 2012.$$

It suffices to show that $(x^2+y^2+z^2)(x^3+y^3+z^3) \leq 2012(x^4+y^4+z^4)$, or $(x^2+y^2+z^2)(x^3+y^3+z^3) \leq (x+y+z)(x^4+y^4+z^4)$. Multiplying out, simplifying and rearranging the terms gives $xy(x-y)(x^2-y^2) + xz(x-z)(x^2-z^2) + yz(y-z)(y^2-z^2) \geq 0$. Since the differences in the brackets in every product have equal signs, the products are non-negative, showing that the necessary inequality holds.

Remark. The inequality that we get after multiplying out and canceling x^5, y^5, z^5, is a special case of Muirhead inequality (with exponent vectors $(4,1,0)$ and $(3,2,0)$).

S6. On an $m \times m$ board, at the midpoints of the unit squares there are some ants. At the time 0 each ant starts moving with speed 1 parallel to some edge of the board until it meets an ant moving in the opposite direction or until it reaches the edge of the board. When two ants moving in the opposite direction meet each other, both turn 90° clockwise and continue moving parallel to another edge of the board. Upon reaching the edge of the board the ant falls off the board.

a) Prove that eventually all the ants will have fallen off the board.

b) Find the latest possible moment for the last ant to fall off the board.

Answer: b) $\frac{3}{2}m - 1$.

Solution 1. Let the lower left corner of the board be the origin. Divide the units of time and space by 2; then the squares are of dimensions 2×2, the coordinates of the midpoints of the squares are odd positive integers, and the speed of the ants is still 1.

We prove by induction that at integer time moments the coordinates of the ants are integers and the sum of the coordinates for any fixed ant has the same parity as the time moment. In addition, the ants can meet only at integer time moments. At time $t = 0$ all coordinates of the ants are odd, so their sum is even. Suppose that at an integer time moment $t = k$ the coordinates of the ants are integers and the sum of the coordinates for any fixed ant has the same parity as the time moment. If two of the ants were to meet each other within the next time unit, they have to move toward each other from time $t = k$, hence one of their coordinates must be the same. Since the parity of the sum of their coordinates was the same at time $t = k$, another of their coordinates had to differ by at least 2. Hence they cannot meet before time $t = k + 1$. Between time moments $t = k$ and $t = k + 1$ every ant has changed only one of its coordinates by 1, hence at time $t = k + 1$ the parity of the sum of the coordinates is again the same as the parity of the time moment.

Next we will prove by induction that for any point with integer coordinates (x, y) there are no collisions at this point after the time moment $t = x + y - 2$. For $x = y = 1$ this is obviously true, since there are no collisions in the middle of the lower left square (otherwise one of the ants has to arrive to this point from the edge of the board). Let (x, y) be arbitrary and suppose that the claim holds for all points with the sum of the coordinates less than $x + y$. Suppose that a collision takes place at point (x, y) at time t. One of the participants had to arrive from a point, where one of the coordinates was smaller; w.l.o.g. we can assume that this was the x-coordinate. If this ant has not collided with anyone before, then $t \leq x - 1 \leq x + y - 2$. If the last collision of this ant occurred at time $t' < t$, then the coordinates of the last collision were $(x - (t - t'), y)$. By the induction assumption $t' \leq x - (t - t') + y - 2$, hence $t \leq x + y - 2$.

By symmetry the claim holds when another corner is chosen as the origin. Let the last collision of a particular ant occur at the point (x, y), where the coordinates are taken with respect to the nearest corner. W.l.o.g., we can assume $x \leq y$. The time from the last collision to the falling off the edge of the ants participating in the collision is at most $2m - x$, hence the time elapsed from the start is at most $x + y - 2 + 2m - x \leq 3m - 2$. By this time all ants have fallen off the edge. With respect to the original units the maximal time is $\frac{3}{2}m - 1$.

For any m the maximal time can be achieved, if in the beginning there are 2 ants at the adjoining corners of the board moving toward each other. At the moment $t = \frac{m-1}{2}$ the pair collides and one of the ants starts moving toward the center, falling off the board at time $t = \frac{3}{2}m - 1$.

Solution 2. Part a) can also be solved as follows. For each ant consider the distance to the edge in the direction of its motion. After an ant falls this distance will remain 0. Observe that as long as an ant moves without collision, this distance decreases with speed 1.

Consider now the sum of all such distances. When a collision happens, the sum of the distances of the two corresponding ants is m, both right before and right after the collision. Thus as long as there are ants left on the board, the total sum decreases with the speed of at least 1. Since in the beginning this sum is a finite number, after some time this sum will become 0 and thus all ants will have fallen off the board.

Remark. This problem, proposed by Estonia, appeared in the IMO 2011 shortlist as C5.

Problems Listed by Topic

Number theory: O1, O7, O12, F1, F5, F9, F10, F15, S2
Algebra: O6, O8, O13, F6, F11, F16, F19, S1, S5
Geometry: O3, O4, O9, O14, F2, F4, F7, F12, F17, S3, S4
Discrete mathematics: O2, O5, O10, O11, O15, F3, F8, F13, F14, F18, S6