Estonian Mathematical Competitions '94

(Selected Problems)

The Estonian Mathematical Olympiad is held annually in three stages: at school, city/district, and national levels. At each level, separate problems are usually given to students of each grade from 9 to 12, in compliance with the school curriculum.

Since 1993, we also have two "open" mathematical competitions each year. At these, any student can participate, and everyone is given the same set of problems.

Here we present a variety of problems from both of these competitions. For each problem, we indicate in brackets the competition, stage (II or III) and grade(s) where it was used. Many of these problems are, in fact, not original but taken from various sources and more or less modified.

1. (II, 9) The sides of a convex quadrangle serve as diameters for four circles. Prove that these circles fully cover the quadrangle.

2. (II, 10) Find the largest natural number n such that $\underbrace{999\ldots99}_{999\text{ digits}}$ is divisible by 9^n.

3. (II, 11) Prove that for any real numbers x and y, we have $|\sin x| > |\cos y|$ if and only if $|\sin y| > |\cos x|$.

4. (II, 12) For a point C on the diameter AB of a semicircle, let P and Q be points on the semicircle such that $\angle ACP = \angle BCQ = \beta$. Prove that, for any given value of β, the length of segment PQ does not depend on the choice of point C.

5. (II, 11–12) The "Top Twenty" of popular songs is selected weekly. It is known that:

   1) the same 20 songs are never selected in the same order on any two consequtive weeks;
   2) once a song has moved backward in the ordering, it cannot improve its position in the future.

   Determine the maximum number of consequtive weeks the same 20 songs can be selected.

6. (III, 9) Label the sides and diagonals of a regular pentagon by natural numbers 1 to 10 and consider all triangles with vertices at the vertices of the pentagon. Is it possible that for each of these triangles the sum of labels at its sides is the same?

7. (III, 10) Find the least natural number having exactly 100 different natural divisors (including 1 and the number itself).

8. (III, 10) Let $\alpha, \beta, \gamma, \delta$ be such that $0^\circ < \alpha, \beta, \gamma, \delta < 90^\circ$, $\sin\alpha = \dfrac{\sin\beta}{\sin\gamma}$ and $\sin\delta = \dfrac{\tan\beta}{\tan\gamma}$. Prove that $\tan\alpha = \dfrac{\tan\delta}{\cos\gamma}$.

9. (III, 10) Three men decided to have a picnic together with their wives. They arrived at the railway-station one by one, each of them greeting by handshake all those already present, except his/her wife or husband. Later one of the men asked all the others how many companions each had shaken hands with upon arrival, and he got five different answers. What would have been his own answer?

10. (III, 11) Compute the sum of all the five-digit natural numbers that do not change when the order of their digits is reversed (we do not accept 0 as the first digit of a natural number).

11. (III, 11) Prove that for any triangle the equality

$$\frac{R}{r} = \frac{\cot\frac{\alpha}{2} + \cot\frac{\beta}{2}}{2\sin\gamma}$$

holds, where α, β, γ are the angles of the triangle and R, r are the radii of its circumcircle and inscribed circle, respectively.

12. (III, 11) For any sequence $A = \{a_1, a_2, \ldots\}$ of real numbers denote $\Delta A = \{b_1, b_2, \ldots\}$ where $b_i = a_{i+1} - a_i$. Determine the value of term a_1 of sequence A, knowing that $a_{19} = a_{94} = 0$ and the sequence $\Delta(\Delta A)$ is constant with all its terms equal to 1.

13. (III, 12) For a triangle ABC, let A_1, B_1 ja C_1 be points on its sides BC, AC and AB respectively, such that the three segments AA_1, BB_1 ja CC_1 intersect in one point O and

$$\frac{AO}{OA_1} + \frac{BO}{OB_1} + \frac{CO}{OC_1} = 1994\,.$$

Find the value of $\dfrac{AO}{OA_1} \cdot \dfrac{BO}{OB_1} \cdot \dfrac{CO}{OC_1}$.

14. (III, 12) Prove that, for any natural number n, the number $n! = 1 \cdot 2 \cdot \ldots \cdot n$ has less than $\frac{n}{4}$ trailing zeros.

15. (III, 12) All three vertices of a regular triangle lie on the sides of a unit square (some of them may coincide with the vertices of the square). Determine the minimum and maximum values for the area of such a triangle.

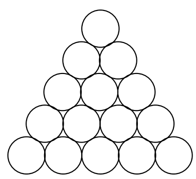
16. (OPEN) The two sides BC and CD of an inscribed quadrangle $ABCD$ are of equal length. Prove that the area of this quadrangle is equal to $S = \frac{1}{2} \cdot AC^2 \cdot \sin \angle A$.

17. (OPEN) Consider the sequence of real numbers $a_0, a_1, a_2, \ldots$:

$$a_0 = x\,, \quad a_1 = y\,, \quad a_i = a_{i-2} + a_{i-1} \ \ (i \geqslant 2)\,.$$

Determine all values of x and y for which this sequence is bounded (i.e. there exists a positive number M such that $|a_i| < M$ for all $i = 0, 1, 2, \ldots$).

18. (OPEN) A number of balls of radius R are laid on the table in the form of a regular triangle with each side containing N balls (see figure for $N = 5$). A frame is put around this triangle and further layers of similar balls are laid on it, each successive layer having one ball less on each of its sides (so the last, N-th layer consists of a single ball). Determine:



a) the total height of the pyramid (for arbitrary N);
b) the total number of balls for $N = 10$;
c) the total number of balls for arbitrary N.

19. (OPEN) Consider a function $f(x)$ such that

$$f(f(x)) = f(x) + 1994\,x$$

for any real value of x.

a) Prove that $f(x) = 0$ if and only if $x = 0$;
b) Find one such function.

20. (OPEN) Prove that if $\frac{AC}{BC} = \frac{AB + BC}{AC}$ in a triangle ABC, then $\angle B = 2 \cdot \angle A$.