PROBLEMS WITH OFFICIAL SOLUTIONS

Mathematical Team Competition *Baltic Way*

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Preface

Mathematical Team Competition *Baltic Way 2025* is organized with the financial support in cooperation with ESF+ project No. 4.2.2.3/1/24/I/001 "Pedagogu profesionālā atbalsta sistēmas izveide".

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Organizers

The competition is organized by University of Latvia Faculty of Science and Technology Department of Mathematics A. Liepa's Correspondence Mathematics School

UNIVERSITY OF LATVIA FACULTY OF SCIENCE AND TECHNOLOGY

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Problems

Problem 1 Does there exist a sequence of integers $a_1, a_2, \ldots$ such that for each integer $d \neq 0$ there are exactly 2025 distinct pairs of indices (i, j) for which $a_i - a_j = d$?

Problem 2 Let $a_1, a_2, \ldots$ be a sequence of real numbers such that

$$\{a_n\}\lfloor a_{n+1}\rfloor = \lfloor a_n\rfloor\{a_{n+1}\}$$

for each positive integer n. Prove that there is a real number λ such that $\{a_m\}\lfloor a_m\rfloor = \lambda$ for infinitely many positive integers m.

Remark: For a real number x, $\lfloor x\rfloor$ denotes the greatest integer that does not exceed x, and $\{x\} = x - \lfloor x\rfloor$.

Problem 3 Let $f\colon \mathbb{Q} \to \mathbb{Q}$ be a function that satisfies

$$f(x) + f(y) \geq f(x + y)$$

for all $x, y \in \mathbb{Q}$. Show that there exists an $\alpha \in \mathbb{R}$ such that $f(x) \geq \alpha x$ for all $x \in \mathbb{Q}$.

Problem 4 Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$(f(a-c) + f(b-d))\,(f(a) + f(b)) = f(ad - bc) + f\,(f(a) + f(b) - ac - bd)$$

for all real numbers a, b, c, d.

Problem 5 Let n be a positive integer and let $x_1 \leq x_2 \leq \cdots \leq x_n$ be positive real numbers satisfying

$$\frac{x_1^3 + x_2^3 + \cdots + x_n^3}{x_1 + x_2 + \cdots + x_n} = \frac{x_1^2 + x_n^2}{2}.$$

Prove that

$$x_1^2 + x_2^2 + \cdots + x_n^2 \geq n x_1 x_n.$$

Problem 6 Let $n \geq 3$ friends play dodgeball. Initially all players are free. When a free player A *knocks out* another free player B, A imprisons B and frees all players currently imprisoned by B. Two knock-outs cannot happen at the same time.

Find the least number of knock-outs after which it is possible that among every two players at least one has freed the other one.

Problem 7 Let n be an even positive integer. An $n \times n \times n$ cube is composed of n^3 unit cubes. A set of n unit cubes that forms a $1 \times 1 \times n$ box is called a *needle*, which can have three orientations. Find the largest integer K, such that it is possible to select $3K$ needles, K in each orientation, such that no two of them share a unit cube.

Problem 8 A sequence of non-negative integers $a_1, \ldots, a_n$ is called *self-descriptive* if

$$a_k = |\{i \mid a_i \geq k\}|$$

for each $k \in \{1, \ldots, n\}$, i.e. a_k equals the number of elements of the sequence that are greater than or equal to k, for each $k \in \{1, \ldots, n\}$.

For each positive integer n, determine the number of self-descriptive sequences of length n.

Problem 9 A *Baltic polyiamond* is an n-gon with side lengths $n, n-1, \ldots, 2, 1$ in exactly this order, and all internal angles 120° or 240°. Prove that for every Baltic polyiamond, n is divisible by 6.

Problem 10 A positive integer with 9 digits is called *nice* if it contains each digit from 1 to 9. A number is called *wonderful* if it is nice and for each $k = 1, \ldots, 9$, it holds that the k-th digit in the number is equal to the position of k in the number. For example, 847296315 is wonderful:

- the first digit is 8, which is the position of 1 in the number,
- the second digit is 4, which is the position of 2 in the number,
- the third digit is 7, which is the position of 3 in the number, and so on.

Show that the number of wonderful numbers is even.

Problem 11 Let $A_1, \ldots, A_n$ be a sequence of points on the Euclidean plane. Some of the points may coincide, but the sequence contains at least two distinct points. A *Baltic Way* is a route defined by a permutation $B_1, \ldots, B_n$ of the points $A_1, \ldots, A_n$. The length of a Baltic Way is

$$|B_1B_2| + |B_2B_3| + \cdots + |B_{n-1}B_n| + |B_nB_1|,$$

where $|PQ|$ denotes the Euclidean distance between points P and Q.

Let K and L be the lengths of some two Baltic Ways on the same sequence of n points. For a given n, determine the maximum possible value of K/L over all possible sequences $A_1, \ldots, A_n$.

Problem 12 In an acute triangle ABC with $AB < AC$, the incentre is I and the circumcircle is ω. The line AI intersects the side BC at D. Let T be the point on ω such that $AT \parallel BC$. The line TI intersects ω again at P and the circumcircle of triangle APD again at Q. Prove that $AI = DQ$.

Problem 13 In a cyclic quadrilateral $ABCD$, the lines AB and CD intersect at E. A point P lies inside $ABCD$ and satisfies

$$\angle BAP = \angle PCB \quad \text{and} \quad \angle CBP = \angle PDC.$$

Prove that $PE \perp BC$.

Problem 14 In an acute triangle ABC with $AB < AC$, the circumcircle is ω and the orthocentre is H. The point P on major arc BC of ω satisfies $\angle PCB + \angle ACB = 90^\circ$. The point R on segment AC satisfies $BR = CR$. Prove that $\angle HPR = \angle ACB$.

Problem 15 In an acute, scalene triangle ABC, the perpendicular bisector of BC intersects AC and AB in A_b and A_c, respectively. Define B_c, B_a, C_a and C_b similarly. Prove that the circumcentres of ABC, $A_bB_cC_a$, and $A_cB_aC_b$ are collinear.

Problem 16 Let a, b, c, n be positive integers satisfying $abc = 10^n - 1$. If $S(x)$ denotes the sum of digits of x in decimal notation, prove that

$$S(a) + S(b^2) + S(c^4) \geq \sqrt[3]{243n}.$$

Problem 17 A positive integer is called *brilliant* if its positive divisors can be partitioned into two sets with an equal number of elements and an equal sum of elements.

Find all possible values of the number of positive divisors of a brilliant number.

Problem 18 Find all functions $f\colon \mathbb{Z}_{>0} \longrightarrow \mathbb{Z}_{>0}$ such that $f(2) = 1$ and

$$\operatorname{lcm}(f(a+b), f(b)) \mid \operatorname{lcm}(a + f(b), b)$$

for all positive integers a, b.

Problem 19 There are 100 positive integers written on a blackboard. Anna plays a game. A move consists of choosing two integers a and b on the blackboard with the property that $a \mid b$, erasing them both and writing the integer b/a. She makes moves until there are no more valid moves, at which point the game ends.

Find the largest N such that for some initial state Anna can both:

- finish the game with only one number remaining, and
- finish the game with N numbers remaining.

Problem 20 Determine all positive integers n such that there exists a permutation $a_1, \ldots, a_n$ of $1, \ldots, n$, where the numbers $a_1, 2a_2, \ldots, na_n$ are all pairwise distinct modulo n.

Solutions

Problem 1 Does there exist a sequence of integers $a_1, a_2, \ldots$ such that for each integer $d \neq 0$ there are exactly 2025 distinct pairs of indices (i, j) for which $a_i - a_j = d$?

Solution. We will prove that there exists a sequence with the desired property.

Note first that if $a_i - a_j = d$ then $a_j - a_i = -d$, so it is enough to prove that for each integer $d > 0$ there exists exactly 2025 pairs (i, j) such that $a_i - a_j = d$.

We construct the sequence inductively; let a_1 be an arbitrary integer.

Suppose we have chosen $a_1, a_2, \ldots, a_n$ such that for all $d > 0$ there are at most 2025 pairs of indices $1 \leqslant i, j \leqslant n$ with $a_i - a_j = d$, and let $l > 0$ be the least integer for which there are stricly fewer than 2025 pairs of indices $1 \leqslant i, j \leqslant n$ with $a_i - a_j = l$. Then let $a_{n+1} \gg a_n$ and $a_{n+2} = a_{n+1} + l$ such that

$$a_{n+2} - a_i > a_{n+1} - a_i > a_j - a_k$$

for all $1 \leqslant i, j, k \leqslant n$. This is clearly possible by taking a_{n+1} large enough, and in doing so, there is still no $d > 0$ such that there are more than 2025 pairs $1 \leqslant i, j \leqslant n + 2$ for which $a_i - a_j = d$

By induction, we obtain a sequence $a_1, a_2, \ldots$ with the property that for each $d > 0$ there are at most 2025 pairs (i, j) with $a_i - a_j = d$. Since we choose the minimal l in each step, it is clear however, that there will be exactly 2025 such pairs for each $d > 0$. By the introductory remark, it is clear that the constructed sequence satisfies the desired property.

Remark There is an alternative solution reliant on the fact that 2025 is a perfect square. In the inductive step of the construction, if we instead choose l as simply the least positive integer not on the form $a_i - a_j$, and proceed as above, we obtain a sequence where each $d \neq 0$ appears exactly once as $a_i - a_j$. Consider now the sequence $a_1, \ldots, a_1, a_2, \ldots, a_2, a_3 \ldots$, where each a_i appears 45 times. For any $d \neq 0$, we may now choose any of the 45 copies of a_i and any of the 45 copies of a_j for $d = a_i - a_j$, giving a total of exactly $45^2 = 2025$ pairs (i, j) satisfying $d = a_i - a_j$.

Problem 2 Let $a_1, a_2, \ldots$ be a sequence of real numbers such that

$$\{a_n\}\lfloor a_{n+1}\rfloor = \lfloor a_n\rfloor\{a_{n+1}\}$$

for each positive integer n. Prove that there is a real number λ such that $\{a_m\}\lfloor a_m\rfloor = \lambda$ for infinitely many positive integers m.
Remark: For a real number x, $\lfloor x\rfloor$ denotes the greatest integer that does not exceed x, and $\{x\} = x - \lfloor x\rfloor$.

Solution 1. First, we note that the problem statement is true if there are infinitely many m with either $\{a_m\} = 0$ or $\lfloor a_m\rfloor = 0$. Thus, after possibly reindexing, we may assume that $\{a_n\}, \lfloor a_n\rfloor \neq 0$ for all n. In this case, we may rewrite the condition as

$$\frac{\lfloor a_{n+1}\rfloor}{\lfloor a_n\rfloor} = \frac{\{a_{n+1}\}}{\{a_n\}}$$

for all n. By telescoping, this gives the identity

$$\frac{\lfloor a_i\rfloor}{\lfloor a_j\rfloor} = \frac{\{a_i\}}{\{a_j\}}$$

for each pair of indices (i, j). Thus, if $\lfloor a_i\rfloor = \lfloor a_j\rfloor$ then $\{a_i\} = \{a_j\}$, and so especially $\{a_i\}\lfloor a_i\rfloor = \{a_j\}\lfloor a_j\rfloor$. Hence we are done if we can prove that some value appears infinitely many times among the $\lfloor a_n\rfloor$. From the identity above, we also get that

$$\lfloor a_n\rfloor = \frac{\lfloor a_1\rfloor}{\{a_1\}}\{a_n\},$$

for all n, which implies that

$$|a_n| < |\lfloor a_n\rfloor| + 1 = \left|\frac{\lfloor a_1\rfloor}{\{a_1\}}\{a_n\}\right| + 1 \leq \left|\frac{\lfloor a_1\rfloor}{\{a_1\}}\right| + 1.$$

Hence the sequence is bounded, and so $\lfloor a_n\rfloor$ takes only finitely many values. By pigeonhole, some value appears infinitely many times, and so we are done.

Solution 2. Rewrite given condition as

$$\frac{\lfloor a_{n+1}\rfloor}{\{a_{n+1}\}} = \frac{\lfloor a_n\rfloor}{\{a_n\}} = \rho$$

Consider arbitrary real number x satisfying $\frac{\lfloor x\rfloor}{\{x\}} = \rho$. There are 2 possible cases to consider:

- If $\rho \geq 0$, then we can write $x = y + \epsilon$, where y is nonnnegative integer and $0 \leq \epsilon < 1$. Notice that
$$\rho = \frac{\lfloor x\rfloor}{\{x\}} = \frac{y}{\epsilon} \implies y = \rho\epsilon$$
This means that $|y| = |\rho|\epsilon < |\rho|$
- If $\rho < 0$, then we can write $x = -(y + \epsilon)$, where x is nonnegative integer and $0 \leq \epsilon < 1$. Notice that
$$\rho = \frac{\lfloor x\rfloor}{\{x\}} = \frac{-(y+1)}{1-\epsilon} \implies y = -\rho(1-\epsilon) - 1$$
This means that $|y| = |-\rho(1-\epsilon) - 1| < |\rho| + 1$

From above 2 observations it follows that sequence $\lfloor a_1 \rfloor, \lfloor a_2 \rfloor, \ldots$ is bounded and hence by pigeonhole principle there are infinitely many positive integers m such that $\lfloor a_m \rfloor = C$, where C is constant. However, notice that $\lfloor a_m \rfloor$ uniquely determines $\{a_m\}$, since $\frac{\lfloor a_m \rfloor}{\{a_m\}} = \rho$, this implies that for infinitely many positive integers m we get that $\{a_m\}\lfloor a_m \rfloor = \lambda$ for some constant λ.

Problem 3 Let $f\colon \mathbb{Q} \to \mathbb{Q}$ be a function that satisfies

$$f(x) + f(y) \geq f(x+y)$$

for all $x, y \in \mathbb{Q}$. Show that there exists an $\alpha \in \mathbb{R}$ such that $f(x) \geq \alpha x$ for all $x \in \mathbb{Q}$.

Solution We'll begin by noticing that $f(0) \geq 0$, since

$$\begin{aligned} f(0) + f(0) &\geq f(0+0) \\ f(0) &\geq f(0) - f(0) \\ f(0) &\geq 0. \end{aligned}$$

Next, let's prove the following lemma:

Lemma: For all $n \in \mathbb{Z}_+$ and $x \in \mathbb{Q}$ we have $nf(x) \geq f(nx)$.
Proof: We'll do induction on n. Since $1 \cdot f(x) \geq f(1 \cdot x)$, the base case $n = 1$ is true. Assuming that $nf(x) \geq f(nx)$ we get that

$$\begin{aligned} (n+1)f(x) &= nf(x) + f(x) \\ &\geq f(nx) + f(x) \\ &\geq f(nx + x) \\ &= f((n+1)x), \end{aligned}$$

which completes the induction step.

Next, we want to prove that for all rationals $x < 0$ and $y > 0$ we have $f(x)/x \leq f(y)/y$. Let $x = -a/b$ and $y = c/d$, where a, b, c, d are positive integers. Now

$$\begin{aligned} f(ac) + f(-ac) &\geq f(ac - ac) \\ f\left(\frac{c}{d} \cdot ad\right) + f\left(-\frac{a}{b} \cdot bc\right) &\geq f(0) \\ f\left(\frac{c}{d} \cdot ad\right) + f\left(-\frac{a}{b} \cdot bc\right) &\geq 0 \\ ad \cdot f\left(\frac{c}{d}\right) + bc \cdot f\left(-\frac{a}{b}\right) &\geq 0 \\ ad \cdot f(y) + bc \cdot f(x) &\geq 0 \\ \frac{ad}{ac} \cdot f(y) + \frac{bc}{ac} \cdot f(x) &\geq \frac{0}{ac} \\ \frac{d}{c} \cdot f(y) + \frac{b}{a} \cdot f(x) &\geq 0 \\ \frac{1}{y} \cdot f(y) - \frac{1}{x} \cdot f(x) &\geq 0 \\ \frac{f(x)}{x} &\leq \frac{f(y)}{y} \end{aligned}$$

and we get what we wanted.

Now, take the sets

$$S = \left\{ \frac{f(x)}{x} \colon x < 0 \right\} \quad \text{and} \quad T = \left\{ \frac{f(y)}{y} \colon y > 0 \right\}.$$

Since for all $p \in S$ and $q \in T$ we have $p \leq q$, there must exist a real number α such that for all $p \in S$ and $q \in T$ we have $p \leq \alpha \leq q$. Choose such a number α. Now, if $x < 0$, we have $f(x)/x \leq \alpha \iff f(x) \geq \alpha x$, if $x = 0$ we have $f(0) \geq 0 = \alpha \cdot 0$ and if $x > 0$ we have $f(x)/x \geq \alpha \iff f(x) \geq \alpha x$. Therefore $f(x) \geq \alpha x$ for all $x \in \mathbb{Q}$.

Problem 4 Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$(f(a-c)+f(b-d))\,(f(a)+f(b)) = f(ad-bc)+f\,(f(a)+f(b)-ac-bd)$$

for all real numbers a, b, c, d.

Solution. The answers are $f(x) = 0$, $f(x) = \frac{1}{2}$ and $f(x) = x^2$.

Let $P(a, b, c, d)$ denote the assertion of given functional equation. If $f(x) = \lambda$ for all $x \in \mathbb{R}$, then from $P(a, b, c, d)$ it follows that $4\lambda^2 = 2\lambda$, hence $\lambda = 0$ or $\lambda = \frac{1}{2}$. Both functions clearly work. From now on assume that $f(x)$ is not constant.

From $P(0, 0, t, t)$ we get that $4f(-t)f(0) = f(0) + f(2f(0))$ for any $t \in \mathbb{R}$. If $f(0) \neq 0$ then the above readily gives that f is a constant function – contradiction. Thus $f(0) = 0$.

Assume that $\alpha \neq 0$ and $f(\alpha) = 0$. From $P(\alpha, 0, 0, t)$ we obtain that $0 = f(\alpha t)$ for any $t \in \mathbb{R}$, which implies that f is constant – contradiction. Hence $f(x) = 0 \iff x = 0$.

Finally, from $P(x, x, x, x)$ we obtain $0 = f(2f(x) - 2x^2)$ for any $x \in \mathbb{R}$, hence $2f(x) - 2x^2 = 0$. In other words, $f(a) = x^2$ for any $x \in \mathbb{R}$. Direct computation shows that such a function works.

Problem 5 Let n be a positive integer and let $x_1 \leq x_2 \leq \cdots \leq x_n$ be positive real numbers satisfying

$$\frac{x_1^3 + x_2^3 + \cdots + x_n^3}{x_1 + x_2 + \cdots + x_n} = \frac{x_1^2 + x_n^2}{2}.$$

Prove that

$$x_1^2 + x_2^2 + \cdots + x_n^2 \geq n x_1 x_n.$$

Solution. For convenience, let's denote $m = x_1$ and $M = x_n$. It follows that for every integer i such that $1 \leq i \leq n$ we have

$$M^2 - x_i^2 \geq 0 \quad \text{and} \quad x_i - m \geq 0.$$

By multiplying these inequalities we get $(M^2 - x_i^2)(x_i - m) \geq 0$ which simplifies to

$$\begin{gathered} M^2 x_i - M^2 m - x_i^3 + m x_i^2 \geq 0 \\ M^2 x_i + m x_i^2 \geq M^2 m + x_i^3 \qquad (1) \end{gathered}$$

Similarly, from $M - x_i \geq 0$ and $x_i^2 - m^2 \geq 0$ we get $(M - x_i)(x_i^2 - m^2) \geq 0$ which is equivalent to

$$M x_i^2 + m^2 x_i \geq M m^2 + x_i^3 \qquad (2)$$

By adding inequalities (1) and (2) we get

$$\begin{gathered} M^2 x_i + m x_i^2 + M x_i^2 + m^2 x_i \geq M^2 m + x_i^3 + M m^2 + x_i^3 \\ (M^2 + m^2) x_i + (M + m) x_i^2 \geq (M + m) M m + 2 x_i^3 \end{gathered}$$

Remember that the inequality above holds for every integer i such that $1 \leq i \leq n$, so by adding these inequalities for each index i we get

$$(M^2+m^2)(x_1+x_2+\cdots+x_n)+(M+m)(x_1^2+x_2^2+\cdots+x_n^2) \geq (M+m)Mmn+2(x_1^3+x_2^3+\cdots+x_n^3)$$

Note that if we multiply both sides of the given condition by $2(x_1 + x_2 + \cdots + x_n)$ we get

$$2(x_1^3 + x_2^3 + \cdots + x_n^3) = (x_1^2 + x_n^2)(x_1 + x_2 + \cdots + x_n) = (m^2 + M^2)(x_1 + x_2 + \cdots + x_n)$$

Using this fact and the inequality we got, we deduce that

$$\begin{gathered} (M + m)(x_1^2 + x_2^2 + \cdots + x_n^2) \geq (M + m) M m n \\ x_1^2 + x_2^2 + \cdots + x_n^2 \geq M m n = n x_1 x_n \end{gathered}$$

Problem 6 Let $n \geq 3$ friends play dodgeball. Initially all players are free. When a free player A *knocks out* another free player B, A imprisons B and frees all players currently imprisoned by B. Two knock-outs cannot happen at the same time.
Find the least number of knock-outs after which it is possible that among every two players at least one has freed the other one.

Solution 1. Answer is $\frac{n(n-1)}{2} + 1$.

If among every two new people, one has freed the other one at least once, then there has been at least $\frac{n(n-1)}{2}$ liberations. Two liberations of the same person must be preceded by having been imprisoned two times, because if a person is freed then it cannot be freed before having been imprisoned again. Liberations of different people obviously must be preceded by having been imprisoned during different knockouts, as one knockout can result in imprisoning only one other person. Thus for fulfilling the condition in question, there must be at least $\frac{n(n-1)}{2}$ knockouts plus one more knockout to free the person imprisoned lastly. Altogether, there must be at least $\frac{n(n-1)}{2} + 1$ knockouts.

Now we show that $\frac{n(n-1)}{2} + 1$ knockouts is enough. Firstly, let n be odd. Label all new people by integers $0, 1, \ldots, n-1$. Let the person $\frac{n-1}{2}$ imprison the people $0, 1, \ldots, \frac{n-1}{2} - 1$ during the first $\frac{n-1}{2}$ knockouts, then let the person $n-1$ imprison the people $\frac{n-1}{2}, \frac{n-1}{2}+1, \ldots, n-2$ during the next $\frac{n-1}{2}$ knockouts, and so on cyclically modulo n. In other words, during every new knockouts, the next person in the cyclic order modulo n is imprisoned, the imprisoning person is changed after every $\frac{n-1}{2}$ knockouts and then it jumps over exactly $\frac{n-1}{2} - 1$ people in the cyclic order. Every new imprisoner imprisons the previous imprisoner during its first knockout and frees all countries imprisoned by the previous imprisoner. This implies that shifting the imprisoner by $\frac{n-1}{2}$ is always possible. After $\frac{n(n-1)}{2}$ knockouts, the person 0 imprisons people $\frac{n+1}{2}, \frac{n+1}{2}+1, \ldots, n-1$ and all other new people are free. At last, let the country $\frac{n-1}{2}$ imprisoner the person 0 again. After that, every new person has once freed all the $\frac{n-1}{2}$ people immediately following it in the cyclic order. Thus among every two new people, one has freed the other one.

Let now n be even. Denote one new person as C. Apply the programme of knockouts described above for odd n to the set of all other new people, but at the end of the series of knockouts organized by each person, add one more knockout during which this person imprisons also C. During the first knockout of the next imprisoner, the latter frees C, so it will be able to imprison it again during its last knockout. This way, there will be $\frac{(n-1)(n-2)}{2} + 1 + (n-1)$ or, equivalently, $\frac{n(n-1)}{2} + 1$ knockouts in total. After that, among every two new people, one has freed the other one.

Solution 2. For the construction part, we follow solution 1. We prove that there must be at least $\frac{n(n-1)}{2} + 1$ knockouts differently. We show that the sum of the number of currently imprisoned people and the number of liberations having taken place so far (different people freed during one knockout contribute independently) increases by exactly 1 during each knockout. Indeed, let u be the number of imprisoned new people before the knockout, let v be the number of liberations having taken place before the knockout and let k be the number of people freed during the knockout. Then after the knockout, the number of imprisoned new people is $u-k+1$ and the number of liberations having taken place is $v+k$, which is $u+v+1$ in total.

As the number of imprisoned new people and the number of liberations are both zeros at the beginning, the sum of these numbers equals the number of knockouts all the time. If among

every two new people, one of them has freed the other one at least once, then the number of liberations is at least $\frac{n(n-1)}{2}$ and the number of imprisoned new people is at least 1 (the person that has been knocked out lastly is imprisoned), which is $\frac{n(n-1)}{2} + 1$ in total. So there must have been at least $\frac{n(n-1)}{2} + 1$ knockouts.

Problem 7 Let n be an even positive integer. An $n \times n \times n$ cube is composed of n^3 unit cubes. A set of n unit cubes that forms a $1 \times 1 \times n$ box is called a *needle*, which can have three orientations. Find the largest integer K, such that it is possible to select $3K$ needles, K in each orientation, such that no two of them share a unit cube.

Solution. We will prove that the answer is $\mathcal{X} = \frac{n^2}{4}$.

For the construction, consider the $2 \times 2 \times 2$ case and then divide each of these large prisms into $\frac{n^2}{4}$ smaller prisms.

Let $n = 2k$. Consider all $1 \times 2k \times 2k$ prisms (there are $6k$ of them). Call them plates. Each plate can only contain needles of a single orientation. Assuming it is possible to do $k^2 + 1$ needles, we will show that each needle orientation is contained in at least $2k + 1$ plates which will give a contradiction by pigeonhole.

Fix the orientation of a needle. Each needle of this orientation can be contained in only two different orientations of plates, call these orientations c_1 and c_2. Additionally, each needle corresponds to a unique pair of a plate of c_1 and c_2. So if there are p_1 plates of c_1 which contain this orientation needle (similarly p_2) then there are at most $p_1 p_2$ needles and hence:

$$(p_1 + p_2)^2 \geq 4p_1 p_2 \geq 4(k^2 + 1) > 4k^2$$

as needed.

Problem 8 A sequence of non-negative integers $a_1, \ldots, a_n$ is called *self-descriptive* if

$$a_k = |\{i \mid a_i \geq k\}|$$

for each $k \in \{1, \ldots, n\}$, i.e. a_k equals the number of elements of the sequence that are greater than or equal to k, for each $k \in \{1, \ldots, n\}$.
For each positive integer n, determine the number of self-descriptive sequences of length n.

Solution. The answer is 2^n.

All elements of a self-descriptive sequence are integers between 0 and n (inclusive), since they're sizes of sets with at most n elements. We should also notice that a self-descriptive sequence is always decreasing (non-increasing). This is because all numbers that are greater than or equal to $k+1$ are also greater than (or equal to) k, and therefore $a_k \geq a_{k+1}$.

Now, take any self-descriptive sequence $a_1, \ldots, a_n$ and draw an $n \times n$ grid. Number the columns $1, \ldots, k$ from left to right and number the rows $1, \ldots, k$ from bottom to top. For each index k, colour the bottom a_k squares of the k-th column gray, and leave the rest white (Figure 1).

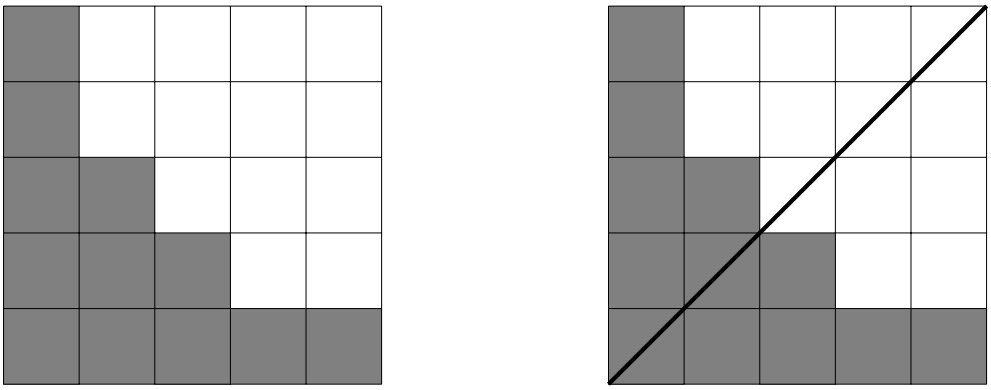


Figure 1: Grid representing the self-descriptive sequence $(5, 3, 2, 1, 1)$. The grid is symmetric across the main diagonal.

Consider any column k with exactly m gray cells. These gray cells are the bottom m cells of the column and $a_k = m$. Since $a_k = m$, there are exactly m numbers in the sequence that have a value greater than or equal to k. As the sequence is decreasing, these are the first m elements of the sequence. Therefore, the first m columns will have their bottom k cells gray (and possibly more above that) and the last $n - m$ columns will have fewer than the bottom k cells gray. Notably, the cell on row k will be gray in the first m columns and white in the last $n - m$ columns. In other words, the leftmost m cells in row k are gray and the rest are white. Since this is true for all indices k, the grid is symmetric across the main (bottom left–top right) diagonal (Figure 1).

It's clear that every distinct self-descriptive sequence gives a unique grid satisfying the previous constraints. The same also works in reverse: every distinct $n \times n$ grid symmetric across the main diagonal, where each column has all its gray cells below its white cells, gives a unique sequence of n non-negative integers, represented by the numbers of gray cells in each column. All of these sequences are self-descriptive, since if row k has exactly m gray cells, then $a_k = m$. Now row k has its leftmost m cells gray and the remaining $n - m$ cells white, and therefore

exactly m of the elements of the sequence are greater than or equal to k. Thus, we have a bijection between self-descriptive sequences and grids.

Now we only need to count the number of such grids. Draw the boundary between the gray and the white cells and connect it to the top left and bottom right corners along the boundary of the grid (Figure 2). Since each row and column switches from gray to white at most once, the boundary will be a down-right path from the top left corner to the bottom right corner. The gray cells are below the boundary and the white cells are above the boundary. The grid is symmetric across the main diagonal, and therefore the boundary is also symmetric. Thus, choosing one half of the boundary (e.g. top left to the main diagonal) will lock in the other half (Figure 2).

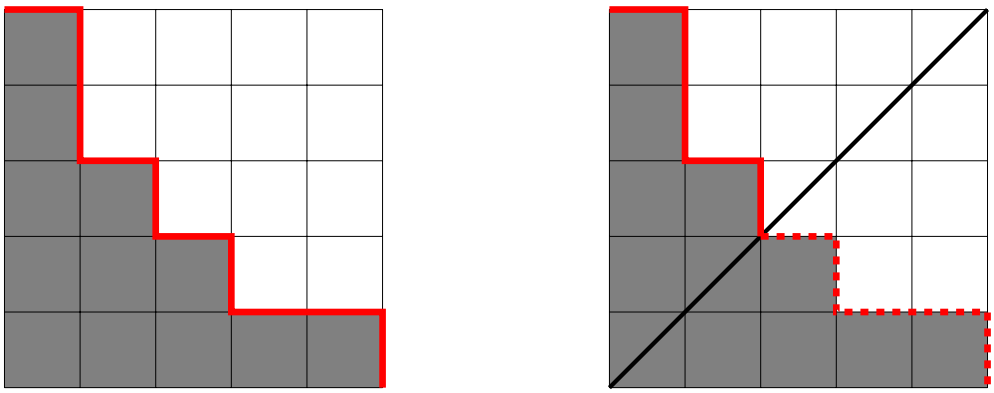


Figure 2: The boundary between the gray and the white cells drawn in red. As the boundary is symmetric across the main diagonal, only one half of it can be chosen freely.

It's clear that every distinct grid satisfying the contstraints gives a unique boundary. The same also works in reverse: every right-down path from the top left to the bottom right, that is symmetric across the main diagonal, drawn as the boundary of the gray and the white cells gives a unique grid satisfying the constraints. Therefore we have a bijection between grids and right-down paths. It's now enough to count the number of such paths. We can freely choose a down-right path from the top left to the main diagonal. In an $n \times n$ grid the path consists of n steps, each having two options: right or down. Therefore the number of such paths is 2^n. This is equal to the number of self-descriptive sequences of length n, which therefore is also 2^n.

Problem 9 A *Baltic polyiamond* is an n-gon with side lengths $n, n-1, \ldots, 2, 1$ in exactly this order, and all internal angles 120° or 240°. Prove that for every Baltic polyiamond, n is divisible by 6.

Solution. We start by proving a lemma.

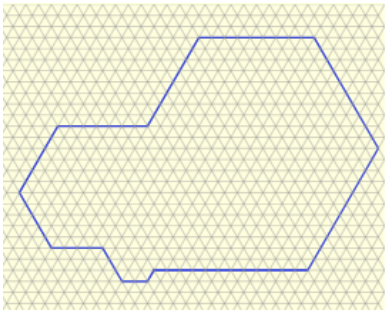
Lemma: If all interior angles of of an n-gon are 120° and 240°, then n is an even number.
Proof: The sum of the interior angles of a polygon is $180° \cdot (n-2)$. For a polygon with a angles equal to 120° and b angles equal to 240°

$$180° \cdot (n-2) = a \cdot 120° + b \cdot 240°$$
$$3(n-2) = 2a + 4b.$$

The right-hand side is an even number, so the left-hand side must also be an even number, so n is even.

Introduce 6 unit vectors $\mathbf{a} = (1;0)$, $\mathbf{b} = (1/2, \sqrt{3}/2)$, $\mathbf{c} = (-1/2, \sqrt{3}/2)$, $\mathbf{d} = (-1, 0)$, $\mathbf{e} = (-1/2, -\sqrt{3}/2)$, $\mathbf{f} = (1/2, -\sqrt{3}/2)$. Every next vector is obtained rotating the previous one by 60° counterclockwise.

As the consecutive side lengths $\{n, n-1, \ldots, 2, 1\}$ are known in advance, each Baltic polyiamond can be specified as a sequence of its side vector directions only.



For example, the polyiamond in the above image is encoded as `ABCDEDEFAFAB`, since its sides (adding up to $(0;0)$) are

$$12\mathbf{a}, 11\mathbf{b}, 10\mathbf{c}, 9\mathbf{d}, 8\mathbf{e}, 7\mathbf{d}, 6\mathbf{e}, 5\mathbf{f}, 4\mathbf{a}, 3\mathbf{f}, 2\mathbf{a}, 1\mathbf{b}.$$

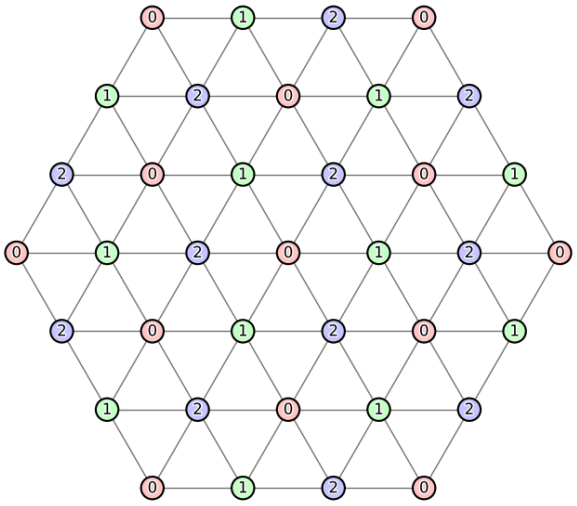
Definition. The vertex where the side of length n meets the side of length 1 will be called the *origin* of the Baltic polyiamond.

Without loss of generality we assume that the longest side has the direction `A` – if necessary, rotate it by a multiple of 60°.

Lemma 2: An encoding of any Baltic polyiamond starting with `A` consists of letter pairs `AB`, `AF`, `CB`, `CD`, `ED`, `EF` in some order. (For example, the 12-gon above can be expressed as pairs like this: `AB.CD.ED.EF.AF.AB`.
Proof: Each of the side directions `A`, `C`, `E` can be followed just by some letter that immediately precedes or immediately follows it. For example, `A` can be followed by `F` or `B` ("slight right" and "slight left" turns), but it cannot be followed by `E` or `C` (which would be "sharp right" and "sharp left" turns causing acute internal angles). `A` cannot be followed by another `A` or by `D` as

then the side does not change the direction at all.
Similarly C can be only followed by B and D and E can only followed by D and F. Each new pair can only start with another letter A, C, E, because the side direction alternates between two sets {A, C, E} and {B, D, F}.

Lemma 3: Color every point in the infinite triangle grid in colors 0, 1 or 2. as shown in the image.



Assume that a Baltic polyiamond is drawn so that the origin is located in a red vertex (color 0). Mark the colors of vertices after traversing every two sides. The colors must follow this sequence:

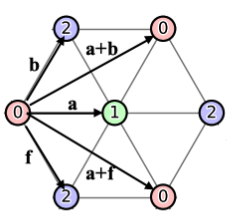
$$0 \to 1 \to 2 \to 0 \to 1 \to 2 \to 0 \to 1 \to 2 \to \dots$$

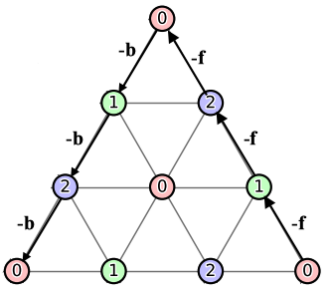
In other words, after traversing every two sides of a Baltic polyiamond having lengths $2k$ and $2k-1$ respectively, color adds 1 (modulo 3).
Proof: According to Lemma 2, each two consecutive sides of lengths $2k$ and $2k-1$ can have any directions AB, AF, CB, CD, ED, EF. We observe that the grid coloring is symmetric against rotations by 120° around any grid point. For this reason it is sufficient to analyze just the directions AB and AF.

Consider the direction AB and assume that the corresponding vector $2k \cdot \mathbf{a} + (2k-1) \cdot \mathbf{b}$ starts from some color $c_k \in \{0, 1, 2\}$. Transform the sum as follows:

$$2k \cdot \mathbf{a} + (2k-1) \cdot \mathbf{b} = 2k(\mathbf{a} + \mathbf{b}) + (-\mathbf{b}).$$

The first vector $2k(\mathbf{a} + \mathbf{b})$ preserves the color c_k, but the second vector adds 1 to the color c_k (modulo 3).





Similarly, assume that the vector $2k \cdot \mathbf{a} + (2k-1) \cdot \mathbf{f}$ starts from color 0. It is a sum of two vectors: $2k(\mathbf{a}+\mathbf{f})$ and $-\mathbf{f}$. In this case $2k(\mathbf{a}+\mathbf{f})$ preserves the color, but $-\mathbf{f}$ advances the color number by 1.

The problem statement follows from Lemma 3: Traversing every 2 sides changes color (adds 1 modulo 3), but after traversing all n sides the polygon must return to its origin having color 0. Therefore the number of side pairs is divisible by 3 and consequently n is divisible by 6.

Problem 10 A positive integer with 9 digits is called *nice* if it contains each digit from 1 to 9. A number is called *wonderful* if it is nice and for each $k = 1, \ldots, 9$, it holds that the k-th digit in the number is equal to the position of k in the number. For example, 847296315 is wonderful:

- the first digit is 8, which is the position of 1 in the number,
- the second digit is 4, which is the position of 2 in the number,
- the third digit is 7, which is the position of 3 in the number, and so on.

Show that the number of wonderful numbers is even.

Solution 1. To any nice number $a_1 a_2 \cdots a_9$ we associate another nice number, $b_1 b_2 \cdots b_9$, called the *sorting number*, uniquely defined by $a_{b_1} a_{b_2} \cdots a_{b_9} = 123456789$. This is well-defined since nice numbers have the same digits as 123456789. Note that a nice number being wonderful is equivalent to the number being its own sorting number.

Let $a_1 \cdots a_9$ be an arbitrary nice number with sorting number $b_1 \cdots b_9$. This means that for every pair of digits $i, j \in \{1, \ldots, 9\}$, we have $b_i = j \iff a_j = i$. By symmetry it follows that $a_1 \cdots a_9$ is the sorting number of $b_1 \cdots b_9$.

Thus all nice numbers that are **not** wonderful can be grouped into pairs of numbers that are each other's sorting number. In particular, there is an even number of nice numbers that are not wonderful. Since the number of nice numbers is even (there are 9! permutations of 123456789), we conclude that the number of wonderful numbers must also be even.

Solution 2. We generalize to *n-wonderful* numbers of length n for $n \in \{1, \ldots, 9\}$. Let T_n denote the number of n-wonderful numbers. We see that $T_1 = 1$ and $T_2 = 2$, corresponding to the numbers $1, 12$ and 21.

Now consider an arbitrary n-wonderful number $A = a_1 \cdots a_n$ for $2 < n \leq 9$.

1. If $a_n = n$, then $a_1 \cdots a_{n-1}$ is an $(n-1)$–wonderful number.
2. If $a_n = k < n$, then we must have $a_k = n$. By removing k and n from the number A and subtracting 1 from each digit greater than k, we obtain an $(n-2)$-wonderful number.

In case (1), the n-wonderful number can be uniquely associated with the $(n-1)$-wonderful number.

In case (2), the corresponding reverse operation is: Take a wonderful number $b_1 \cdots b_{n-2}$ and add 1 to all digits greater than or equal to k, then insert n in the kth position and insert k at the end. As this can be done for each $k \in \{1, \ldots, n-1\}$, there are $n-1$ different n-wonderful numbers that map to the same $(n-2)$-wonderful number.

We conclude that $T_n = T_{n-1} + (n-1)T_{n-2}$. Therefore $T_3 = 2 + 2 = 4$, which is even, and it follows from the recursion that T_n is even for every $2 \leq n \leq 9$, and in particular for $n = 9$.

Problem 11 Let $A_1, \ldots, A_n$ be a sequence of points on the Euclidean plane. Some of the points may coincide, but the sequence contains at least two distinct points. A *Baltic Way* is a route defined by a permutation $B_1, \ldots, B_n$ of the points $A_1, \ldots, A_n$. The length of a Baltic Way is

$$|B_1B_2| + |B_2B_3| + \cdots + |B_{n-1}B_n| + |B_nB_1|,$$

where $|PQ|$ denotes the Euclidean distance between points P and Q.
Let K and L be the lengths of some two Baltic Ways on the same sequence of n points. For a given n, determine the maximum possible value of K/L over all possible sequences $A_1, \ldots, A_n$.

Solution. The answer is $\lfloor n/2 \rfloor$.

Let's number the points such that the order $A_1, \ldots, A_n$ is the Baltic Way of length L. Consider the tour $A_1, \ldots, A_n$ cyclically as a circle, where A_i and A_{i+1} are adjacent and A_n and A_1 are adjacent, and the indices increase when going clockwise.

Now consider some other Baltic Way with the order $A_{i_1}, \ldots, A_{i_n}$. If $i_1 < i_2$, going clockwise around the circle from A_{i_1} to A_{i_2} we get by the triangle inequality that

$$|A_{i_1}A_{i_2}| \leq |A_{i_1}A_{i_1+1}| + \cdots + |A_{i_2-1}A_{i_2}|$$

and by going counterclockwise around the circle we get that

$$|A_{i_1}A_{i_2}| \leq |A_{i_1}A_{i_1-1}| + \cdots + |A_1A_n| + \cdots + |A_{i_2+1}A_{i_2}|.$$

If $i_1 > i_2$, we get that when going clockwise

$$|A_{i_1}A_{i_2}| \leq |A_{i_1}A_{i_1+1}| + \cdots + |A_nA_1| + \cdots + |A_{i_2-1}A_{i_2}|$$

and when going counterclockwise

$$|A_{i_1}A_{i_2}| \leq |A_{i_1}A_{i_1-1}| + \cdots + |A_{i_2+1}A_{i_2}|.$$

Adding the first two inequalities together gives $2|A_{i_1}A_{i_2}| \leq L$ and adding the last two inequalities together also gives $2|A_{i_1}A_{i_2}| \leq L$. Therefore $2|A_{i_1}A_{i_2}| \leq L$ is always true. This works for all pairs of adjacent points (A_{i_k}, A_{i_k+1}): we always have $2|A_{i_k}A_{i_k+1}| \leq L$. Adding together the inequalities of all pairs of points gives $2K \leq nL$.

Now, take all pairs of adjacent points $(A_{i_k}, A_{i_{k+1}})$ and (A_{i_n}, A_{i_1}) on the tour of length K. Adding together the inequalities always going clockwise around the circle gives $K \leq aL$, where a is an integer, since we've ended at the same point we started at. Similarly, adding together all inequalities going around the circle counterclockwise gives $K \leq bL$ with b similarly being an integer. Adding these two inequalities together gives $2K \leq aL + bL$.

We now added together the same inequalities in two different orders and got $2K \leq nL$ and $2K \leq aL + bL$. Since the order of addition is irrelevant, we must have $a + b = n$. Since $K \leq aL$ and $K \leq bL$ hold, then also $K \leq \min(a, b)L$ holds. Since a and b are integers, we must have $\min(a, b) \leq \lfloor n/2 \rfloor$ and therefore $K \leq \lfloor n/2 \rfloor L$ and $K/L \leq \lfloor n/2 \rfloor$.

Let's now construct a set of points with two Baltic Ways that achieves the ratio $K/L \leq \lfloor n/2 \rfloor$.

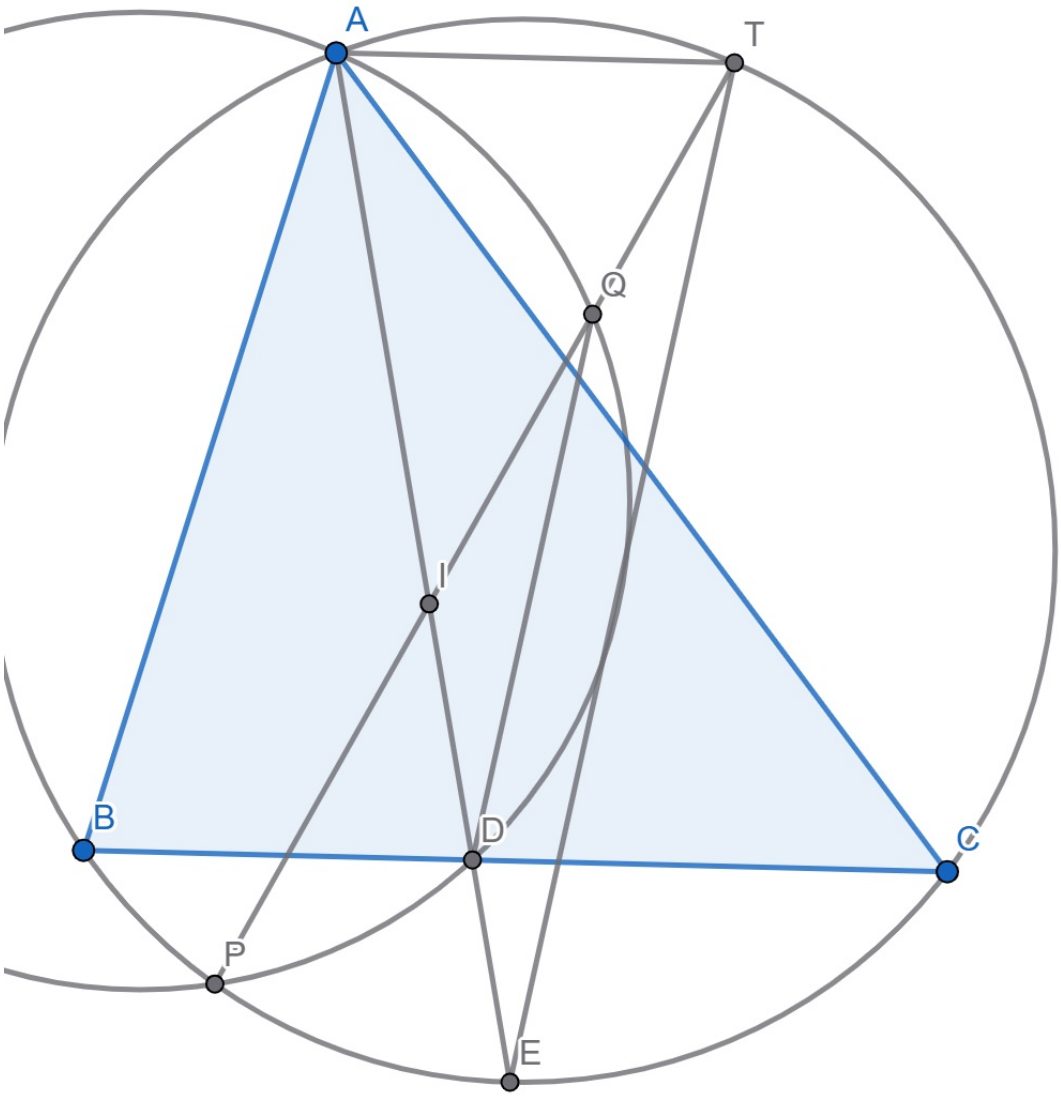
If n is even, we have $n = 2k$. Let $A_1 = \cdots = A_k = (0,0)$ and $A_{k+1} = \cdots = A_n = (0,1)$. Now, $L = 0 + \cdots + 0 + 1 + 0 + \cdots + 0 + 1 = 2$. Let $(i_1, \ldots, i_n) = (1, k+1, 2, k+2, \ldots, k, 2k)$ and therefore $K = 1 + \cdots + 1 = n$ and we get that $K/L = n/2$.

If n is odd, we have $n = 2k+1$. Let $A_1 = \cdots = A_{k+1} = (0,0)$ and $A_{k+2} = \cdots = A_{2k+1} = (0,1)$. Now, $L = 0+\cdots+0+1+0+\cdots+0+1 = 2$. Let $(i_1, \ldots, i_n) = (1, k+1, 2, k+2, \ldots, k+1, 2k, 2k+1)$ and therefore $K = 1 + \cdots + 1 + 0 + 1 = 2k = n - 1$ and we get that $K/L = \frac{n-1}{2} = \lfloor n/2 \rfloor$.

Problem 12 In an acute triangle ABC with $AB < AC$, the incentre is I and the circumcircle is ω. The line AI intersects the side BC at D. Let T be the point on ω such that $AT \parallel BC$. The line TI intersects ω again at P and the circumcircle of triangle APD again at Q. Prove that $AI = DQ$.

Solution Let $E = AI \cap (ABC)$. It follows E lies on AI and $EA = ET$.

We have $DQ \parallel ET$ since $\angle ADQ = \angle APQ = \angle APT = \angle AET$. Therefore triangles IDQ and IET are similar, yielding $\frac{ID}{IE} = \frac{DQ}{ET}$. It is now sufficient to show that $\frac{ID}{IE} = \frac{AI}{AE}$, which after subtracting 1 gives $\frac{ED}{IE} = \frac{EI}{AE}$. The latter rearranges is equivalent to $EI^2 = ED \cdot EA$, which is a well-known fact, since $EI = EB$ and EB is tangent to (ABD).

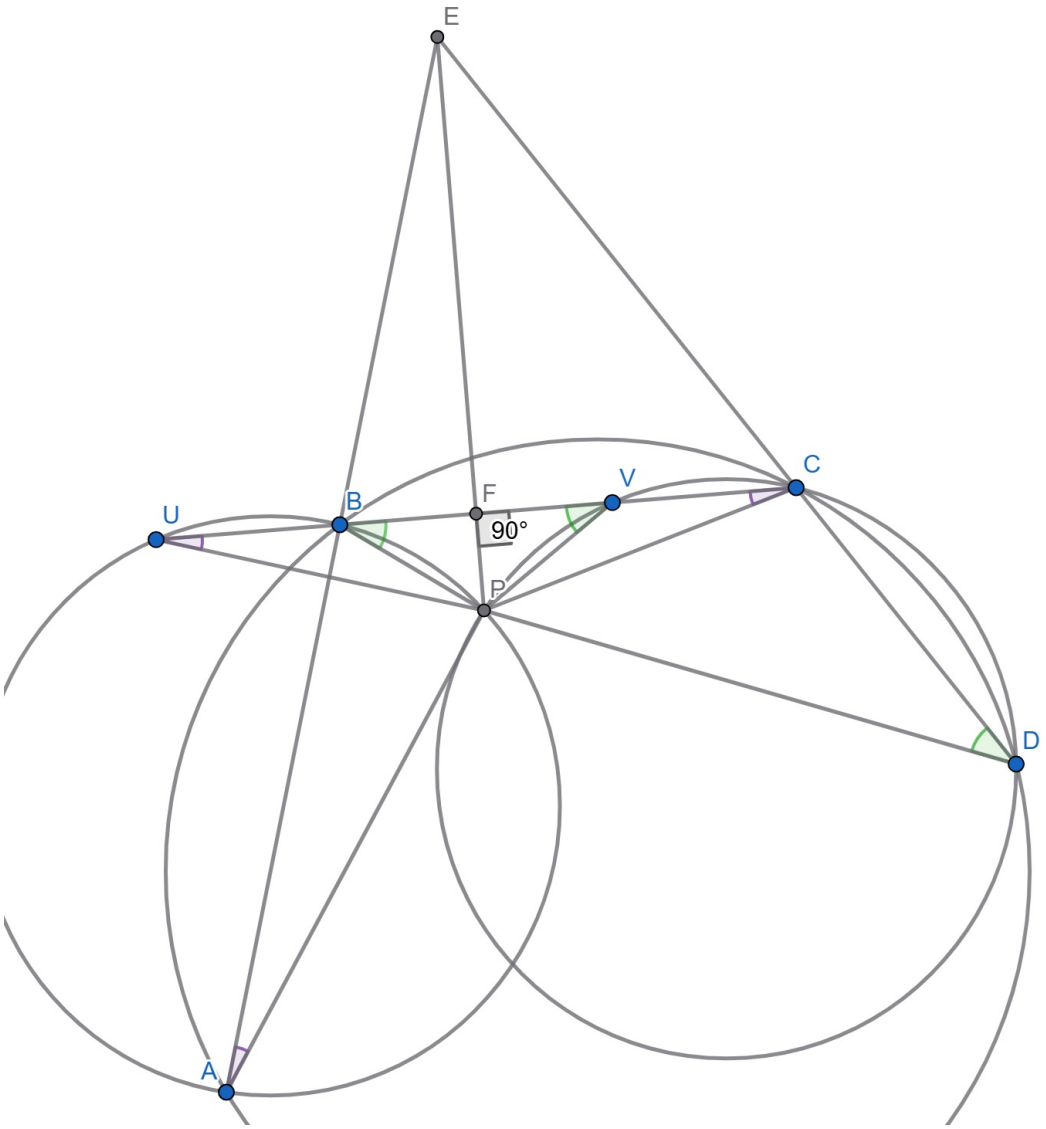


Problem 13 In a cyclic quadrilateral $ABCD$, the lines AB and CD intersect at E. A point P lies inside $ABCD$ and satisfies

$$\angle BAP = \angle PCB \quad \text{and} \quad \angle CBP = \angle PDC.$$

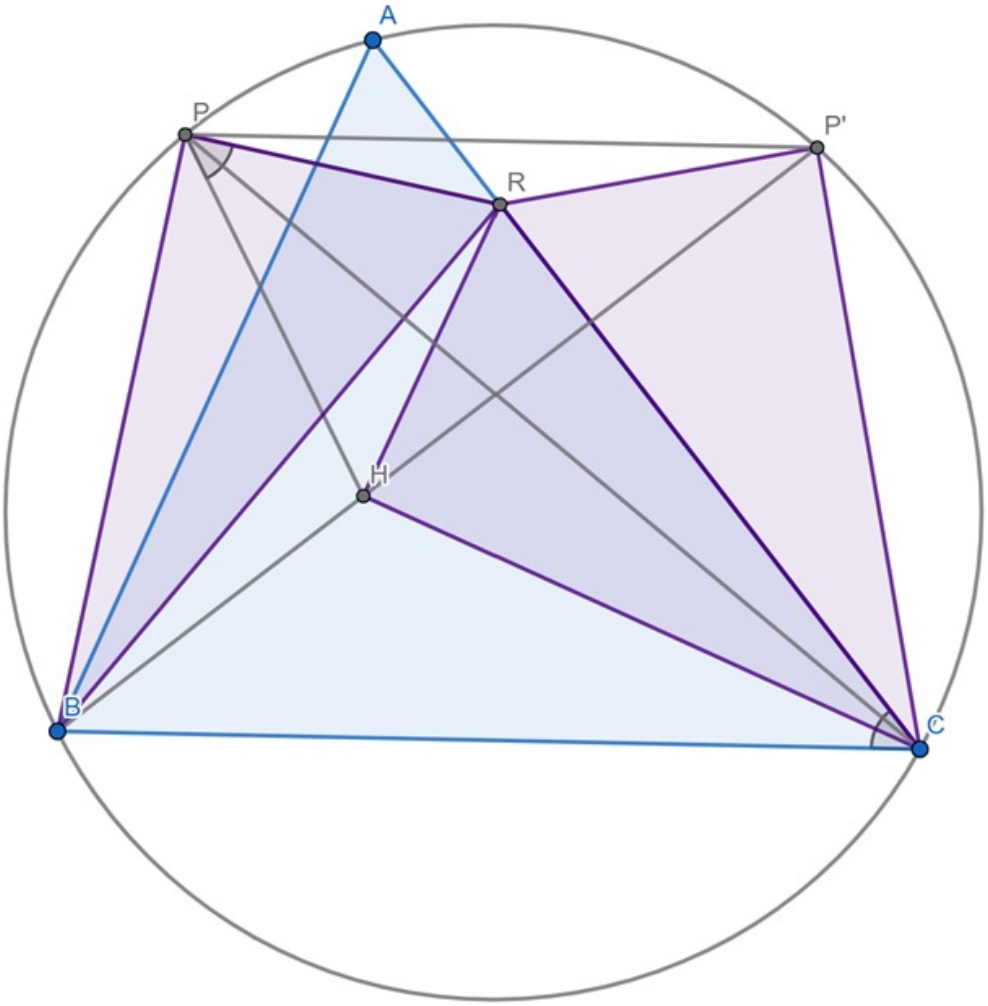
Prove that $PE \perp BC$.

Solution. Let $\Omega_1 = (ABP)$, $\Omega_2 = (CDP)$ and let BC cut Ω_1 and Ω_2 at points $U \neq B$ and $V \neq C$, respectively. We find $\angle CUP = \angle BAP = \angle PCB = \angle PCU$, so $\triangle UPC$ is iscoceles with $PU = PC$. Similarly, $PB = PV$. Let F be the midpoint of BV, which is also the midpoint of CU, and the feet of the perpendicular from P to BC. By symmetry on line BC, F has equal power of point with respect to Ω_1 and Ω_2, hence it belongs to the radical axis of these circles. Since $EB \cdot EA = EC \cdot ED$, E also belongs to this radical axis, and by obvious reasons point P also lies on it, hence P, F and E are collinear and the conclusion follows.



Problem 14 In an acute triangle ABC with $AB < AC$, the circumcircle is ω and the orthocentre is H. The point P on major arc BC of ω satisfies $\angle PCB + \angle ACB = 90°$. The point R on segment AC satisfies $BR = CR$. Prove that $\angle HPR = \angle ACB$.

Solution Let BH intersect ω again in P'. Then $\angle CBP' = 90° - \angle ACB = \angle PCB$, meaning that $PBCP'$ is an isosceles trapezoid. By symmetry, triangles BPR and $CP'R$ are equal, and also triangles CRH and CRP' are symmetrical with respect to AC. Hence there exist rotation mapping $\triangle BRP$ into $\triangle CRH$, hence triangles $\triangle PRH$ and $\triangle CBR$ are similar, meaning that $\angle HPR = \angle ACB$, finishing the proof.

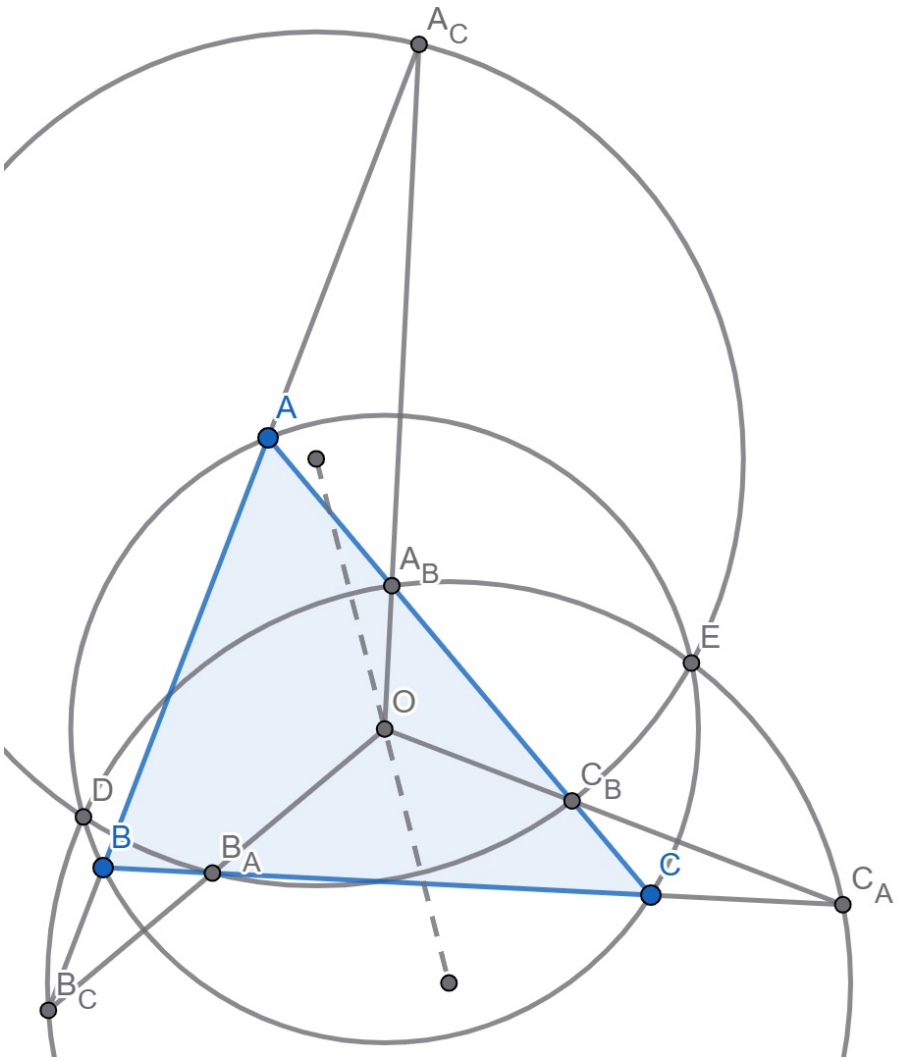


Problem 15 In an acute, scalene triangle ABC, the perpendicular bisector of BC intersects AC and AB in A_b and A_c, respectively. Define B_c, B_a, C_a and C_b similarly. Prove that the circumcentres of ABC, $A_bB_cC_a$, and $A_cB_aC_b$ are collinear.

Solution. Let O and ω denote the circumcentre and circumcircle of ABC, respectively. Without loss of generality, suppose O, A_B and A_C are collinear in this order. Then we have

$$\angle OAA_B = 90^\circ - \angle ABC = \angle AA_CO$$

so $\triangle A_BAO \sim \triangle AA_CO$. Thus, $\frac{OA}{OA_B} = \frac{OA_C}{OA}$, or equivalently, $OA^2 = OA_B \cdot OA_C$, which implies that A_B and A_C are swapped under inversion in ω. By symmetry, the same holds for B_C and B_A, and for C_A and C_B, and so the circumcircles of $A_BB_CC_A$ and $A_CB_AC_B$ are swapped under inversion in ω. This implies the desired result.



Problem 16 Let a, b, c, n be positive integers satisfying $abc = 10^n - 1$. If $S(x)$ denotes the sum of digits of x in decimal notation, prove that

$$S(a) + S(b^2) + S(c^4) \geq \sqrt[3]{243n}.$$

Solution. We will need the following 3 lemmas.

Lemma 1: For any two positive integers p and q, it holds $S(p+q) \leq S(p) + S(q)$.
Proof: In fact, $S(p+q) = S(p) + S(q) - 9\{\#\text{carries when adding } p \text{ and } q\}$, which instantly follows from the addition formula.

Lemma 2: For any positive integers p and q, it holds $S(pq) \leq S(p)S(q)$.
Proof: Let $p = p_m 10^m + p_{m-1}10^{m-1} + \cdots + p_0$, where $0 \leq p_i \leq 9$. Now

$$S(qp) = S(qp_m 10^m + qp_{m-1}10^{m-1} + \cdots + qp_0) \leq S(qp_m 10^m) + S(qp_{m-1}10^m) + \cdots + S(qp_0) =$$

$$= S(qp_m) + S(qp_{m-1}) + \cdots + S(qp_0) \leq \underbrace{S(q) + d(q) + \ldots S(q)}_{p_m} + \cdots + \underbrace{S(q) + S(q) + S(q)}_{p_0} =$$

$$S(q)(p_m + p_{m-1} + \cdots + p_0) = S(q)S(p)$$

Lemma 3: For any positive integer m, any multiple of $10^m - 1$ has sum of digits at least $9m$.
Proof: We will prove this result by induction. Obviously $S(10^m - 1) = 9m$. Let p be a multiple of $10^m - 1$ and assume the inequality holds for all smaller multiples. Write $p = 10^m p_1 + p_0$, where $p_0 \leq 0$. Note that $10^m p_1 + p_0 \equiv p_1 + p_0 (\text{mod } 10^m - 1)$, so $p_1 + p_0$ is divisible by $10^m - 1$ and since $p_1 + p_0 < p$, by induction it follows

$$S(p) = S(p_1) + S(p_0) \geq S(p_1 + p_0) \geq 9m$$

Going back to the original problem, note that ab^2c^4 is a multiple of $abc = 10^n - 1$, so has sum of digits at least $9n$. Now, by AM-GM:

$$S(a) + S(b^2) + S(c^4) \geq 3\sqrt[3]{S(a)S(b^2)S(c^4)} \geq 3\sqrt[3]{S(ab^2c^4)} \geq \sqrt[3]{243n}$$

Problem 17 A positive integer is called *brilliant* if its positive divisors can be partitioned into two sets with an equal number of elements and an equal sum of elements. Find all possible values of the number of positive divisors of a brilliant number.

Solution. The answer is all even numbers starting from 8.

Obviously, the number of divisors of a brilliant number must be even. Moreover, as the number itself must belong to one group, the sum of all proper divisors must be at least as large as the number. Hence primes and prime powers are not brilliant as the divisors of a number p^k are $1, p, \ldots, p^{k-1}, p^k$ and

$$1 + p + \ldots + p^{k-1} = \frac{p^k - 1}{p - 1} \leq p^k - 1 < p^k.$$

Next we show that there are no brilliant numbers with 4 or 6 positive divisors.

- If a brilliant number had 4 divisors, it would be representable as pq for some primes p and q. As one proper divisor should join pq in one group, the sum of members of either group would be at least $pq+1$. Hence $pq+q+p+1 \geq 2(pq+1) = 2pq+2$, or equivalently, $0 \geq pq - p - q + 1 = (p-1)(q-1)$, which is impossible.

- If a brilliant number had 6 divisors, it would be representable as p^2q for some primes p and q. As two distinct proper divisors should join p^2q in one group, the sum of members of either group would be at least p^2q+3. Hence $p^2q+pq+q+p^2+p+1 \geq 2(p^2q+3) = 2p^2q+6$. This inequality is equivalent to

  $$2p^2 - 6 \geq (p^2 - p - 1)(q + 1).$$

  As $q + 1 \geq 3$, this implies the inequality $2p^2 - 6 \geq 3(p^2 - p - 1)$ which is equivalent to $p^2 - 3p + 3 \leq 0$. As the discriminant of the quadratic equation $p^2 - 3p + 3 = 0$ is negative and the leading coefficient is positive, the inequality cannot be satisfied.

Finally we show that all numbers in the form $2^{n-1} \cdot 3$, where $n \geq 4$, are brilliant. Such number has $2n$ positive divisors:

$$1, 2, 4, \ldots, 2^{n-1},$$
$$3 \cdot 1, 3 \cdot 2, 3 \cdot 4, \ldots, 3 \cdot 2^{n-1}.$$

In an expected partition into two groups, either group must contain n divisors and the sum of members of each group must equal the arithmetic mean of the two row sums above. Note that both the desired number of members and the desired sum of members could be achieved by numbers

$$2 \cdot 1, 2 \cdot 2, 2 \cdot 4, \ldots, 2 \cdot 2^{n-1},$$

if all they were among the divisors of $2^{n-1} \cdot 3$. The only one that is not a divisor of $2^{n-1} \cdot 3$ is the last number $2 \cdot 2^{n-1}$, because the others are equal to $2, 4, \ldots, 2^{n-1}$, respectively. Now remove the two largest numbers, i.e., 2^{n-1} and 2^n, and add $3 \cdot 2^{n-1}$. Then only divisors of $2^{n-1} \cdot 3$ remain and also the sum of members is maintained, but the number of members decreases by 1. To achieve the right number of members, replace 4 with 1 and 3. This proves that every number of the form $2n$, where $n \geq 4$, is equal to the number of positive divisors of a brilliant number.

Problem 18 Find all functions $f\colon \mathbb{Z}_{>0} \longrightarrow \mathbb{Z}_{>0}$ such that $f(2) = 1$ and

$$\operatorname{lcm}(f(a+b), f(b)) \mid \operatorname{lcm}(a + f(b), b)$$

for all positive integers a, b.

Solution Answer is

$$f(x) = \begin{cases} 1, \text{ if } x \text{ is odd or } x = 2, \\ 1 \text{ or } 2, \text{ otherwise.} \end{cases}$$

Let $P(a, b)$ be the assertion. Throughout the solution we will be using that $\operatorname{lcm}(a, b) \mid n \implies a \mid n$ and also $\operatorname{lcm}(n, 1) = n$.

From $P(1, 1)$ we get

$$\begin{gathered} \operatorname{lcm}\big(f(2), f(1)\big) \mid \operatorname{lcm}(1 + f(1), 1) = 1 + f(1), \\ f(1) \mid 1 + f(1), \\ f(1) \mid 1 \implies f(1) = 1. \end{gathered}$$

From $P(a, 1)$ it follows that $f(a + 1) \mid a + 1$, so $f(n) \mid n$ for all $n \in \mathbb{Z}^+$. However, from $P(1, a)$ we deduce that

$$f(a + 1) \mid \operatorname{lcm}\big(1 + f(a), a\big).$$

On the other hand from $P(1, a)$ it follows that $f(a+1) \mid \operatorname{lcm}\big(1 + f(a), a\big)$. As $f(a+1) \mid a+1$, $\gcd\big(f(a+1), a\big) = 1$, so $f(a+1) \mid 1 + f(a)$. In particular $f(a+1) \leq 1 + f(a) \leq 2 + f(a-1) \leq \cdots \leq a - 1 + f(2) = a$.

Also note that the only time the function increases, it increases by 1. Therefore if at any point $f(a) = n$, then all of the values $\leq n$ are achieved with $x \leq a$.

We will prove that the only values the function can have is 1 and 2. Assume the contrary, that there exists $a \in \mathbb{Z}^+$ such that $f(a) = 3$ and choose the smallest such a. We know that $a \geq 6$ as $f(a+1) \leq a$ and $f(a) \mid a$ Then $f(a-1) \geq 2$ but $f(a-1) < 3$ as we chose the smallest a. So $f(a-1) = 2$. But as $f(a) \mid a$, we know that $a \mid 3$ and $(a-1) \mid 2$, therefore $(a-3) \mid 6$. But then $\gcd(a - 4, 6) = 1$, also $f(a - 4) \mid (a - 4)$ and $f(a - 4) < 3$. So the only possible option is $f(a-4) = 1$. Considering $P(4, a-4)$ we get $f(a) = 3 \mid \operatorname{lcm}\big(4+f(a-4), f(a-4)\big) = \operatorname{lcm}(5, 1) = 5$ – contradiction.

Therefore f can never have value 3 and so can never have any value ≥ 3. So $f(a) = 1$ or $f(a) = 2$ for each a. $f(a) \mid a$ guarantees that $f(a) = 1$ for odd a. There are no restrictions for even a and in fact it is easy to check that whatever $f(a)$ is with even a, that satisfies the original equations.

Problem 19 There are 100 positive integers written on a blackboard. Anna plays a game. A move consists of choosing two integers a and b on the blackboard with the property that $a \mid b$, erasing them both and writing the integer b/a. She makes moves until there are no more valid moves, at which point the game ends.
Find the largest N such that for some initial state Anna can both:

- finish the game with only one number remaining, and
- finish the game with N numbers remaining.

Solution The answer is $N = 99$.

We solve the problem for the more general case with $n \geq 5$ integers on the blackboard.

Since Anna necessarily can reach a state with 1 integer on the board, he must have a first move. Thus, we have $N \leq n - 1$.

For $N \geq n - 1$ we give the following construction. Let $q, p_1, \ldots, p_{n-2}$ be distinct primes and suppose the integers on the board are $qp_1, p_1, p_1p_2, p_2p_3, \ldots, p_{n-3}p_{n-2}$ and $p_{n-2}p_1$. On the one hand, Anna has the moves

$$(p_1, p_1p_2) \mapsto p_2, \quad (p_2, p_2p_3) \mapsto p_3, \quad \ldots, \quad (p_{n-2}, p_{n-2}p_1) \mapsto p_1, \quad (p_1, qp_1) \mapsto q$$

after which only q is left on the board. On the other hand, Anna has the move

$$(p_1, qp_1) \mapsto q$$

after which the integers on the board are $q, p_1p_2, \ldots, p_{n-3}p_{n-2}, p_{n-2}p_1$, none of which are divisible by each other.

Remark. There might exist some slightly different constructions. For example, take $q, p_1, \ldots, p_{n-2}$ be distinct primes and suppose the integers on the board are

$$qp_1, qp_2, \ldots, qp_{n-2}, q^{n-3}, q^{n-3}p_1p_2 \ldots p_{n-2}.$$

On the one hand, Anna has the moves

$$\begin{aligned}
(qp_1,\, q^{n-3}p_1p_2 \cdots p_{n-2}) &\mapsto q^{n-4}p_2 \cdots p_{n-2}, \\
(qp_2,\, q^{n-4}p_2 \cdots p_{n-2}) &\mapsto q^{n-5}p_3 \cdots p_{n-2}, \\
&\vdots \\
(qp_{n-3},\, qp_{n-3}p_{n-2}) &\mapsto p_{n-2}, \\
(p_{n-2},\, qp_{n-2}) &\mapsto q, \\
(q,\, q^{n-3}) &\mapsto q^{n-4}
\end{aligned}$$

after which only q^{n-4} is left on the board. On the other hand, Anna has the move

$$(q^{n-3}, q^{n-3}p_1p_2 \ldots p_{n-2}) \mapsto p_1p_2 \ldots p_{n-2}$$

after which the integers on the board are $qp_1, qp_2, \ldots, qp_{n-2}, p_1p_2 \ldots p_{n-2}$, none of which are divisible by each other.

Problem 20 Determine all positive integers n such that there exists a permutation $a_1, \ldots, a_n$ of $1, \ldots, n$, where the numbers $a_1, 2a_2, \ldots, na_n$ are all pairwise distinct modulo n.

Solution. The answer is $n = 1$ and $n = 2$.

First, note that $\gcd(x, n) = \gcd(x + kn, n)$ for any $x, k, n \in \mathbb{N}$, $n > 0$. Therefore the sequences

$$(\gcd(1, n), \ldots, \gcd(n, n)), \quad (\gcd(a_1, n), \ldots, \gcd(a_n, n)) \quad \text{and} \quad (\gcd(a_1, n), \gcd(2a_2, n), \ldots, \gcd(na_n, n)$$

are all permutations of each other. Also note that $\gcd(xy, n) \geq \gcd(x, n)$ for all $x, y, n \in \mathbb{N}$, $n > 0$. Therefore

$$\gcd(k, n) \leq \gcd(ka_k, n) \quad \text{and} \quad \gcd(a_k, n) \leq \gcd(ka_k, n).$$

Since the sequences are permutations of each other, they have the same sum, and therefore

$$\gcd(k_n, n) = \gcd(ka_k, n) = \gcd(a_k, n)$$

for all $1 \leq k \leq n$. Leet

$$S_x = \{k \mid \gcd(k, n) = x\}.$$

Since the maps $k \mapsto a_k$ and $k \mapsto ka_k$ modulo n are bijections, and they preserve the gcd with n, the maps are bijections from each S_x to itself.

Now, assume there exists a prime p such that $p^2 \mid n$. Since $\gcd(p, n) = p$, we must have $\gcd(a_p, n) = p$ and $\gcd(pa_p, n) = p$, but since $p \mid a_p$ we have $p^2 \mid pa_p$ and therefore $p^2 \mid \gcd(pa_p, n)$, which is a contradiction. Therefore all primes that divide n divide it at most once.

Now, assume there exists a prime $p > 2$ such that $p \mid n$. Let $q = \frac{n}{p}$ and consider the set S_q. Every element of S_q must be a multiple of q, and therefore $S_q \subseteq \{q, 2q, \ldots, pq\}$. Since $pq = n$, we have $\gcd(pq, n) = n \neq q$ and $pq \notin S_q$. Since q has all prime factors of n except for p, multiplying q with any number from $\{1, \ldots, p-1\}$ will not increase the gcd with n, and therefore $S_q = \{q, 2q, \ldots, (p-1)q\}$. Let $P = q \cdot 2q \cdot \ldots \cdot (p-1)q$. Since $p \nmid q$, we have $p \nmid s$. Therefore also $s = a_q \cdot a_{2q} \cdot \ldots \cdot a_{(p-1)q}$ and

$$\begin{aligned}
s &\equiv qa_q \cdot 2qa_{2q} \cdot \ldots \cdot (p-1)qa_{(p-1)q} && \pmod{n} \\
s &\equiv qa_q \cdot 2qa_{2q} \cdot \ldots \cdot (p-1)qa_{(p-1)q} && \pmod{p} \\
s &\equiv (q \cdot 2q \cdot \ldots \cdot (p-1)q) \cdot (a_q \cdot a_{2q} \cdot \ldots \cdot a_{(p-1)q}) && \pmod{p} \\
s &\equiv (q \cdot 2q \cdot \ldots \cdot (p-1)q) \cdot s && \pmod{p} \\
1 &\equiv q \cdot 2q \cdot \ldots \cdot (p-1)q && \pmod{p} \\
1 &\equiv (p-1)! \cdot q^{p-1} && \pmod{p} \\
1 &\equiv -1 \cdot 1 && \pmod{p} \\
2 &\equiv 0 && \pmod{p}
\end{aligned}$$

and therefore $p \mid 2$, which is a contradiction. Therefore the only prime that may divide n is 2.

If $n = 1$, the condition is trivially satisfied. If $n = 2$, we can choose $(a_1, a_2) = (1, 2)$ and now $a_1 = 1 \equiv 1 \pmod{2}$ and $2a_2 = 4 \equiv 0 \pmod{2}$, so $n = 2$ also works.