Estonian Math Competitions 2015/2016

The Gifted and Talented Development Centre
Tartu 2016

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WE THANK:

Estonian Ministry of Education and Research

University of Tartu

Problem authors: Juhan Aru, Matija Bucić (Croatia), Urve Kangro, Härmel Nestra, Heiki Niglas, Uve Nummert, Erik Paemurru, Ago-Erik Riet, Triinu Veeorg

Translators: Urve Kangro, Härmel Nestra, Kaur Aare Saar, Laur Tooming

Editors: Härmel Nestra, Reimo Palm

Estonian Mathematical Olympiad

http://www.math.olympiaadid.ut.ee/

Mathematics Contests in Estonia

The Estonian Mathematical Olympiad is held annually in three rounds: at the school, town/regional and national levels. The best students of each round (except the final) are invited to participate in the next round. Every year, about 110 students altogether reach the final round.

In each round of the Olympiad, separate problem sets are given to the students of each grade. Students of grade 9 to 12 compete in all rounds, students of grade 7 to 8 participate at school and regional levels only. Some towns, regions and schools also organize olympiads for even younger students. The school round usually takes place in December, the regional round in January or February and the final round in March or April in Tartu. The problems for every grade are usually in compliance with the school curriculum of that grade but, in the final round, also problems requiring additional knowledge may be given.

The first problem solving contest in Estonia took place in 1950. The next one, which was held in 1954, is considered as the first Estonian Mathematical Olympiad.

Apart from the Olympiad, open contests are held twice a year, usually in September and in December. In these contests, anybody who has never been enrolled in a university or other higher education institution is allowed to participate. The contestants compete in two separate categories: Juniors and Seniors. In the first category, students up to the 10th grade can participate; the other category has no restriction. Being successful in the open contests generally assumes knowledge outside the school curriculum.

Based on the results of all competitions during the year, about 20 IMO team candidates are selected. IMO team selection contest for them is held in April or May; in recent years experimentally in two rounds. Each round is an IMO-style two-day competition with 4.5 hours to solve 3 problems on both days. Some problems in our selection contest are at the level of difficulty of the IMO but easier problems are usually also included.

The problems of previous competitions can be downloaded at the Estonian Mathematical Olympiads website.

Besides the above-mentioned contests and the quiz "Kangaroo" some other regional and international competitions and matches between schools are held.

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This booklet presents the problems of the open contests, the final round of national olympiad and the team selection contest. For the open contests and the final round, selection has been made to include only problems that have not been taken from other competitions or problem sources and seem to be interesting enough. The team selection contest is presented entirely.

Selected Problems from Open Contests

O1. *(Juniors.)* A positive integer n is *interesting*, if for some positive integer m and positive integers a, b that are smaller than m, $\frac{m^2}{ab} = n$. For example, 10 is interesting because $\frac{20^2}{4 \cdot 10} = 10$. Find the smallest interesting integer.

Answer: 2.

Solution. For $n = 2$ we can take $m = 12$, $a = 8$ and $b = 9$, because $\frac{12^2}{8 \cdot 9} = \frac{144}{72} = 2$. On the other hand, 1 is not interesting, because if $\frac{m^2}{ab} = 1$, or $m^2 = ab$, then a and b cannot both be less than m at the same time.

Remark. Every number greater than 1 is interesting. The construction in the solution for $n = 2$ generalizes to any $n > 1$, if we define $m = n^2(n+1)$, $a = n^3$ and $b = (n+1)^2$.

O2. *(Juniors.)* Is there a two-digit number n that does not end with zero such that

a) all numbers that can be formed by adding one or more zeros between the two-digit number's digits are its multiples?

b) none of the numbers that can be formed by adding one or more zeros between the two-digit number's digits are its multiples?

c) some numbers that can be formed by adding one or more zeros between the two-digit number's digits are its multiples and some are not?

Answer: a) yes, b) yes, c) yes.

Solution. a) One such number is $n = 15$. All numbers formed by adding zeros between its digits are divisible by 3 and 5.

b) One such number is $n = 12$. Adding zeros between the digits, the last two digits will always be 02. Hence no such number will be divisible by 4.

c) One such number is $n = 11$, because 101 is not divisible by 11 but 1001 is.

O3. *(Juniors.)* A right triangle ABC has the right angle at vertex A. Circle c passes through vertices A and B of the triangle ABC and intersects the sides AC and BC correspondingly at points D and E. The line segment CD has the same length as the diameter of the circle c. Prove that the triangle ABE is isosceles.

Solution 1. Since $\angle BAD = 90°$ (Fig. 1), BD is the diameter of circle c and therefore $CD = BD$. Since BD is diameter, also $\angle BED = 90°$, so DE is an altitude of the isosceles triangle BDC, bisecting its base BC. Hence E is the midpoint of the hypotenuse BC of the triangle ABC. Since the midpoint of the hypotenuse is the circumcentre of a right triangle, it follows $EA = EB$. This means that ABE is an isosceles triangle.

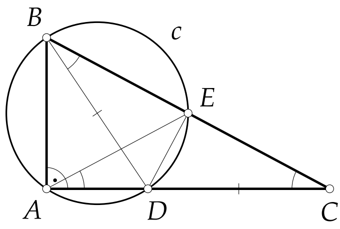

Fig. 1

Solution 2. As in the previous solution, we show that $CD = BD$. Hence $\angle ECD = \angle EBD$. From the equality of the inscribed angles subtending the arc ED it also follows $\angle EBD = \angle EAD$. From the triangle ABC we get $\angle ABC = 90^\circ - \angle BCA$, or $\angle EBA = 90^\circ - \angle ECD$. On the other hand, $\angle EAB = \angle DAB - \angle EAD = 90^\circ - \angle EBD = 90^\circ - \angle ECD$. Consequently $\angle EBA = \angle EAB$. So the triangle ABC is isosceles.

O4. *(Juniors.)* Let d be a positive number. On the parabola, whose equation has the coefficient 1 at the quadratic term, points A, B and C are chosen in such a way that the difference of the x-coordinates of points A and B is d and the difference of the x-coordinates of points B and C is also d. Find the area of the triangle ABC.

Answer: d^3.

Solution 1. Without loss of generality assume that equation of the parabola is $y = x^2$ (Fig. 2). Let the abcissas of the points A, B, and C be a, b, and c. Let A', B', and C' be the projections of A, B and C onto the x-axis. Denoting the area of a region $\mathcal{K}$ by $S_{\mathcal{K}}$ we have $S_{ABC} = S_{ACC'A'} - S_{ABB'A'} - S_{BCC'B'}$. Since $S_{ACC'A'} = 2d \cdot \frac{a^2+c^2}{2}$, $S_{ABB'A'} = d \cdot \frac{a^2+b^2}{2}$, $S_{BCC'B'} = d \cdot \frac{b^2+c^2}{2}$, it follows that $S_{ABC} = \frac{1}{2}d(2a^2 + 2c^2 - a^2 - b^2 - b^2 - c^2) = \frac{1}{2}d(a^2 - b^2 - b^2 + c^2) = \frac{1}{2}d\big(d(a+b) - d(b+c)\big) = \frac{1}{2}d^2(a-c) = \frac{1}{2}d^2 \cdot 2d = d^3$.

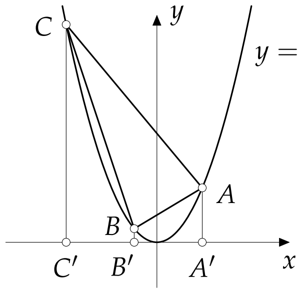

Fig. 2

Solution 2. Without loss of generality we can assume that point B lies at the origin. Then the equation of the parabola is $y = x^2 + px$ with some p. The coordinates of A and C are then $(d, d^2 + pd)$ and $(-d, d^2 - pd)$. Let D be the midpoint of the segment AC. Its coordinates are $(0, d^2)$, so BD is perpendicular to the x-axis, hence the lengths of the altitudes of both triangles ABD and BCD with the base BD are d. So both the triangles have area $\frac{d^3}{2}$, hence the area of the triangle ABC is d^3.

O5. *(Juniors.)* On the plane three different points P, Q, and R are chosen. It is known that however one chooses another point X on the plane, the point P is always either closer to X than the point Q or closer to X than the point R. Prove that the point P lies on the line segment QR.

Solution. We show that if the point P lies outside the segment QR, then the conditions of the problem are not satisfied.

If P lies on the line QR but outside the segment QR (Fig. 3), then we can

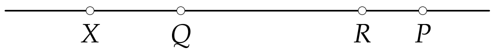

Fig. 3

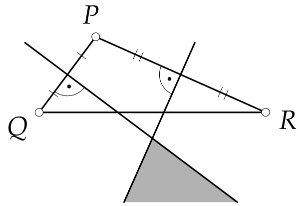

Fig. 4

take the point X on the line QR on the other side of the segment QR. Then the points Q and R are closer to the point X than the point P.

If P lies outside the line QR (Fig. 4), then the perpendicular bisectors of the segments PQ and PR intersect. Choose the points X in the region, which lies towards Q from the perpendicular bisector of PQ and towards R from the perpendicular bisector of PR (the dark region on the figure). Then the points Q and R are closer to the point X than the point P.

O6. *(Seniors.)* a) Find the largest number that is the greatest common divisor of some four different two-digit numbers. b) Find the largest number that is the least common multiple of some four different two-digit numbers.

Answer: a) 24; b) $99 \cdot 98 \cdot 97 \cdot 95$.

Solution. a) Let d be the greatest common divisor of some four different two-digit numbers. Since all these numbers are divisible by d, the least possible candidates of these four numbers are $d, 2d, 3d, 4d$. Hence $4d < 100$, thus $d \leqslant 24$. On the other hand, the greatest common divisor of 24, 48, 72 and 96 is 24.

b) The numbers 99, 98, 97, and 95 are pairwise relatively prime, hence $\operatorname{lcm}(99, 98, 97, 95) = 99 \cdot 98 \cdot 97 \cdot 95$. To show that this is the largest possible, consider four different two-digit numbers a_1, a_2, a_3, a_4; assume without loss of generality that $a_1 > a_2 > a_3 > a_4$. If $a_4 \leqslant 95$, then $\operatorname{lcm}(a_1, a_2, a_3, a_4) \leqslant a_1a_2a_3a_4 \leqslant 99 \cdot 98 \cdot 97 \cdot 95$. If $a_4 = 96$, then the four numbers can only be 99, 98, 97, 96, but $\operatorname{lcm}(99, 98, 97, 96) = \frac{99 \cdot 98 \cdot 97 \cdot 96}{2 \cdot 3} < 99 \cdot 98 \cdot 97 \cdot 95$.

O7. *(Seniors.)* Angles α and β are such that $\frac{\tan\alpha}{\tan\beta} = k \neq 1$. Express $\frac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)}$ in terms of k.

Answer: $\frac{k+1}{k-1}$.

Solution. We have $k = \frac{\tan\alpha}{\tan\beta} = \frac{\sin\alpha \cdot \cos\beta}{\cos\alpha \cdot \sin\beta}$, or $\sin\alpha\cos\beta = k \cdot \cos\alpha\sin\beta$. Therefore

$$\frac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)} = \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\sin\alpha\cos\beta - \cos\alpha\sin\beta} = \frac{(k+1)\cdot\cos\alpha\sin\beta}{(k-1)\cdot\cos\alpha\sin\beta} = \frac{k+1}{k-1}.$$

O8. *(Seniors.)* Let n be a natural number such that $n+1$, $n+3$, $n+7$ and $n+9$ are prime numbers, and $n+31$, $n+33$, $n+37$ and $n+39$ are also prime numbers. Find the remainder of n divided by 210.

Answer: 190.

Solution. Consider the remainders of n when divided by 2, 3, 5, and 7. In the following tables the left column shows the remainder and the right column shows, which of the given eight numbers cannot be prime, if $n \geqslant 7$:

n mod 2	divisible by 2
1	$n+1$

n mod 3	divisible by 3
0	$n+3$
2	$n+1$

n mod 5	divisible by 5
1	$n+9$
2	$n+3$
3	$n+7$
4	$n+1$

n mod 7	divisible by 7
0	$n+7$
2	$n+33$
3	$n+39$
4	$n+3$
5	$n+9$
6	$n+1$

Thus if $n \geqslant 7$ then the remainder of n when divided by 2 and 5 is 0 and when divided by 3 and 7 is 1; hence $n-1$ is a multiple of 3 and 7. Consequently n is a multiple of 10 and $n-1$ is a multiple of 21. This implies that $n+20$ is a multiple of 21 and a multiple of 10, hence a multiple of 210. Consequently the remainder of n when divided by 210 is 190. The numbers $n = 1, 2, 3, 4, 5, 6$ do not satisfy the conditions of the problem.

Remark. The smallest numbers satisfying the conditions of the problem are $n = 1006300, 2594950, 3919210, 9600550, \ldots$

O9. *(Seniors.)* Find all real-valued functions f defined on real numbers which satisfy $f(f(x)+f(y)) = f(x)+y$ for all real x, y.

Answer: $f(x) = x$.

Solution 1. Let z_1, z_2 be real numbers for which $f(z_1) = f(z_2)$. Substituting $y = z_1$ and $y = z_2$ into the given equation we get $f(f(x)+f(z_1)) = f(x)+z_1$, and $f(f(x)+f(z_2)) = f(x)+z_2$. Since the left hand sides are equal, we have $f(x)+z_1 = f(x)+z_2$, whence $z_1 = z_2$. Hence f is one-to-one. Substituting $y = 0$ into the given equation we get $f(f(x)+f(0)) = f(x)$ for any real x. Since f is one-to-one, we have $f(x)+f(0) = x$. If $x = 0$, then the last equation gives $2f(0) = 0$, or $f(0) = 0$. So this equation simplifies to $f(x) = x$. The function $f(x) = x$ satisfies the original equation.

Solution 2. Interchanging x and y in the given equation we get $f(f(y) + f(x)) = f(y)+x$. Since the left hand side is the same as in the original equation, we have $f(x)+y = f(y)+x$. Substituting $y = 0$ into this we get $f(x) = x+f(0)$. Substituting into the original equation all applications of f according to the last equality, we get $x+f(0)+y+f(0)+f(0) = x+f(0)+y$. This gives $f(0) = 0$ and from $f(x) = x+f(0)$ we get $f(x) = x$.

O10. *(Seniors.)* The bisector of the angle A of the triangle ABC intersects the side BC at D. A circle c through the vertex A touches the side BC at D. Prove that the circumcircle of the triangle ABC touches the circle c at A.

Solution 1. Let the centres of the circumcircle ABC and the circle c be respectively O and P (Fig. 5). Denote $\angle CAB = \alpha$, $\angle ABC = \beta$ and $\angle BCA = \gamma$. Then the bisector of angle A creates the angles $\angle ADB = 180^\circ - \beta - \frac{\alpha}{2}$ and $\angle ADC = 180^\circ - \gamma - \frac{\alpha}{2}$ at point D. Without loss of generality, we can assume $\gamma \geqslant \beta$ (otherwise exchange the roles of points B and C). Due to this assumption, $\angle ADB \geqslant \angle ADC$, so $\angle ADB \geqslant 90^\circ$. Since $\angle PDB = 90^\circ$ and $PA = PD$, we have $\angle DAP = \angle ADP = \angle ADB - \angle PDB = \angle ADB - 90^\circ = 90^\circ - \beta - \frac{\alpha}{2}$, from which $\angle CAP = \angle CAD + \angle DAP = 90^\circ - \beta$. On the other hand, from the same assumption we get $\beta < 90^\circ$, and hence $\angle AOC = 2\angle ABC = 2\beta$, from which $\angle CAO = \frac{180^\circ - \angle AOC}{2} = 90^\circ - \beta$. It follows that $\angle CAP = \angle CAO$, or the lines AP and AO coincide. This means that the tangent lines to the circumcircle of the triangle ABC and the circle c at their common point A are both perpendicular to the same line, which means that these tangent lines also coincide. Hence, these circles are tangent to each other at point A.

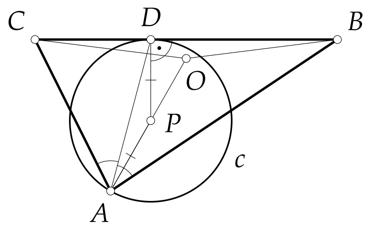

Fig. 5

Solution 2. The claim holds if $AB = AC$, because then the centres of the circle c and the circumcircle of the triangle ABC are on the line AD, which means that the tangents of these circles at point A are both perpendicular to the line AD and hence coincide (Fig. 6). In the following, we assume w.l.o.g that $AC < BC$. Let L be the intersection point of the tangent line to circle c at point A and the line BC (Fig. 7). Since LA and LD are both tangent lines, we have $LA = LD$ and $\angle LAD = \angle LDA$. Since AD is an angle bisector, $\angle CAD = \angle DAB$. Now $\angle CBA = \angle LDA - \angle DAB = \angle LAD - \angle CAD = \angle LAC$. Using the tangent-chord theorem, we conclude that LA is also a tangent line to the circumcircle of the triangle ABC.

O11. *(Seniors.)* On a southern island, there are n fortresses lying on one line ($n > 0$). Each fortress is guarded by two elephants, both watching along the line of fortresses but in the opposite directions. For a fortress A

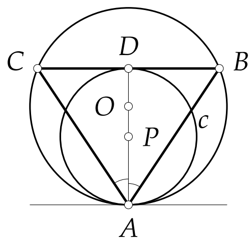

Fig. 6

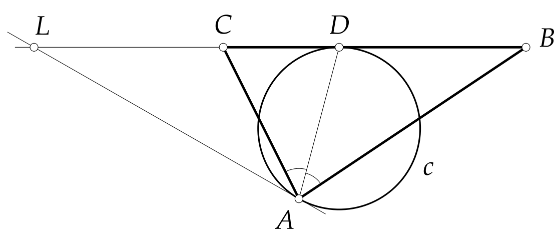

Fig. 7

being able to conquer a fortress B, the elephant from A who is watching towards B must go along the line of fortresses to B, fight with all elephants who stand on his way and are watching opposite to him (including that of fortress B but not the other elephant from A), and win them all. Given that all elephants have different constant weights and a heavier elephant always wins a lighter elephant, prove that there exists exactly one fortress that cannot be conquered by any other fortress.

Solution 1. The elephants from the outermost fortresses who watch in the direction where there are no more fortresses never take part in any fight, so we can discount them. Until only one fortress remains on the island, repeat the following: remove the heaviest elephant together with all fortresses and elephants on his watching direction. Thanks to the initial assumption, at least one fortress is removed on each step. It is easy to see that the fortress guarded by the heaviest elephant can conquer all fortresses that are subject to removal on the current step, while no fortress subject to removal can conquer none of the remaining fortresses since that would require winning the heaviest elephant. Hence the fortress that remains after all other fortresses are removed can be conquered by none of the other fortresses, while all other fortresses can be conquered by at least one of the other fortresses.

Solution 2. A fortress being able to conquer another fortress implies the former fortress also being able to conquer all fortresses between the two fortresses. Let $F_1, \ldots, F_n$ be the fortresses along the line. There definitely exists a fortress which cannot be conquered by any fortress with a smaller number (for instance, F_1 is such). Let F_i be the one with the largest number among those. We show that this fortress is the one we are looking for.

Suppose that F_j can conquer F_i. Then $j > i$ and F_j can conquer also F_{i+1}, $\ldots$, F_{j-1}. As $j > i$, there must exist $l < j$ such that F_l can conquer F_j. If $l < i$, then F_l can conquer F_i which contradicts the choice of i. If $i \leqslant l$, then because of $l < j$, F_j can conquer F_l, while F_l can conquer F_j as well. This is impossible, since in both cases, the two elephants from F_j and F_l looking in the opposite direction must fight against each other.

It remains to show that there are no more unconquerable fortresses. Suppose, some F_j, $j \neq i$, cannot be conquered. If $j > i$, then this contradicts the choice of i. Hence assume that $j < i$. Let among the elephants of $F_{j+1}, \ldots, F_i$ watching towards F_j, that of F_k be the heaviest. As even F_k cannot conquer F_j, there must exist l such that $j \leqslant l < k$ and the elephant from F_l watching towards F_k is heavier than the elephant from F_k watching towards F_l. But this means that F_l can conquer F_i which is impossible.

Solution 3. We proceed by induction on n. The claim obviously holds if $n = 1$. Consider a situation with $n + 1$ fortresses and assume that the claim holds for n fortresses. Let A be the fortress on one end of the line and let B be the next fortress. Let x be the weight of the elephant from A watching towards B, let y be the weight of the elephant from B watching towards A and let z be the weight of the other elephant from B. Consider three cases.

1) The case $x < y$. When excluding A together with its two elephants, there must exist exactly one invincible fortress K among the others by the induction hypothesis. As B can conquer A and the elephant from A cannot pass B, fortress K is the only invincible also in the presence of A.

2) The case $y < x < z$. When excluding A together with its two elephants, there must exist exactly one invincible fortress K among the others. If $K \neq B$ then the fortress that can conquer B can conquer also A while A cannot conquer K (if A could conquer K, also B could). Consequently, K is the only invincible fortress also in the presence of A. If $K = B$ then in the presence of A, A would be the only invincible fortress since it can conquer B while other fortresses cannot pass B.

3) The case $y < x$, $z < x$. Excluding now B together with its two elephants, there must exist a unique invincible fortress L among the others. Note that adding B does not change the correlation of forces among the other fortresses. Indeed, if A can conquer another fortress in the situation without B then it can do it in the presence of B, too, and if some fortress can conquer A in the situation without B then it can do it in the presence of B, too. But B can be conquered by A and if B could conquer L then also A could. As a consequence, L is the only invincible fortress also in the presence of B.

Remark. This problem, proposed by Estonia, appeared in the IMO 2015 shortlist as C1.

O12. *(Seniors.)* Find all positive real solutions of the system of equations $x + \frac{1}{x} - w = 2, y + \frac{1}{y} - w = 2, z + \frac{1}{x} + w = 2, y + \frac{1}{z} + w = 2$.

Answer: $x = 2$, $y = w = \frac{1}{2}$, $z = 1$.

Solution. Subtracting the second equation from the first and multiplying by xy we get $x^2y + y - xy^2 - x = 0$, which gives either $x = y$ or $x = \frac{1}{y}$.

If $x = y$, then subtracting the fourth equation from the third and multiplying by xz, we similarly get either $x = z$ or $x = -\frac{1}{z}$. If $x = z$, then subtracting the third equation from the first gives $-2w = 0$, hence $w = 0$, which is not positive. If $x = -\frac{1}{z}$, then x and z cannot be both positive.

If $x = \frac{1}{y}$, then subtracting the fourth equation from the third gives $z - \frac{1}{z} = 0$, hence $z = 1$. Adding the first equation with the third gives $x + \frac{2}{x} = 3$, or $x^2 - 3x + 2 = 0$. This has a solution $x = 1$, which leads to $x = y = z$ already considered. The second solution $x = 2$ gives $y = \frac{1}{2}$ and $w = \frac{1}{2}$.

O13. *(Seniors.)* Denote by $f^n(x)$ the result of applying the function f n times to x (e.g. $f^1(x) = f(x)$, $f^2(x) = f(f(x))$, $f^3(x) = f(f(f(x)))$ etc). Find all functions from real numbers to real numbers which satisfy $f^d(x) = 2015 - x$ for all divisors d of 2015, which are greater than 1, and for all real x.

Answer: $f(x) = 2015 - x$.

Solution. Since 5 is a divisor of 2015, we have for any real z:

$$f^{25}(z) = f^5(f^5(f^5(f^5(f^5(z))))) = 2015 - f^5(f^5(f^5(f^5(z)))) =$$
$$2015 - (2015 - f^5(f^5(f^5(z)))) = f^5(f^5(f^5(z))) = \ldots = f^5(z) = 2015 - z.$$

Since 13 is also a divisor of 2015, we have for any real z:

$$f^{26}(z) = f^{13}(f^{13}(z)) = 2015 - f^{13}(z) = 2015 - (2015 - z) = z.$$

Consequently $z = f^{26}(z) = f(f^{25}(z)) = f(2015 - z)$. Any real number x can be written as $2015 - z$ for $z = 2015 - x$. Hence $z = f(2015 - z)$ implies $f(x) = 2015 - x$ for any real x.

Finally check that the function $f(x) = 2015 - x$ satisfies the conditions of the problem. Let d be a divisor of 2015 greater than 1. Then d is odd, i.e. $d = 2c + 1$ for a positive integer c. Since $f^2(x) = 2015 - (2015 - x) = x$, we have $f^{2c}(x) = f^2(f^2(\ldots f^2(x)\ldots)) = x$, which implies $f^d(x) = f(f^{2c}(x)) = f(x) = 2015 - x$.

O14. *(Seniors.)* An inventor presented to the king a new exciting board game on a 9×10 squared board. The king promised to reward him one rice grain for the first square, one rice grain for the second square, and for each following square the same number of grains as for the two preceding squares together. Prove that for the last square the inventor gets at least 2015^4 grains.

Solution. Enumerate all squares with $1, \ldots, 90$. Let the number of rice grains promised for the n-th square be F_n; then according to the problem $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for all $n > 2$. Notice that $F_n > F_{n-1}$ if $n > 2$, hence $F_{2(n+1)} = F_{2n+2} = F_{2n+1} + F_{2n} > F_{2n} + F_{2n} = 2 \cdot F_{2n}$ for all n. This implies $F_{2 \cdot 4} > 2 \cdot F_{2 \cdot 3} = 2 \cdot 8 = 2^4$ and by mathematical induction $F_{2n} > 2^n$ for all $n > 3$. Therefore $F_{90} > 2^{45} > 2^{44} = (2^{11})^4 = 2048^4 > 2015^4$.

O15. *(Seniors.)* The circumcentre of an acute triangle ABC is O. Line AC intersects the circumcircle of AOB at a point X, in addition to the vertex A. Prove that the line XO is perpendicular to the line BC.

Solution 1. By the properties of inscribed angles $\angle CXO = \angle ABO$ (Fig. 8) independent of whether the point X lies on the side CA or on the extension of CA over A. Since O is the circumcentre of ABC, we have $\angle ABO = \angle BAO$. Let the intersection of lines XO and BC be Y and let the point on the circumcircle of ABC lying diametrically opposite A be Z. Since $\angle CXY = \angle ZAB$, and $\angle YCX = \angle BZA$ as inscribed angles subtending the same arc AB, the triangles CXY and ZAB are similar. Hence $\angle CYX = \angle ZBA = 90°$, as $\angle ZBA$ is subtended by a diameter.

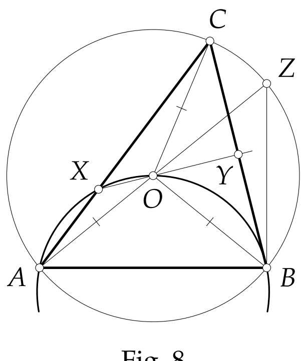

Fig. 8

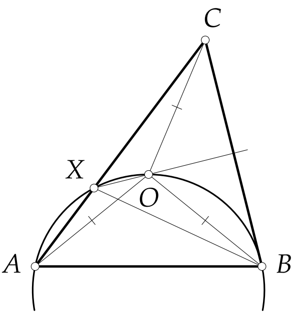

Fig. 9

Solution 2. Obviously $\angle OBC = \angle OCB$. If X lies on segment AC (Fig. 9), then $\angle OBX = \angle OAX = \angle OAC = \angle OCA$. If X lies on the extension of AC over A, then $\angle OBX = 180° - \angle OAX = \angle OAC = \angle OCA$. In both cases we have $\angle OBX = \angle OCA$ and $\angle XBC = \angle OBX + \angle OBC = \angle OCA + \angle OCB = \angle ACB = \angle XCB$. Therefore XBC is an isosceles triangle with the apex X, hence X lies on the perpendicular bisector of BC. Since O lies on the same line, the line XO is the perpendicular bisector of the side BC.

O16. *(Seniors.)* On a switchboard there are nm lamps arranged in an $n \times m$ array. In the beginning all lamps are off. At each step one can switch three consecutive lamps in one row or in one column, changing the state of each lamp to the opposite. For which pairs of positive integers (n, m) is it possible to achieve the situation, where all the lamps are switched on?

Answer: all pairs (n, m), where either n or m is a multiple of 3.

Solution. If n (or m) is a multiple of 3, then we can divide all lamps in each column (or row) into groups of 3 and switch the lamps on by the groups.

If neither n nor m is a multiple of 3, then color all lamps by diagonals with three colors (Fig. 10). Then each switching changes the state of exactly one lamp of each color, therefore each switching changes the parity of the number of lamps switched on for each color. Since in the beginning all lamps are off, the parity of the lamps switched on for each color always stays the same. Let $n = 3a + b$ and $m = 3c + d$, where a and c are nonnegative integers and b and d are either 1 or 2. In regions of sizes $3a \times 3c$, $b \times 3c$ and $3a \times d$ the numbers of lamps of each color are equal and hence their parities are equal, but in the remaining $b \times d$ region there is either 1 lamp (Fig. 11), 2 lamps of different colors (Fig. 12 and 13), or 4 lamps, of which 2 are of one color and two others of the two other colors (Fig. 14), hence the parities of the numbers of lamps of each color are not equal. Consequently, it is not possible to switch on all lamps.

1	2	3	1	2	3
2	3	1	2	3	1
3	1	2	3	1	2
1	2	3	1	2	3
2	3	1	2	3	1

Fig. 10

1

Fig. 11

1	2

Fig. 12

1
2

Fig. 13

1	2
2	3

Fig. 14

Selected Problems from the Final Round of National Olympiad

F1. *(Grade 9.)* Non-equilateral triangle ABC has a $60°$ angle at vertex A. Let the angle bisector drawn from vertex A intersect the opposite side at point D, and let Q and R be the feet of the altitudes drawn from vertices B and C, respectively. Prove that lines AD, BQ and CR intersect in three distinct points that are vertices of an equilateral triangle.

Solution. If line AD passed through the point of intersection of lines BQ and CR, the line segment AD would be an altitude of triangle ABC. As AD is also the angle bisector, triangle ABC would be isosceles with $AB = AC$. As $\angle BAC = 60°$, triangle ABC would be equilateral, contradicting the assumption. Hence the lines AD, BQ, and CR meet in three distinct points.

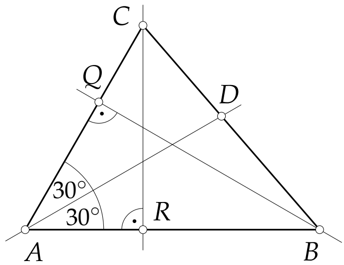

Fig. 15

By assumptions, $\angle QAD = \angle RAD = 30°$ and $\angle AQB = \angle ARC = 90°$ (Fig. 15). Hence AD and BQ intersect at angle $60°$, as well as AD and CR. Thus two angles of the triangle whose vertices are the three intersection points of lines AD, BQ and CR have size $60°$. Such a triangle is equilateral.

F2. *(Grade 9.)* There are 8 white pawns on the squares at one edge of an 8×8 chessboard and 8 black pawns on the squares at the opposite edge. On each move, a player shifts one of his pawns by one or more squares forward (toward the opponent's piece) or backward, but moving a pawn to a square containing the opponent's pawn or over such a square is prohibited. Moves are performed alternately, with white starting. The player who cannot make a move loses. Which player has a winning strategy?

Answer: black player.

Solution. Black can use the following strategy. If white moves his kth pawn counting from his left, by n squares forward, black moves his kth pawn counting from his left, by n squares forward. If white moves his pawn by n squares backward, black moves his pawn on the same file by n squares forward. After each move made by black, the distance between two pawns on each file is the same as that on the file symmetric w.r.t. the midpoint of the board. Thus whenever a white pawn has moved forward, the black can make the move determined by the strategy described. As the black pawns move forward only, white will be paralyzed sooner or later.

F3. *(Grade 9.)* Determine the largest possible number of primes among 100 consecutive natural numbers.

Answer: 26.

Solution. There are 25 primes among the numbers from 1 to 100. The number of primes in the next a few number intervals with length 100 are shown in the table:

Interval	change w.r.t. previous		Number
	out	in	of primes
$1, \ldots, 100$			25
$2, \ldots, 101$	1	101	26
$3, \ldots, 102$	2	102	25
$4, \ldots, 103$	3	103	25
$5, \ldots, 104$	4	104	25
$6, \ldots, 105$	5	105	24
$7, \ldots, 106$	6	106	24

The largest number of primes in these number intervals is 26.

We show now that there cannot be more primes among 100 consecutive natural numbers. Consider arbitrary 100 consecutive natural numbers, the least of which is larger than 7. None of the numbers under consideration that is divisible by one of numbers 2, 3, 5 and 7 is a prime. We show in the rest that there are at least 74 such numbers. As every second number is divisible by 2, there are 50 even numbers. As every third number is divisible by 3, there are at least 33 such numbers. Every second among them has been counted as an even number, thus there are at least 16 new numbers divisible by 3. As every fifth number is divisible by 5, there are at least 20 such numbers. Every second among them is even, thus the number of odd numbers divisible by 5 is 10. Among them in turn, every third is divisible by 3, thus there are at least 6 numbers divisible by 5 not counted yet. As every seventh number is divisible by 7, there are at least 14 such numbers. Every second among them is even, thus there are at least 7 odd numbers divisible by 7. Every third among them is divisible by 3, which eliminates at most 3 numbers, and every fifth is divisible by 5, which eliminates at most 2 numbers. Hence at least 2 numbers not counted before are divisible by 7. Altogether, we have at least $50 + 16 + 6 + 2 = 74$ composite numbers.

F4. *(Grade 10.)* Find all four-digit numbers which are exactly by 2016 larger than the four-digit number obtained by moving the first digit to the end.

Answer: 3109, 4220, 5331, 6442, 7553, 8664, and 9775.

Solution. Let the first digit of the number be a and the number formed by the remaining digits be k. By the conditions, $1000a + k = 10k + a + 2016$, whence $111a - k = 224$. Hence $a \geqslant 3$, implying the solutions $a = 3$, $k = 109$; $a = 4$, $k = 220$; $a = 5$, $k = 331$; $a = 6$, $k = 442$; $a = 7$, $k = 553$; $a = 8$, $k = 664$; $a = 9$, $k = 775$. The corresponding four-digit numbers satisfying the conditions of the problem are 3109, 4220, 5331, 6442, 7553, 8664, and 9775.

F5. *(Grade 10.)* Find all pairs (a, b) of integers satisfying the equality

$$3(a^2 + b^2) - 7(a + b) = -4.$$

Answer: $(0, 1)$, $(1, 0)$, $(2, 2)$.

Solution 1. The given equation is equivalent to $(6a - 7)^2 + (6b - 7)^2 = 50$. Number 50 can be represented as the sum of two squares as $25 + 25$ or $1 + 49$. Hence both $6a - 7$ and $6b - 7$ must be among the numbers $7, 5, 1, -1, -5$ and -7. As both a and b are integers, only 5, -1 and -7 fit. We obtain the following cases: $6a - 7 = 5$, $6b - 7 = 5$; $6a - 7 = -1$, $6b - 7 = -7$; $6a - 7 = -7$, $6b - 7 = -1$. The corresponding solutions are $a = 2$, $b = 2$; $a = 1$, $b = 0$; and $a = 0$, $b = 1$.

Solution 2. Consider the equation as a quadratic equation w.r.t. b. In order to have solutions, its discriminant must be non-negative, i.e., $49 - 12(3a^2 - 7a + 4) \geqslant 0$. This condition is equivalent to the quadratic inequality $36a^2 - 84a - 1 \leqslant 0$, whose solutions are $(7 - \sqrt{50})\,/\,6 \leqslant a \leqslant (7 + \sqrt{50})\,/\,6$. As a is an integer, its only suitable values are 0, 1, and 2, the corresponding values of b are 1, 0, and 2.

F6. *(Grade 10.)* Call a convex polygon on a plane *correct* if for its every side there exists a unique vertex of the polygon that lies farther from that side than any other vertex of the polygon. Call the perpendicular drawn from the vertex farthest from side XY to side XY an *altitude* of the correct polygon. Find all natural numbers n for which there exists a correct n-gon whose all n altitudes meet in one point.

Answer: all natural numbers $n \geqslant 3$.

Solution. Let one of the vertices be $O(0, 0)$ and let the other vertices $A_1, \ldots, A_{n-1}$ lie on a circle with radius 1 and centre O in such a way that $A_1(1, 0)$, $A_{n-1}(0, 1)$ and $A_2, \ldots, A_{n-2}$ are all on the shorter arc A_1A_{n-1} (Fig. 16 depicts the case $n = 6$). The vertex farthest from line OA_1 is A_{n-1}, the vertex farthest from line OA_{n-1} is A_1. The vertex farthest from any other line determined by a side of the polygon is O because the line passing through O parallel to such a side lies in II and IV quarters while the other vertices of the polygon lie above it in I quarter. Thus the polygon is correct. The altitudes drawn to sides OA_1 and OA_{n-1} are OA_{n-1} and OA_1, respectively, they meet at point O. As O is the vertex farthest from any other side, all other altitudes meet in O, too.

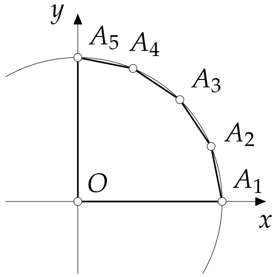

Fig. 16

Remark. If n is odd, then a regular n-gon is correct with its all altitudes meeting in the circumcentre.

F7. *(Grade 10.)* Manni and Miku play the following game with rooks on an 8×8 chessboard. At the beginning of the game, Miku places 8 rooks to the squares of the board according to his will. Then both players make moves alternately, Manni starts. On any move, each player shifts exactly one rook along a rank or file (i.e. row or column) by one or more squares in one direction. If a rook moves to a square that contains another rook, the latter is removed from the board; it is not allowed to move a rook over another. A player who is the first to remove a rook from the board wins; however, neither moving nor removing a rook that was moved by the opponent on his last move is allowed. Does either of the players have a winning strategy and if yes then which of them?

Answer: neither does.

Solution. We show at first that Miku can play in such a way that Manni can never remove a rook. Let there be one rook in each rank and file in the initial configuration. Suppose that Manni moves a rook from square (x, y) to square (x, z). As a consequence, each file contains one rook but there are no rooks in rank y and two rooks in rank z; let a rook be on square (t, z), $t \neq x$. Let Miku move the rook from square (t, z) to square (t, y). After that, there is one rook in each rank and file again. In the case of Manni's move in the perpendicular direction, Miku's reply would be analogous. In such a way, Miku can reply to all Manni's following moves.

Now we show that also Manni can play so that Miku never wins. If there are two or more rooks on one rank or file in the initial configuration then Manni can remove one of them and win immediately. Therefore assume that initially there is one rook on each rank and file. Let there be rooks on squares $(1, u)$ and $(2, v)$. Manni can move the rook from $(2, v)$ to square $(2, u)$. As Miku is not allowed to remove this rook but Manni threatens to remove the rook on $(1, u)$ on the next move, Miku must move this rook to some square $(1, w)$. If $w \neq v$, Manni can win by moving the rook from square $(2, u)$ to square $(2, w)$ and removing either of the other rooks on rank w on the next move. If $w = v$, then there is one rook in each rank and file again and Manni can continue with the same strategy.

Remark. In Manni's strategy, it is important to move one rook immediately besides another rook (otherwise Miku would win by moving a third rook between them) and the attacked rook lies by the edge of the board (otherwise Miku would win by moving the attacked rook farther away so that the rook moved by Manni would itself remain under attack).

F8. *(Grade 11.)* Find the largest natural number n for which $3^{2016} - 1$ is divisible by 2^n.

Answer. 7.

Solution. We have $3^{2016} - 1 = (3^{63} - 1)(3^{63} + 1)(3^{126} + 1)(3^{252} + 1) \cdot$ $\cdot (3^{504} + 1)(3^{1008} + 1)$. Numbers 3^{126}, 3^{252}, 3^{504} and 3^{1008} are squares of odd numbers, hence congruent to 1 modulo 8. Thus $3^{126} + 1$, $3^{252} + 1$, $3^{504} + 1$ and $3^{1008} + 1$ are congruent to 2 modulo 8. Consequently, these four factors are divisible by 2 but not by 4. As $3^{62} \equiv 1 \pmod 8$, we have $3^{63} \equiv 3$

(mod 8). Hence $3^{63}-1$ and $3^{63}+1$ are congruent to 2 and 4 modulo 8, respectively. The former thus is divisible by 2 but not by 4 and the latter is divisible by 4 but not by 8. Putting it all together, the exponent of 2 in the product is $1+2+1+1+1+1=7$.

F9. *(Grade 11.)* Three workers must do a work completely. At first, one of them works as long as the other two would work together to complete one half of the work. Then another worker works as long as the other two would work together to complete one half of the work. Finally the third worker works as long as the other two would work together to complete one half of the work. With this, the whole work becomes completed. How many times faster would the work become completed if all the workers worked together?

Answer: 2.5.

Solution 1. Let the contributions of the first, second and third worker per a time unit be x, y, z, measured as percentages of the whole work. One half of the work would be done by the second and third worker together within $\frac{1}{2(y+z)}$ time units, by the third and first worker together within $\frac{1}{2(z+x)}$ time units and, by the first and second worker together, within $\frac{1}{2(x+y)}$ time units. By working in the way described in the problem text, they spend $\frac{1}{2(y+z)}+\frac{1}{2(z+x)}+\frac{1}{2(x+y)}$ time units, but if they worked all together, they would spend $\frac{1}{x+y+z}$ time units. We are asked the ratio of these numbers. Since by assumption $\frac{x}{2(y+z)}+\frac{y}{2(x+z)}+\frac{z}{2(x+y)}=1$, we get

$$\frac{\frac{1}{2(y+z)}+\frac{1}{2(z+x)}+\frac{1}{2(x+y)}}{\frac{1}{x+y+z}}=\frac{x+y+z}{2(y+z)}+\frac{x+y+z}{2(z+x)}+\frac{x+y+z}{2(x+y)}=$$
$$=\frac{1}{2}+\frac{x}{2(y+z)}+\frac{1}{2}+\frac{y}{2(z+x)}+\frac{1}{2}+\frac{z}{2(x+y)}=\frac{3}{2}+1=2.5.$$

Solution 2. Suppose that the first worker worked a time units, the second worker worked b time units and the third worker worked c time units. Altogether, they spend $a+b+c$ time units to perform the whole work. If the second worker worked a time units and the third also a time units, they would perform half of the work. Similarly, if the third worker worked b time units and the first also b time units then they would complete half of the work, and if the first worker worked c time units and the second also c time units, they would complete half of the work. Consequently, if the first worker worked $b+c$ time units, the second worker worked $a+c$ time units and the third worked worked $a+b$ time units, they would complete one and a half such works. Including the work really done, it turns out that if all workers worked $a+b+c$ time units then they would perform 2.5 such works. Thus the three workers together would act 2.5 times faster than in the situation of the problem.

F10. *(Grade 11.)* In the space, rays k, l, m originate from a point O. Let the size of the angle between k and l be α, the size of the angle between l and m be β and the size of the angle between m and k be γ, whereby $\alpha + \beta \leqslant 180^\circ$. Ray r bisects the angle between k and l; ray s bisects the angle between l and m. Is it sure that the size of the angle between r and s is $\frac{\gamma}{2}$?

Answer: no.

Solution. Let O be a vertex of a cube and let rays k, l and m be directed along the edges of the cube. Then $\alpha = \beta = \gamma = 90^\circ$. Rays r and s are directed from vertex O along the angle bisectors of the faces that are diagonals at the same time. The other endpoints of the diagonals, R and S, are endpoints of the diagonal of one and the same face of the cube. The three diagonals form an equilateral triangle ORS, whence $\angle ROS = 60^\circ$. Thus the size of the angle between rays r and s is not equal to half of the size of the angle between rays k and m.

F11. *(Grade 11.)* Each point at the sides of an equilateral triangle is colored either red or blue. Is it sure that there exists a right triangle whose all vertices are of the same colour?

Answer: yes.

Solution 1. Let the equilateral triangle be XYZ. We show that there exists a point on some side that has the same colour as its projection to another side. For that, take points P, Q and R on sides XY, YZ and ZX, respectively, in such a way that $XP : XY = YQ : YZ = ZR : ZX = 1 : 3$ (Fig. 17). Then $PQ \perp YZ$ because, denoting the midpoint of YZ by T, $TQ : TY = (\frac{1}{2} - \frac{1}{3}) : \frac{1}{2} = 1 : 3 = XP : XY$, implying $PQ \parallel XT$. Hence Q is the projection of P to YZ. Analogously, R is the projection of Q to ZX and P is the projection of R to XY. As at least two points among P, Q and R must have the same colour, a point and its projection have the same colour.

W.l.o.g., let P and its projection Q both be red. Let M be the projection of Q to XY and N the projection of M to YZ. If M is red, then PQM is a right triangle with all vertices red. If N is red, then PQN is a right triangle with all vertices red. If Y is red, then PQY is a right triangle with all vertices red. Otherwise, MNY is a right triangle with all vertices blue.

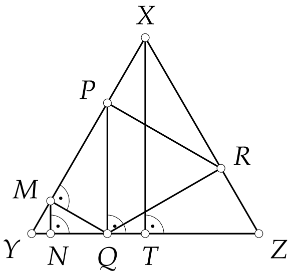

Fig. 17

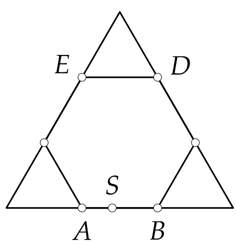

Fig. 18

Solution 2. Consider a regular hexagon whose vertices lie on the sides of the triangle (Fig. 18). Suppose that two opposite vertices of the hexagon are of the same colour. If there is one more vertex of the same colour among the other four vertices, there is a right triangle with all vertices being of that colour. Otherwise, any three vertices among the four remaining ones form a right triangle with all vertices being of the other colour. If any two opposite vertices are of different colours, then there exist two neighbouring vertices of different colours. The corresponding opposite vertices are of different colours, too. One pair of these differently coloured vertices lies on a side of the initial triangle; let these be A and B (red and blue, resp.) and their opposite vertices D and E (blue and red, resp.). Angles ABD and BAE are right. Let S be any point on AB that is not a vertex of the hexagon. If S is red, then SAE is a right triangle with all vertices red; if S is blue, then SBD is a right triangle with all vertices blue.

F12. *(Grade 11.)* In the beginning, there are two positive integers on a blackboard. On each step, one chooses numbers a and b such that $a \leqslant b$ from the numbers on the blackboard in all possible ways (equality means that one may take the same number twice), finds all corresponding sums $a + b + \gcd(a, b)$ and replaces all the numbers on the blackboard instantly with these sums. Prove that at some step at least one number will occur more than once on the blackboard.

Solution 1. If the numbers chosen from the blackboard are x and y, then the number $x + y + \gcd(x, y)$ will be on the blackboard on the next step. If x is chosen together with itself, the number $x + x + \gcd(x, x) = 3x$ will be on the blackboard on the next step. We show that there will be two equal numbers on the blackboard after the second step at latest. Assume that the initial numbers are n and m and the numbers $n + m + \gcd(n, m)$, $3n$ and $3m$ appearing on the first step are all distinct. Choosing number $n + m + \gcd(n, m)$ together with itself, we obtain $3(n + m + \gcd(n, m)) = 3n + 3m + 3\gcd(n, m)$. Choosing $3n$ and $3m$, we obtain $3n + 3m + \gcd(3n, 3m)$. As $\gcd(3n, 3m) = 3\gcd(n, m)$, the same number will appear twice after the second step.

Solution 2. Let the largest number on the blackboard after step s and the number of numbers on the blackboard after step s be $m(s)$ and $n(s)$, respectively. Obviously $m(s + 1) = m(s) + m(s) + \gcd(m(s), m(s)) = 3m(s)$, as $x \leqslant m(s)$ and $y \leqslant m(s)$ imply $\gcd(x, y) \leqslant m(s)$. Thus $m(s) = 3^s m(0)$ for all s. Suppose that there will never be equal numbers on the blackboard. Then $n(s + 1) = \frac{n(s)\cdot(n(s)+1)}{2} > \frac{n(s)^2}{2}$ for every s. Since $n(0) = 2$, we get $n(1) = 3$ and $n(2) = 6$. An easy induction shows that $n(s) > 2 \cdot 3^{2^{s-2}}$ for every $s \geqslant 2$. Thus $\frac{n(s)}{m(s)} > \frac{2}{m(0)} \cdot 3^{2^{s-2}-s}$. If $s \to \infty$, the number $3^{2^{s-2}-s}$ becomes larger than $\frac{m(0)}{2}$, so $n(s) > m(s)$. On the other hand, $n(s) \leqslant m(s)$ since the numbers are positive, a contradiction.

F13. *(Grade 12.)* Denote the number of all positive divisors of a positive integer n by $\delta(n)$ and the sum of all positive divisors of a positive integer n by $\sigma(n)$. Prove that $\sigma(n) > \frac{\delta(n)^2}{2}$.

Solution 1. Let $a_1, a_2, \ldots, a_{\delta(n)}$ be the positive divisors of n in the increasing order. We obtain $\sigma(n) = a_1 + a_2 + \ldots + a_{\delta(n)} \geqslant 1 + 2 + \ldots + \delta(n) = \frac{\delta(n)\cdot(\delta(n)+1)}{2} > \frac{\delta(n)^2}{2}$.

Solution 2. For every positive divisor d of a positive integer n, $d + \frac{n}{d} \geqslant 2\sqrt{d \cdot \frac{n}{d}} = 2\sqrt{n}$. Thereby if $d \neq \sqrt{n}$, the inequality is strict. If n is not a perfect square, then by adding all these inequalities for divisors $d < \sqrt{n}$ leads to $\sigma(n) > \frac{\delta(n)}{2} \cdot 2\sqrt{n} = \delta(n)\sqrt{n}$, since every divisor of n occurs exactly once in a pair $(d, \frac{n}{d})$. If n is a perfect square, then analogously $\sigma(n) \geqslant \frac{\delta(n)-1}{2} \cdot 2\sqrt{n} + \sqrt{n} = \delta(n)\sqrt{n}$. On the other hand, if n is not a perfect square then the number of pairs $(d, \frac{n}{d})$ where $d < \sqrt{n}$, is less than $\sqrt{n}$, whence $\sqrt{n} > \frac{\delta(n)}{2}$. If n is a perfect square then the number of such pairs is at most $\sqrt{n} - 1$, which gives $\sqrt{n} - 1 \geqslant \frac{\delta(n)-1}{2}$ implying $\sqrt{n} > \frac{\delta(n)}{2}$ again. Hence, $\sigma(n) \geqslant \delta(n)\sqrt{n} > \delta(n) \cdot \frac{\delta(n)}{2} = \frac{\delta(n)^2}{2}$.

Remark. Denote by $\sigma_k(n)$ the sum of kth powers of the divisors of positive integer n. Both solutions directly generalize to show the inequality $\sigma_k(n) > \frac{\delta(n)^{k+1}}{2^k}$ for all $k \geqslant 1$ (in the first solution, one has to apply the inequality between the kth power mean and the arithmetic mean).

F14. *(Grade 12.)* Some cities of a country are connected with roads. We say that a city A *belongs to a cycle of length* n if one can travel from A through exactly $n - 1$ other cities and return back to A. It is known that each city of the country belongs to a cycle of length 4 and also to a cycle of length 5. Is it sure that a) at least one city belongs to a cycle of length 3? b) each city belongs to a cycle of length 3?

Answer: a) no; b) no.

Solution. Let there be 10 cities and let cities be connected as in the Fig. 19. Then every city belongs to a cycle of length 5 and also to a cycle of length 4. On the other hand, no cycles of length 3 exist.

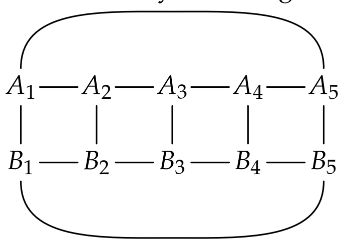

Fig. 19

F15. *(Grade 12.)* There are 125 distinct positive integers in a row in such a way that among every three consecutive numbers the second one is larger than the arithmetic mean of the first and the third one. Find the largest number in the row, given that it is as small as possible under such conditions.

Answer: 2016.

Solution. Let the numbers in a row be $a_1, a_2, \ldots, a_{125}$. By conditions, we have $a_{i+1} > \frac{a_i+a_{i+2}}{2}$ for every $i = 1, 2, \ldots, 123$, which is equivalent to $a_{i+1} - a_i > a_{i+2} - a_{i+1}$. Denoting $d_i = a_{i+1} - a_i$, we have $d_1 > d_2 > \ldots > d_{124}$. Let a_m be the largest among $a_1, \ldots, a_{125}$; then $d_1, d_2, \ldots, d_{m-1}$ are positive and $d_m, d_{m+1}, \ldots, d_{124}$ are negative. If both 1 and -1 occurred among the differences $d_1, d_2, \ldots, d_{124}$, then they should be consecutive, i.e., $d_i = 1$ and $d_{i+1} = -1$ for some i, whence $a_{i+1} = a_{i-1}$. Contradiction to the assumption that the given numbers are distinct shows that either 1 or -1 is missing among the differences. W.l.o.g., assume that 1 is missing (if -1 is missing, we can reverse the numeration of the given numbers). Then $a_m = a_1 + (d_1 + d_2 + \ldots + d_{m-1}) \geqslant 1 + (m + (m-1) + \ldots + 2) = 1 + 2 + \ldots + m$ and $a_m = a_{125} - (d_m + d_{m+1} + \ldots + d_{124}) \geqslant 1 + (1 + 2 + \ldots + (125 - m))$. Among numbers m and $125 - m$, one is at least 63, whence the previously established inequalities imply $a_m \geqslant 1 + 2 + \ldots + 63$.

On the other hand, taking $a_1 = 1$ and $d_i = 64 - i$, $1 \leqslant i \leqslant 62$, and $d_i = 62 - i$, $63 \leqslant i \leqslant 124$, the largest number a_{63} equals $1 + 2 + \ldots + 63$, while all numbers a_i are positive, as $1 = a_1 < a_2 < \ldots < a_{63}$ and $a_{63} > a_{64} > \ldots > a_{125} = (1 + 2 + \ldots + 63) - (1 + 2 + \ldots + 62) = 63$. All these numbers are distinct, since for every $i = 64, 65, \ldots, 125$, $a_i = (1 + 2 + \ldots + 63) - (1 + 2 + \ldots + (i - 63)) = (i - 62) + (i - 61) + \ldots + 63 = a_{127-i} - 1$, whence a_i lies strictly between a_{126-i} and a_{127-i}. Consequently, the largest number written in the row is $1 + 2 + \ldots + 63$, i.e., 2016.

IMO Team Selection Contest I

S1. There are k heaps on the table, each containing a different positive number of stones. Jüri and Mari make moves alternatingly; Jüri starts. On each move, the player making the move has to pick a heap and remove one or more stones in it from the table; in addition, the player is allowed to distribute any number of remaining stones from that heap in any way between other non-empty heaps. The player to remove the last stone from the table wins. For which positive integers k does Jüri have a winning strategy for any initial state that satisfies the conditions?

Answer: for any k.

Solution 1. Call a position *balanced*, if the non-empty heaps can be divided into pairs, with an equal number of stones in both heaps of each pair. We show that in a balanced position, the player who moves second has a winning strategy. If there are no heaps left, the second player has already won. In the general case, suppose that the first player picks the heap H, takes n stones from it off the table, and moves a_1 stones to the first heap, a_2 stones to the second, etc. If any stones are left in H after that, the second player can then pick the heap H' that is paired with H, remove n stones from it, and move a_1 stones to the heap paired with the first heap, a_2 stones to the heap paired with the second heap, etc. On the other hand, if the first player empties the heap H, the second player does the same as in the first case, with the following exception: if the first player moved any stones to H', the second player takes this many additional stones off the table instead of moving them back to H, ensuring that the heap H' also becomes empty. In both cases, the position after the second player's move is balanced again. Since the number of stones on the table decreases with each move and the second player can always make a move, the second player eventually wins.

Finally, we show that Jüri can move into a balanced position with his first move, giving him a winning strategy. Number the heaps $1, 2, \ldots, k$ in decreasing order of size, and let the sizes of the heaps be $a_1 > a_2 > \cdots > a_k$. Jüri should pick heap 1 and move $a_2 - a_3$ stones to heap 3, $a_4 - a_5$ stones to heap 5, etc. If k is odd, Jüri should take all stones remaining in heap 1 off the table; if k is even, he should leave a_k stones in heap 1. It remains to verify that this move is possible. If $k = 1$, the game will be over after Jüri's move. For $k > 1$, $(a_2 - a_3) + (a_4 - a_5) + \cdots + (a_{k-1} - a_k) < (a_1 - a_2) + (a_2 - a_3) + (a_3 - a_4) + (a_4 - a_5) + \cdots + (a_{k-1} - a_k) = a_1 - a_k$, which shows that after redistributing stones to odd-numbered heaps, there are still more than a_k stones left in heap 1, so the move is possible for both odd and even k. Furthermore, the strictness of the inequality ensures that some stones will be left to take off the table as required.

Solution 2. Let a *balanced position* be defined as in solution 1. First we show that from any non-balanced position, there exists a move to a balanced position. This can be done by ignoring all existing pairs of equal-sized heaps and proceeding as in solution 1 (distributing stones from the biggest remaining heap). Second, we show that there is no move from a balanced position to another balanced position. Suppose for the sake of contradiction that such a move exists. For any positive integer i, let u_i and v_i be the number of heaps with at least i stones before and after that move. Since both positions are balanced, all numbers u_i and v_i are even. Since the number of stones decreases in only one heap, $v_i \geq u_i - 1$; parity now implies $u_i \leq v_i$ for all i.

Before the move, let's number all stones in each heap with consecutive integers starting from 1. Then for each i, there are u_i stones numbered i on the table, and the total number of stones before the move is therefore $u_1 + u_2 + \ldots$. Analogously, the number of stones after the move is $v_1 + v_2 + \ldots$. The inequality $u_i \leq v_i$ now implies that the number of stones on the table does not decrease with the move, a contradiction with the rules of the game. Since the initial position is not balanced and the number of stones decreases with each move, always moving into a balanced position is a winning strategy for Jüri.

S2. Let p be a prime number. Find all triples (a, b, c) of integers (not necessarily positive) such that $a^b b^c c^a = p$.

Answer: if $p > 2$ then $(p, 1, 1)$ and $(-p, 1, -1)$ together with cyclic permutations; if $p = 2$ then $(2, 1, 1)$, $(2, 1, -1)$, $(2, 2, -1)$ and $(-2, 2, -1)$ together with cyclic permutations.

Solution. Suppose a, b, c satisfy the equation. As p is positive, this implies that $|a|^b|b|^c|c|^a = p$. Clearly none of a, b, c can be zero.

Observe that $\gcd(a, b, c) = 1$. Indeed, if $d \mid a$, $d \mid b$, $d \mid c$, then the exponent of p in the canonical representation of each of the positive rational numbers $|a|^b$, $|b|^c$, $|c|^a$ is divisible by d. Hence the exponent of p in the canonical representation of the product $|a|^b|b|^c|c|^a$ is divisible by d. As this product equals to p, we get $|d| = 1$.

Consider now arbitrary prime number q different from p. Let α, β, γ be the exponents of q in the canonical representation of the positive integers $|a|$, $|b|$, $|c|$, respectively. Then $\alpha b + \beta c + \gamma a = 0$ whereby not all exponents α, β, γ are positive because $\gcd(a, b, c) = 1$. Consequently, if some of α, β, γ is positive, then there must be exactly two positive exponents among α, β, γ. W.l.o.g., assume $\alpha > 0$, $\beta > 0$, $\gamma = 0$. Then $\alpha b + \beta c = 0$, implying $\alpha|b| = \beta|c|$. Hence $|b|$ divides $\beta|c|$. As q^β divides $|b|$ while q^β is relatively prime to $|c|$, this implies $q^\beta \mid \beta$ and $q^\beta \le \beta$ which is impossible. This means that actually $\alpha = \beta = \gamma = 0$ and $|a|$, $|b|$, $|c|$ are all powers of p.

Hence the equation rewrites to $p^{\alpha b}p^{\beta c}p^{\gamma a} = p$ where α, β, γ are now the exponents of p in the canonical representation of $|a|$, $|b|$, $|c|$, respectively. This is equivalent to $\alpha b + \beta c + \gamma a = 1$. By $\gcd(a, b, c) = 1$, one of α, β, γ must be zero, and clearly, one of the summands αb, βa, γa must be positive. W.l.o.g., let $\alpha b > 0$, i.e., $\alpha > 0$ and $b > 0$. Now there are three cases.

If $\beta = 0$ and $\gamma = 0$ then $b = 1$ and $|c| = 1$. Furthermore, $\alpha b + \beta c + \gamma a = 1$ reduces to $\alpha = 1$, whence $|a| = p$. If $p > 2$ then the exponents of a and c in the original equation, b and a, are both odd, whence a and c must have the same sign to make the product $a^b b^c c^a$ positive. Both triples $(p, 1, 1)$ and $(-p, 1, -1)$ satisfy the original equation. If $p = 2$ then c^a is positive anyway, hence a must be positive. Both triples $(2, 1, 1)$ and $(2, 1, -1)$ satisfy the original equation.

If $\beta = 0$ and $\gamma > 0$ then $b = 1$. Furthermore, $\alpha b + \beta c + \gamma a = 1$ reduces to $\alpha + \gamma a = 1$, whence $a < 0$. We obtain $p^\alpha \le \gamma p^\alpha = \gamma|a| = \alpha - 1 < \alpha$ which is impossible.

If $\beta > 0$ and $\gamma = 0$ then $|c| = 1$. Furthermore, $\alpha b + \beta c + \gamma a = 1$ reduces to $\alpha b + \beta c = 1$, which gives $c = -1$ and $\alpha p^\beta = 1 + \beta$ as the only possibility. If $p > 2$ then this leads to contradiction similar to the previous case. If $p = 2$ then $\alpha = \beta = 1$ is the only solution. This leads to triples $(2, 2, -1)$ and $(-2, 2, -1)$ which both satisfy the original equation.

S3. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the equality $f(2^x + 2y) = 2^y f(f(x)) f(y)$ for every $x, y \in \mathbb{R}$.

Answer: $f(x) = 0$ and $f(x) = 2^x$.

Solution. Substituting $y = -2^{x-1}$ into the original equation gives

$$f(0) = \frac{1}{2^{2^{x-1}}} f(f(x)) f(-2^{x-1}). \tag{1}$$

So, if $f(-2^x) = 0$ for at least one x then also $f(0) = 0$. Then taking $x = 0$ and arbitrary y in the original identity gives $f(1 + 2y) = 0$, i.e., $f \equiv 0$.

Assume in the rest that $f(-2^x) \neq 0$ for every x. Substituting $y = -2^x$ into the original equation gives $f(-2^x) = \frac{1}{2^{2^x}} f(f(x)) f(-2^x)$. Hence, for every x,

$$f(f(x)) = 2^{2^x} \tag{2}$$

Substituting (2) into the original equation and taking $y = 0$, we obtain $f(2^x) = f(f(x))f(0) = 2^{2^x} f(0)$ for all x, which implies

$$f(x) = 2^x f(0) \tag{3}$$

for all positive x. On the other hand, applying (2) to (1) gives

$$f(-2^{x-1}) = \frac{2^{2^{x-1}}}{2^{2^x}} \cdot f(0) = 2^{-2^{x-1}} f(0)$$

for all x, which implies (3) also for all negative x.

We have shown above that $f(0) = 0$ implies $f(x) = 0$ for all x. Hence we may assume that $f(0)$ is either positive or negative. By taking $x = 0$ in (2) and applying (3), we obtain $2 = 2^{2^0} = f(f(0)) = 2^{f(0)} \cdot f(0)$. Both $f(0) < 1$ and $f(0) > 1$ would lead to contradiction, hence $f(0) = 1$ and the only non-zero solution is thus $f(x) = 2^x$.

S4. Prove that for any positive integer n, $2 \cdot \sqrt{3} \cdot \sqrt[3]{4} \cdot \ldots \cdot \sqrt[n-1]{n} > n$.

Solution 1. For $2 \leq k \leq n$, the GM-HM inequality for the numbers $k, \ldots, k, 1$, with k repeated $k - 2$ times, gives

$$k - 1 = \frac{(k-1)^2}{k-1} = \frac{k(k-2)+1}{k-1} \geqslant \sqrt[k-1]{k^{k-2}} = \sqrt[k-1]{\frac{k^{k-1}}{k}} = \frac{k}{\sqrt[k-1]{k}}.$$

Hence $\sqrt[k-1]{k} \geqslant \frac{k}{k-1}$ for every $k = 2, 3, \ldots, n$, with equality only for $k = 2$. Therefore

$$2 \cdot \sqrt{3} \cdot \sqrt[3]{4} \cdot \ldots \cdot \sqrt[n-1]{n} > \frac{2}{1} \cdot \frac{3}{2} \cdot \frac{4}{3} \cdot \ldots \cdot \frac{n}{n-1} = n.$$

Solution 2. For $2 \leq k \leq n$, the HM–GM inequality for the numbers $1, \ldots, 1, k$ (with 1 repeated $k - 2$ times) gives

$$\sqrt[k-1]{k} \geqslant \frac{k-1}{k-2+\frac{1}{k}} = \frac{(k-1)k}{(k-2)k+1} = \frac{(k-1)k}{(k-1)^2} = \frac{k}{k-1},$$

with equality only for $k = 2$. We continue as in solution 1.

Solution 3. From the binomial theorem,

$$\left(\frac{k}{k-1}\right)^{k-1} = \left(1 + \frac{1}{k-1}\right)^{k-1} = \frac{\binom{k-1}{0}}{(k-1)^0} + \frac{\binom{k-1}{1}}{(k-1)^1} + \ldots + \frac{\binom{k-1}{k-1}}{(k-1)^{k-1}}.$$

There are k summands, of which $\frac{\binom{k-1}{0}}{(k-1)^0} = \frac{\binom{k-1}{1}}{(k-1)^1} = 1$, and for $1 < i \leqslant k-1$

$$\frac{\binom{k-1}{i}}{(k-1)^i} = \frac{(k-1)\cdot \ldots \cdot (k-i)}{i! \cdot (k-1)^i} < \frac{(k-1)\cdot \ldots \cdot (k-i)}{(k-1)^i} < 1.$$

Therefore $\left(\frac{k}{k-1}\right)^{k-1} \leqslant \underbrace{1+1+\ldots+1}_{k \text{ times}} = k$, whence $\sqrt[k-1]{k} \geqslant \frac{k}{k-1}$, with equality only for $k=2$. We continue as in solution 1.

Solution 4. We show that $k^k > (k+1)^{k-1}$ for $k \geqslant 2$. The case $k=2$ is obvious. Suppose now that $k^k > (k+1)^{k-1}$ holds for some k, and let's prove $(k+1)^{k+1} > (k+2)^k$. Note that $(k+1)^{k+1} \cdot (k+1)^{k-1} = (k+1)^{2k} = \left(k^2+2k+1\right)^k > \left(k^2+2k\right)^k = (k(k+2))^k = k^k \cdot (k+2)^k$, giving $\frac{(k+1)^{k+1}}{k^k} > \frac{(k+2)^k}{(k+1)^{k-1}}$. This combined with the induction assumption gives

$$(k+1)^{k+1} = \frac{(k+1)^{k+1}}{k^k} \cdot k^k > \frac{(k+2)^k}{(k+1)^{k-1}} \cdot (k+1)^{k-1} = (k+2)^k.$$

We have proven $k^k > (k+1)^{k-1}$, which is equivalent to $\sqrt[k-1]{k} > \sqrt[k]{k+1}$. Therefore $\sqrt[k-1]{k} > \sqrt[n-1]{n}$ for $k = 2, 3, \ldots, n-1$, and

$$2 \cdot \sqrt{3} \cdot \sqrt[3]{4} \cdot \ldots \cdot \sqrt[n-1]{n} > \left(\sqrt[n-1]{n}\right)^{n-1} = n.$$

Solution 5. We give another proof for the inequality $k^k > (k+1)^{k-1}$. It is equivalent to $k \cdot \left(\frac{k}{k+1}\right)^{k-1} > 1$, or $k \cdot \underbrace{\frac{k}{k+1} \cdot \ldots \cdot \frac{k}{k+1}}_{k-1 \text{ times}} > 1$. Note that for $x < k+1$, $x - x \cdot \frac{k}{k+1} = x \cdot \left(1 - \frac{k}{k+1}\right) = x \cdot \frac{1}{k+1} < 1$. Therefore, multiplication with each factor $\frac{k}{k+1}$ decreases the product by less than 1; cumulatively the product becomes smaller by less than $k-1$. Therefore $k \cdot \left(\frac{k}{k+1}\right)^{k-1} > k - (k-1) = 1$.

S5. Let O be the circumcentre of the acute triangle ABC. Let c_1 and c_2 be the circumcircles of triangles ABO and ACO. Let P and Q be points on c_1 and c_2 respectively, such that OP is a diameter of c_1 and OQ is a diameter of c_2. Let T be the intesection of the tangent to c_1 at P and the tangent to c_2 at Q. Let D be the second intersection of the line AC and the circle c_1. Prove that the points D, O and T are collinear.

Solution. Since $\angle OAP = \angle OAQ = 90^\circ$, the points P, A and Q are collinear. Since $\angle OPT = \angle OQT = 90^\circ$, $OPTQ$ is cyclic. Since $OA = OB$, the diameter OP of c_1 is perpendicular to the chord AB. Therefore PT and AB are parallel. Now $\angle TOQ = \angle TPQ = \angle TPA = \angle BAP = \angle BOP = 90^\circ - \angle ABO$. On the other hand, equality of inscribed angles subtending the arc AO of circle c_1 gives $\angle CDO = \angle ABO$ (figures 20 and 21 show two possible situations). Therefore $\angle DOQ = 90^\circ - \angle CDO = 90^\circ - \angle ABO$. In summary, $\angle TOQ = \angle DOQ$, whence D, O and T are collinear.

Remark. This problem originally appeared in the 1st Selection Examination of the Slovenian IMO team in 2015.

S6. A circle is divided into arcs of equal size by n points ($n \geq 1$). For any positive integer x, let $P_n(x)$ denote the number of possibilities for colouring all those points, using colours from x given colours, so that any rotation of the colouring by $i \cdot \frac{360^\circ}{n}$, where i is a positive integer less than n, gives a colouring that differs from the original in at least one point. Prove that the function $P_n(x)$ is a polynomial with respect to x.

Solution 1. Call a colouring of the n points *permissible* if it satisfies the conditions of the problem (is not invariant under any non-full rotation). Call two colourings *equivalent* if any two points are coloured the same by the first colouring iff they are coloured the same by the second. Clearly, any two equivalent colourings use the same number of colours, and if one is permissible, so is the other. Consider an equivalence class whose colourings use exactly y colours. The number of colourings in this class that use colours from a given set of x colours is $x(x-1)\dots(x-y+1)$. This holds also for $x < y$, the product then being zero. $P_n(x)$ is equal to the sum of those products over all equivalence classes of permissible colourings, and is therefore a polynomial with respect to x.

Solution 2. Consider all colourings of the n points with colours from among x given colours. The total number of the colourings is x^n. Let the *period* of a colouring be the least positive number d such that d/n of a full rotation gives the same colouring. $P_n(x)$ is therefore the number of colourings with period n. A standard argument gives that the period of any colouring is a divisor of n.

Let's count the number of colourings with period d. Any such colouring is determined by the colouring of the first d points, and there may be no smaller period among those points; therefore, there are $P_d(x)$ such

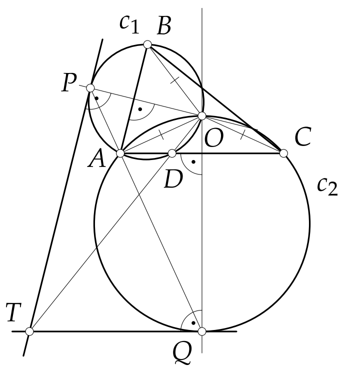

Fig. 20

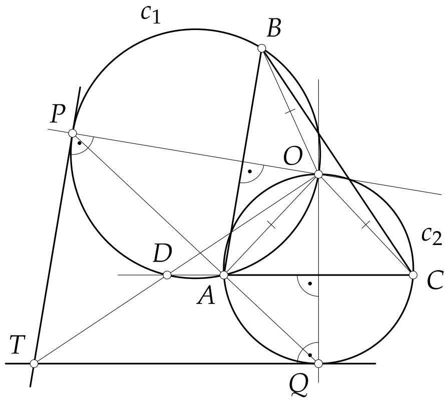

Fig. 21

colourings. Now we get $x^n = \sum_{d|n} P_d(x)$ and therefore $P_n(x) = x^n - \sum_{d|n,\, d<n} P_d(x)$. Since $P_1(x) = x$, induction by n now gives that $P_n(x)$ is a polynomial for any n.

Remark. The Möbius inversion formula gives $P_n(x) = \sum_{d|n} \mu\left(\frac{n}{d}\right) x^d$, where μ is the Möbius function. For example, $P_{15}(x) = x^{15} - x^5 - x^3 + x$ and $P_{16}(x) = x^{16} - x^8$.

The set of colourings with period n can be divided into groups of size n, each group consisting of the rotations of a single colouring. Therefore, $P_n(x)$ is divisible by n. For a prime p, $P_p(x) = x^p - x$, so we get Fermat's little theorem as a special case.

IMO Team Selection Contest II

S7. On the sides AB, BC and CA of triangle ABC, points L, M and N are chosen, respectively, such that the lines CL, AM and BN intersect at a common point O inside the triangle and the quadrilaterals $ALON$, $BMOL$ and $CNOM$ have incircles. Prove that

$$\frac{1}{AL\cdot BM}+\frac{1}{BM\cdot CN}+\frac{1}{CN\cdot AL}=\frac{1}{AN\cdot BL}+\frac{1}{BL\cdot CM}+\frac{1}{CM\cdot AN}.$$

Solution. $ALON$ is a circumscribed quadrilateral, hence $AL + ON = AN + OL$. Similarly $BM + OL = BL + OM$ and $CN + OM = CM + ON$. By adding the equations we obtain $AL + BM + CN = AN + BL + CM$. Lines CL, AM and BN intersect in one point, we get from Ceva's theorem $AL \cdot BM \cdot CN = AN \cdot BL \cdot CM$. By dividing the left hand sides of the last two equations with the right hand sides we obtain the required equation.

S8. Let x, y and z be positive real numbers such that $x+y+z = \frac{1}{x}+\frac{1}{y}+\frac{1}{z}$. Prove that $xy + yz + zx \geq 3$.

Solution 1. Using the given equation and inequality of arithmetic and geometric means, we obtain

$$xy+yz+zx = xyz\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) = \frac{xyz\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^2}{x+y+z} =$$

$$= \frac{xyz\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right)+2x+2y+2z}{x+y+z} \geqslant \frac{xyz\left(\frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx}\right)}{x+y+z}+2 = 3.$$

Solution 2. Multiply both sides of the given equation with xyz:

$$x^2yz + xy^2z + xyz^2 = yz + xz + xy. \tag{4}$$

By the AM-GM inequality, $x^2y^2 + y^2z^2 \geqslant 2\sqrt{x^2y^4z^2} = 2xy^2z$, and analogously $y^2z^2 + z^2x^2 \geqslant 2yz^2x$ and $z^2x^2 + x^2y^2 \geqslant 2zx^2y$. By adding the three equations and using equation (4) we get $2(x^2y^2+y^2z^2+z^2x^2) \geqslant 2(x^2yz+xy^2z+xyz^2)=2(xy+yz+zx)$ or

$$x^2y^2+y^2z^2+z^2x^2 \geqslant xy+yz+zx. \tag{5}$$

By multiplying the original inequality with $xy+yz+zx$ we get an equivalent inequality $x^2y^2+y^2z^2+z^2x^2+2(x^2yz+xy^2z+xyz^2) \geqslant 3(xy+yz+zx)$. This is true because of (4) and (5).

Solution 3. Given inequality is equivalent to $(\frac{1}{x}+\frac{1}{y}+\frac{1}{z})(xy+yz+zx) \geqslant 3(x+y+z)$. By opening the brackets we can see that this in turn is equivalent to $\frac{1}{x}\cdot yz+\frac{1}{y}\cdot zx+\frac{1}{z}\cdot xy \geqslant x+y+z$ and also to $\frac{1}{x}\cdot yz+\frac{1}{y}\cdot zx+\frac{1}{z}\cdot xy \geqslant \frac{1}{y}\cdot xy+\frac{1}{z}\cdot yz+\frac{1}{x}\cdot xz$. The latter inequality is a variant of rearrangement inequality.

S9. Let n be a positive integer such that there exists a positive integer that is less than $\sqrt{n}$ and does not divide n. Let $(a_1,\ldots,a_n)$ be an arbitrary permutation of 1, …, n. Let $a_{i_1}<\ldots<a_{i_k}$ be its maximal increasing subsequence and let $a_{j_1}>\ldots>a_{j_l}$ be its maximal decreasing subsequence. Prove that tuples $(a_{i_1},\ldots,a_{i_k})$ and $(a_{j_1},\ldots,a_{j_l})$ altogether contain at least one number that does not divide n.

Solution 1. The first phase of the solution consists in showing that $kl \geqslant n$. For every $i=1,\ldots,n$, let $f(i)$ denote the length of the longest increasing subsequence ending with a_i, and let $g(i)$ be the length of the longest decreasing subsequence ending with a_i. For distinct indices $i<j$, if $a_i<a_j$ then $f(i)<f(j)$, and if $a_i>a_j$ then $g(i)<g(j)$. Hence pairs of the form $(f(i),g(i))$ where $i=1,\ldots,n$ are all distinct, i.e., there are n different such pairs in total. By the conditions of the problem, the largest number of the form $f(i)$ is k and the largest number of the form $g(i)$ is l. Thus the number of pairs of the form $(f(i),g(i))$ is at most kl. Consequently, $n\leqslant kl$.

From this result, we deduce $k+l\geqslant 2\sqrt{kl}\geqslant 2\sqrt{n}$ by AM-GM. At most one number can belong to an increasing and a decreasing subsequence simultaneously. Thus subsequences $(a_{i_1},\ldots,a_{i_k})$ and $(a_{j_1},\ldots,a_{j_l})$ together contain at least $2\sqrt{n}-1$ different natural numbers in total. By assumptions, number n has at most $\lfloor\sqrt{n}\rfloor-1$ divisors that are not larger than $\sqrt{n}$, the total number $\delta(n)$ of divisors of n satisfies the inequality $\delta(n)\leqslant 2\lfloor\sqrt{n}\rfloor-2$. Consequently, subsequences $(a_{i_1},\ldots,a_{i_k})$ and $(a_{j_1},\ldots,a_{j_l})$ together contain at least one number that does not divide n.

Solution 2. The inequality $kl\geqslant n$ can be proven also in the following way. Let us partition the permutation $(a_1,\ldots,a_n)$ into decreasing subsequences using the following algorithm. The first element of each new subsequence is the first unused element in the original permutation, the next is the first following to it in the original permutation unused element smaller than it etc., until no more elements can be chosen this way. Let these subsequences be $K_1,\ldots,K_x$ in the order of forming.

For every $z = x, x-1, \ldots, 2$ and every element a_j of K_z, there exists an element a_i in K_{z-1} such that $i < j$ and $a_i < a_j$. Indeed, suppose the contrary. Then all elements a_i of K_{z-1} such that $i < j$ are greater than a_j. This means that a_j should have been chosen into K_{z-1}, a contradiction.

Hence, starting from an arbitrary element b_x of K_x, we can choose an element b_{x-1} from K_{x-1}, an element b_{x-2} from K_{x-2}, etc, until b_1 from K_1, in such a way that $b_1 < \ldots < b_{x-1} < b_x$. This is an increasing subsequence of length x of the original permutation. As every element of the original permutation belongs to one of $K_1, \ldots, K_x$, there exists a decreasing subsequence of length at least $\frac{n}{x}$. Now $k \geqslant x$ and $l \geqslant \frac{n}{x}$ together give $kl \geqslant x \cdot \frac{n}{x} = n$.

Solution 3. Another algorithm can be used for partitioning the permutation $(a_1, \ldots, a_n)$ into decreasing subsequences in such a way that there exists an increasing subsequence with each element representing a different part. Let the first element of each new subsequence be the largest among the unused elements, the next be the largest following to it in the original sequence unused element etc., until no more elements can be chosen this way. Let these subsequences be $L_1, \ldots, L_y$ in the order of forming.

For every $z = y, y-1, \ldots, 2$ and every a_i from L_z, there exists an element a_j from L_{z-1} such that $i < j$ and $a_i < a_j$. Indeed, suppose the contrary. Then all elements a_j in L_{z-1} such that $i < j$ are smaller than a_i. This means that a_i should have been chosen into L_{z-1}, contradiction.

Hence, starting from an arbitrary element c_y of L_y, we can choose an element c_{y-1} from L_{y-1}, an element c_{y-2} from L_{y-2} etc., until c_1 from L_1, in such a way that $c_y < c_{y-1} < \ldots < c_1$. The rest is as in Solution 2.

Remark 1. For proving the auxiliary claim $kl \geqslant n$, it is straightforward to apply Erdős-Szekeres theorem that tells that every vector of length $(k-1)(l-1)+1$ of real numbers contains either an increasing subsequence of length k or a decreasing subsequence of length l. Indeed, assume that $kl < n$; then the subsequence of $kl+1$ initial elements one can find either an increasing subsequence of length $k+1$ or a decreasing subsequence of length $l+1$, contradiction. The proof of $kl \geqslant n$ presented in Solution 1 is a variant of the proof of Erdős-Szekeres theorem given in Wikipedia.

Remark 2. Using Chebyshev's theorem, one can show that for every natural number $n \geqslant 25$, there exists a positive integer less than $\sqrt{n}$ by which n is not divisible. Hence the claim of the problem holds for all natural numbers n except for a certain finite number of them. A case study shows that the only numbers for which the claim of the problem is not valid are 1, 2, 4, 6 and 12.

S10. Let m be an integer, $m \geqslant 2$. Each student in a school is practising m hobbies the most. Among any m students there exist two students who have a common hobby. Find the smallest number of students for which there must exist a hobby which is practised by at least 3 students.

Answer: m^2.

Solution. If the number of students is $m^2 - 1 = (m-1)(m+1)$, let us split the students to $m-1$ groups, each with $m+1$ students. Assume that each student has a unique common hobby with every other student in the same group and none of the students is practising any other hobby. Then each student has exactly m hobbies but among each m students at least two of them belong to the same group i.e have a common hobby. Hence the conditions of the problem are satisfied but there exists no hobby which is practised by more than 2 students. If there are less than $m^2 - 1$ students at the school, similar construction can be used with a suitable number of students omitted.

Let us show that if there are at least m^2 students at school, then there must exist a hobby that is practised by at least 3 students. Assume the contrary that every hobby is practised by 2 students at most. Let us form a group of students in which each pair of students have no hobby in common by adding in each iteration to initially empty group one student who does not have a common hobby with any of the students already assigned to the group. As every student has m hobbies the most and each hobby can be common with only one other student, after i iterations there are at most $i(m+1)$ students who cannot be added to the group in the subsequent iterations. As $(m-1)(m+1) < m^2$, it is still possible to add another student to a group of $m-1$ students. Therefore, there exists a group of m students, in which there exists no pair who have a common hobby. That contradicts the conditions of the problem statement.

Remark. The problem can be generalised by still limiting the number of hobbies for each student to m but requiring a pair of students with a common hobby among each group of k students. Then the maximum number of students is $(k-1)(m+1)+1 = km - m + k$. The proof is analogical to the one in the solution.

S11. Find all positive integers n such that $(n^2 + 11n - 4) \cdot n! + 33 \cdot 13^n + 4$ is a perfect square.

Answer: $n = 1$ and $n = 2$.

Solution. Let us denote $a_n = (n^2 + 11n - 4) \cdot n! + 33 \cdot 13^n + 4$. If $n \geqslant 4$, then 8 divides $n!$. Hence $a_n \equiv 33 \cdot 13^n + 4 \equiv 5^n + 4 \pmod 8$. As $5^2 \equiv 1 \pmod 8$, then $5^n \equiv 1 \pmod 8$ for all even n. Therefore, $a_n \equiv 5 \pmod 8$, if $n \geqslant 4$ and n is even. But perfect squares leave remainders 0, 1 or 4 when dividing by 8.

Secondly, when $n \geqslant 7$, then 7 divides $n!$. So $a_n \equiv 33 \cdot 13^n + 4 \equiv 5 \cdot (-1)^n + 4 \pmod 7$. Therefore, for odd $n \geqslant 7$ $a_n \equiv -5 + 4 = -1 \pmod 7$. But perfect squares leave remainders 0, 1, 4 or 2 when dividing by 7.

We are left with possible candidates $n = 1$, $n = 2$, $n = 3$ and $n = 5$. For $n = 5$, a_n is not a perfect square because $a_5 \equiv 33 \cdot 13^5 + 4 \equiv 3^6 - 1 \equiv 3 \pmod 5$, but perfect squares leave remainder 0, 1 or 4 when divided by 5. Also, a_n is not a perfect square for $n = 3$, because $a_3 = (9 + 33 - 4) \cdot 6 + 33 \cdot$

$13^3+4 \equiv 3+3^4-1 \equiv 3 \pmod 5$. Finally, we can check that $a_1 = (1+11-4)\cdot 1+33\cdot 13+4 = 441 = 21^2$ and $a_2 = (4+22-4)\cdot 2+33\cdot 169+4 = 5625 = 75^2$.

Remark. This problem originally appeared in the 3rd Selection Examination of the Slovenian IMO team in 2014.

S12. The circles k_1 and k_2 intersect at points M and N. The line l intersects with the circle k_1 at points A and C and with circle k_2 at points B and D, so that points A, B, C and D are on the line l in that order. Let X be a point on line MN such that the point M is between points X and N. Lines AX and BM intersect at point P and lines DX and CM intersect at point Q. Prove that $PQ \parallel l$.

Solution 1. Let Y be the second intersection of line AX with the circle k_1 and Z be the second intersection of line DX with the circle k_2 (Fig. 22). As X is on the radical axis of k_1 and k_2, we have $XY\cdot XA = XZ\cdot XD$ from which the points A, Y, Z and D lie on the same circle. Therefore $\angle XZY = \angle XAD$ and $\angle XYZ = \angle XDA$.

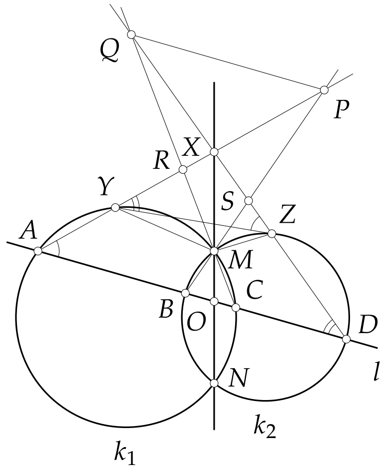

Fig. 22

Let R be the intersection of CM and AX. Let S be the intersection BM and DX. Points R and X lie on the same side of points A and Y, points S and X lie on the same side of points D and Z. Points R and Q lie on the same side of points C and M, points S and P lie on the same side of points B and M. As A, Y, M and C lie on the same circle k_1, we have $\angle QMY = \angle RMY = \angle RAC = \angle XAD = \angle XZY = \angle QZY$. As Q and Y lie on the same side of line MN but Z on the other side, M and Z lie on the same side of line QY. Hence the equation $\angle QMY = \angle QZY$ implies that Q, Y, M and Z lie on the same circle. Similarly we get $\angle PMZ = \angle SMZ = \angle SDB = \angle XDA = \angle XYZ = \angle PYZ$, which can be used to analogically show that P, Z, M and Y lie on the same circle. Finally, P, Q, Y and Z have to lie on the same circle in that order (Q and Y lie on the same side of line MN, P and Z on the other side). Therefore, $\angle QPA = \angle QPY = \angle QZY = \angle XZY = \angle XAD = \angle PAD$, which implies $PQ \parallel l$ (because Q and D are on different sides of AP).

Solution 2. Let O be the intersection of AD and MN. Let's use coordinate system with an origin O, x-axis along the line AD and y-axis along the line MN. Then the coordinates of points A, B, C, D, M and X are respectively $(a,0)$, $(b,0)$, $(c,0)$, $(d,0)$, $(0,u)$ and $(0,v)$ for some real numbers a, b, c, d, u, v. As O is on the radical axis of the circles k_1 and k_2, we have $ac = bd = p$, where p is the power of point O with respect to circles k_1 and k_2.

The equation for line AX in the coordinate system is $\frac{x}{a} + \frac{y}{v} = 1$ and the equation for line BM is $\frac{x}{b} + \frac{y}{u} = 1$. The y-coordinate of the intersection of those lines hence satisfies $a\left(1 - \frac{y_P}{v}\right) = b\left(1 - \frac{y_P}{u}\right)$, from which $y_P = \frac{(b-a)uv}{bv-au}$. Analogically the equations for the lines DX and CM, i.e. $\frac{x}{d} + \frac{y}{v} = 1$ and $\frac{x}{c} + \frac{y}{u} = 1$, give the y-coordinate of Q to be $y_Q = \frac{(c-d)uv}{cv-du}$. Substituting here $c = \frac{p}{a}$ and $d = \frac{p}{b}$ yields to

$$y_Q = \frac{\left(\frac{p}{a} - \frac{p}{b}\right)uv}{\frac{p}{a}v - \frac{p}{b}u} = \frac{\frac{b-a}{ab}uv}{\frac{bv-au}{ab}} = \frac{(b-a)uv}{bv-au}.$$

As $y_P = y_Q$, the line PQ is parallel to x-axis and the line l.

Remark. This problem originally appeared in the Croatian Mathematical Olympiad in 2015.

Problems Listed by Topic

Number theory: O1, O2, O6, O8, F3, F4, F8, F12, F13, S2, S9, S11
Algebra: O7, O9, O12, O13, F5, S3, S4, S8
Geometry: O3, O4, O5, O10, O15, F1, F6, F10, S5, S7, S12
Discrete mathematics: O11, O14, O16, F2, F7, F9, F11, F14, F15, S1, S6, S10