Estonian Mathematical Olympiad '93

(Selected Problems)

The mathematical olympiad in Estonia is held annually in three stages: at school, city/district, and all-Estonian levels. At each level, separate problems are usually given to students of each grade from 9 to 12, in compliance with the school curriculum. Younger students also take part at the first two levels.

Below, after each problem number, the numbers in brackets indicate the grade(s) and stage (II or III) where it was used.

Problem 6 originates from Iceland '92; problem 4 is due to an oral contribution from Sweden; problems 2, 3 and 5 are taken from various Russian sources; the rest seem to be original.

Problems

1. (9, II) Find all triples of pairwise different real numbers (a, b, c), such that the system of equations

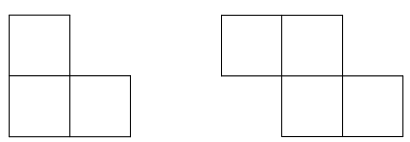
$$\begin{cases} ax = b \\ bx = c \\ cx = a \end{cases}$$

   has a solution.

2. (10, II) The point of osculation of the hypotenuse of a right triangle with its inscribed circle divides the hypotenuse into intervals of lenghts a and b. Find the area of the triangle.

3. (10, III) Natural numbers m, n and k have the properties: m^n divides n^m, and n^k divides k^n. Prove that m^k divides k^m.

4. (10-11, III) For which natural numbers n is it possible to cover a rectangle of size $3 \times n$ unit squares with shapes shown on the figure, without overlapping? (Each shape is available in unlimited quantity.)



5. (10, III) Let r be the radius of the inscribed circle of a right triangle, and let h be the height of the triangle, drawn to its hypotenuse. This height divides the original triangle into two smaller triangles, let r_1 and r_2 be the radii of the circles inscribed in these triangles. Prove the following equalities:

   a) $r_1 + r_2 + r = h$;

   b) $r_1^2 + r_2^2 = r^2$.

6. (11, II) In a right triangle ABC, the medians drawn to its side AC and to the hypotenuse are perpendicular to each other. Find the area of the triangle if the side AB has length 1.

7. (11, III) Find all functions $f(n)$ having the following three properties:

   a) for any natural number n, $f(n)$ is also a natural number;

   b) $f(n+m) = f(n)f(m)$ holds for any natural numbers n and m;

   c) the equation $f(f(n)) = (f(n))^2$ has some natural number n_0 as its solution.

8. (12, II) Prove that, for any natural number $k \geqslant 2$, there exist k pairwise different natural numbers n_1, n_2, ..., n_k such that

$$\frac{1}{n_1} + \frac{1}{n_2} + \cdots + \frac{1}{n_k} = \frac{3}{17}.$$

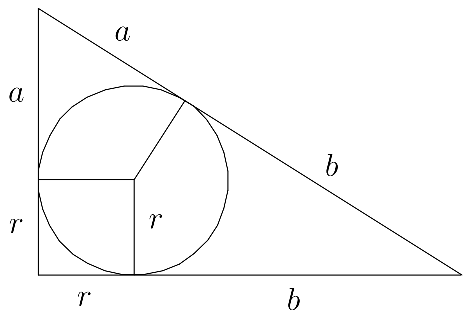
9. (12, III) A rectangle is cut into five rectangular pieces having equal areas. Prove that at least two of the pieces are identical.

10. (12, III) Let us call a natural number "beautiful" if it is a perfect square itself, and its presentation in the decimal system is obtained by writing side by side two or more squares of natural numbers. (For example, 169 is a "beautiful" number, as $169 = 13^2$, $16 = 4^2$ and $9 = 3^2$.) How many "beautiful" numbers divisible by 1993 are there?

Hints and Solutions

1. There are no such triples. It is easy to see that $abc \neq 0$, and $a + b + c = 0$ if a solution $x \neq 1$ exists. Now, one of the numbers a, b, c must have a sign opposite to the two others, and thus x should be both positive and negative.

2. From the Pythagorean Theorem we find



$$(r+a)^2 + (r+b)^2 = (a+b)^2\,,$$

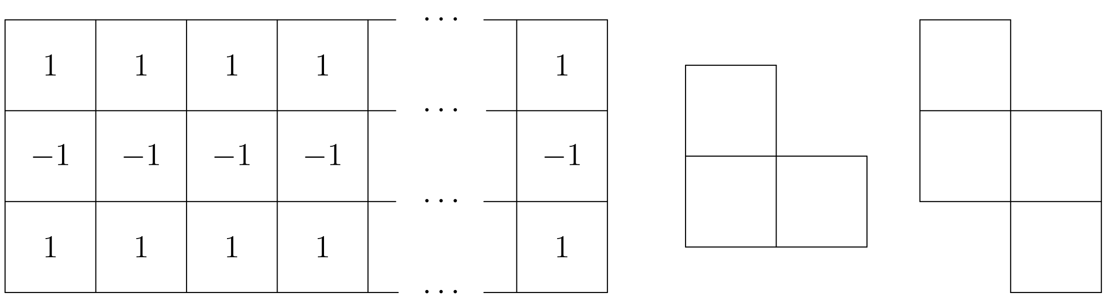
and

$$r = \frac{-a-b+\sqrt{a^2+6ab+b^2}}{2}\,.$$

So, the area is $S = \dfrac{(r+a)(r+b)}{2} = ab$.

3. By the assumptions, $(m^n)^k = (m^k)^n$ divides $(n^k)^m = (n^m)^k$ which divides $(k^m)^n = (k^n)^m$. Thus, $(m^k)^n$ divides $(k^m)^n$, and this implies m^k dividing k^m .

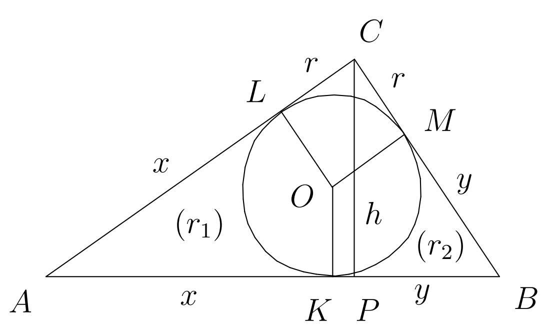
4. Assign numbers to the unit squares as shown on the figure:



(A) (B)

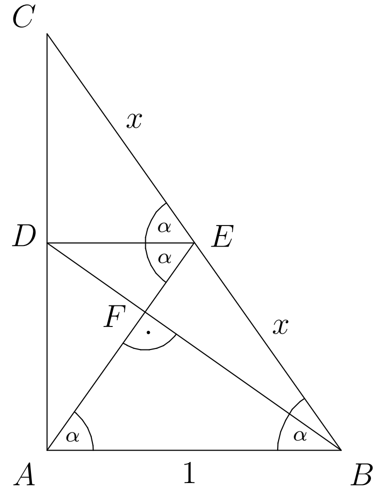
The sum of numbers covered by shape (B) is always equal to zero; for shape (A) this sum is either 1 or -1 . As the sum of all numbers assigned to the squares is n , we have to use at least n shapes, that is, we cannot use shape (B) at all. Now, it is easy to see that covering is possible if and only if n is an even number.

5. Calculating the area of the triangle ABC in two different ways, we obtain the equality $h(x+y) = 2(r^2+xr+yr)$; thus $h = \dfrac{2r(x+y+r)}{x+y}$. As the triangles APC, CPB and ACB are all similar, we get



$\dfrac{r_1}{x+r} = \dfrac{r_2}{y+r} = \dfrac{r}{x+y}$, and $r_1+r_2+r = \dfrac{r}{x+y}(2x+2y+2r) = h$. Now, we also have $r_1^2 + r_2^2 = \dfrac{r^2}{(x+y)^2}((x+r)^2+(y+r)^2) = r^2$, as $(x+y)^2 = (x+r)^2+(y+r)^2$ by the Pythagorean Theorem.

6. Denote $\angle CED = \angle AED = \angle EAB = \alpha$ and $CE = EB = EA = x$. Then we have $\frac{FE}{DE} = \cos\alpha = \frac{FA}{AB}$, implying $\frac{FE}{FA} = \frac{1}{2}$ and $FE = \frac{x}{3}$. As $\frac{DE}{CE} = \cos\alpha = \frac{FE}{DE}$, we get $x^2 = \frac{3}{4}$ and $AC = \sqrt{4x^2-1} = \sqrt{2}$. The area is equal to $\frac{AB \cdot AC}{2} = \frac{\sqrt{2}}{2}$.



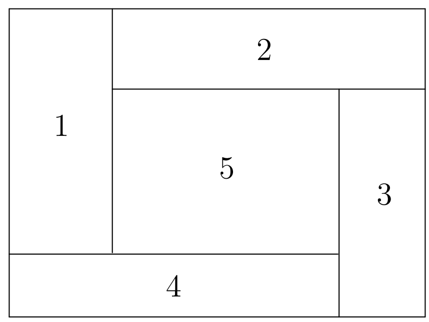
7. Condition b) implies $f(n+1) = f(n)f(1)$ for any natural number n, and thus $f(n) = (f(1))^n$. Denote $f(1) = k$, then by condition c) there is a natural number n_0 such that $k^{k^{n_0}} = (k^{n_0})^2$. This implies either $k = 1$ or $k^{n_0} = 2n_0$; in the latter case we have $k = 2$ and $n_0 = 1$ or $n_0 = 2$. Therefore, the only functions having the required properties are $f(n) \equiv 1$ and $f(n) = 2^n$.

8. For $k = 2$, take $n_1 = 6$, $n_2 = 102$. Now, use the identity

$$\frac{1}{m} = \frac{1}{m+1} + \frac{1}{m(m+1)}$$

taking the greatest of the denominators as m, at each step.

9. Quite obviously, the assertion holds when we cut a rectangle in two, three, or four pieces. The only way to cut a rectangle in five rectangular pieces so that no two, three or four of them make up a rectangle is shown on the figure. By comparing the areas, it is easy to show that pieces 1, 3 and 2, 4 must be identical.



10. Denote $x = 1993$ and $y = 2 \cdot 1993$, then the number

$$(10^8 \cdot x + y)^2 = x^2 \cdot 10^{16} + 2xy \cdot 10^8 + y^2$$

is “beautiful” (it is obtained by writing side by side the perfect squares x^2, $2xy$ ja y^2), and it is divisible by 1993. Now, for any “beautiful” number n divisible by 1993, check that $100n$ has the same properties.