

Estonian math competitions 2000/2001

We thank the IMO community for many of these problems which have been taken from various materials distributed at the recent IMO-s.

Autumn Open Contest: October 2000

Juniors (up to 10th grade)

- How many positive integers less than 20002001 and not containing digits other than 0 and 2 are there?
- Find the two last digits of the number $1! + 2! + 3! + \dots + 2000!$.
- Consider points C_1, C_2 on the side AB of a triangle ABC , points A_1, A_2 on the side BC and points B_1, B_2 on the side CA such that these points divide the corresponding sides to three equal parts. It is known that all the points A_1, A_2, B_1, B_2, C_1 and C_2 are concyclic. Prove that triangle ABC is equilateral.
- Real numbers x and y satisfy the system of equations

$$\begin{cases} x + y + \frac{x}{y} = 10 \\ \frac{x(x+y)}{y} = 20 \end{cases}.$$

Find the sum of all possible values of the expression $x + y$.

- Let $m^* = m + 3$ for any odd integer m and $m^* = \frac{m}{2}$ for any even integer m .
 - Find all integers k such that $k^{***} = 1$.
 - Prove that, for every odd integer K , there exist precisely three different integers k such that $k^{***} = K$.
 - How many different integers k with the property $k^{***} = K$ exist for an even integer K ?

Seniors (grades 11 and 12)

- Points A, B, C, D, E and F are given on a circle in such a way that the three chords AB, CD and EF intersect in one point. Express angle EFA in terms of angles ABC and CDE (find all possibilities).

- Find the largest real number K having the following property: for any positive real numbers a, b, c satisfying the inequality $a + b + c \leq K$, the inequality $abc \leq K$ also holds.
- Prove that, for any integer $n \geq 0$, the number $\underbrace{11\dots1}_{3^n \text{ digits}}$ is divisible by 3^n , but is not divisible by 3^{n+1} .
- The terms of the sequence a_1, a_2, a_3, \dots satisfy the condition $a_n = a_{n-1} - a_{n-2}$ for any $n \geq 3$. Find the sum of the first 2000 terms of this sequence, if the sum of the first 1997 terms is 2002 and the sum of the first 2002 terms is 1997.
- On a plane n points are given, no three of them collinear. At most how many line segments it is possible to draw between these points in such a way that the line segments form no triangle with vertices at the given points?

Solutions of Autumn Open Contest

J1. *Answer:* 136.

The set of integers under consideration consists of all integers with up to 7 digits containing only digits 0 and 2, all 8-digit integers of the form 20000*** and the integer 20002000. There are 2^{k-1} integers with exactly k digits 0 and 2, and 2^3 integers of the form 20000***. So the required number of integers is

$$(2^0 + 2^1 + \dots + 2^6) + 8 + 1 = (2^7 - 1) + 9 = 136.$$

J2. *Answer:* 13.

The product $1 \cdot 2 \cdot \dots \cdot 10$ has 2, 5 and 10 as factors, therefore being divisible by 100. Hence the last two digits of $n!$ are zeros for any $n \geq 10$ and it suffices to find two last digits of $1! + 2! + \dots + 9!$. The two last digits of the summands are 01, 02, 06, 24, 20, 20, 40, 20 and 80, yielding 13 as the answer.

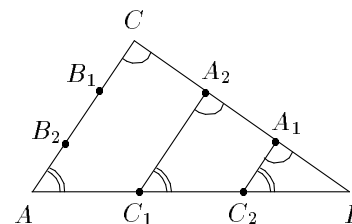


Figure 1

- Label the points on the sides of the triangle so that $|AC_1| = |C_1C_2| = |C_2B|$, $|BA_1| = |A_1A_2| = |A_2C|$ and $|CB_1| = |B_1B_2| = |B_2A|$ (see Fig. 1). Then we have

$\angle BA_1C_2 = \angle BA_2C_1 = \angle BCA$ and $\angle BC_2A_1 = \angle BC_1A_2 = \angle BAC$. Since points A_1, A_2, C_1 and C_2 are concyclic, we get $\angle BA_2C_1 = 180^\circ - \angle AC_2A_1 = \angle BC_2A_1$, which gives $\angle BCA = \angle BAC$. The equality $\angle BAC = \angle CBA$ follows by symmetry.

J4. *Answer:* 10.

By Viète's theorem, the possible values of $x + y$ are included in the set of roots of the quadratic equation

$$a^2 - 10a + 20 = 0.$$

This equation has two different roots because $D = 10^2 - 4 \cdot 20 > 0$. Viète's formulae give 10 to be the sum of these roots. It remains to check that 11 is not among the roots (as $y = \frac{x+y}{11-(x+y)}$ from the first equation, $x+y \neq 11$ enables us to find the corresponding values for x and y).

J5. *Answer:* a) 1, -2 and 8; c) 5.

a), b) Observe that if m^* is odd, then both m and m^{**} are even. Hence if $K = k^{***}$ is odd, then $k^{**} = 2K$, and k and k^* are not both odd. This gives the following three possibilities.

- 1) If both k and k^* are even, then $k = 2k^* = 4k^{**} = 8K$.
- 2) If k is odd and k^* is even, then $k = k^* - 3 = (2k^{**} - 3) = 4K - 3$.
- 3) If k is even and k^* is odd, then $k = 2k^* = 2 \cdot (k^{**} - 3) = 2 \cdot (2K - 3) = 4K - 6$.

The numbers $8K$, $4K - 3$ and $4K - 6$ are pairwise distinct since modulo 4 they are congruent to 0, 1 and 2, respectively. For a), $K = 1$ gives $k \in \{1, -2, 8\}$.

c) Let now K be even. If k^{**} is even, then we get the same three possibilities for k as above. If k^{**} is odd, then k^* is even and k can be either even or odd.

- 4) If k is even, then $k = 2k^* = 4k^{**} = 4(K - 3) = 4K - 12$.
- 5) If k is odd, then $k = k^* - 3 = 2k^{**} - 3 = 2(K - 3) - 3 = 2K - 9$.

Since K is even, the numbers $8K$, $4K - 3$, $4K - 6$ and $4K - 12$ are congruent to 0, 5, 2 and 4, respectively, modulo 8. Moreover, $2K - 9$ is congruent to either 3 or 7 modulo 8. Hence these five numbers are pairwise distinct.

S1. *Answer:* Angle EFA is equal to either $\angle ABC + \angle CDE$, or $\angle ABC - \angle CDE$, or $\angle CDE - \angle ABC$, or $180^\circ - \angle ABC - \angle CDE$.

Given the chords AB and CD , the chord EF can be drawn in four essentially different ways — point E can lie on the circle between points D and A , between points A and C , between points C and B or between points B and D (see Fig. 2).

Let us find $\angle EFA$ for case (c). Since EFC and CDE are angles subtended by the same chord EC , we have $\angle EFC = \angle CDE$; similarly $\angle CFA = \angle ABC$. Hence

$$\angle EFA = \angle CFA + \angle EFC = \angle ABC + \angle CDE.$$

In cases (a) and (b) similar arguments give $\angle EFA = \angle CDE - \angle ABC$ and $\angle EFA = \angle ABC - \angle CDE$, respectively.

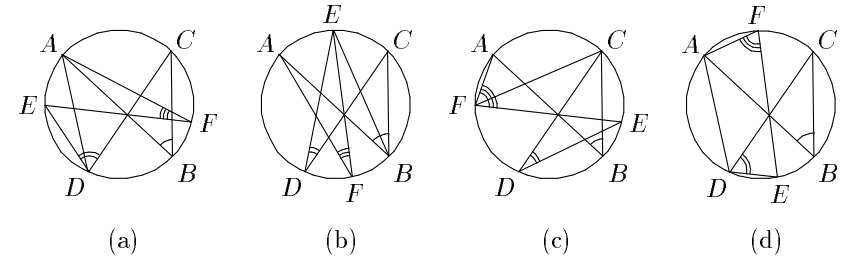


Figure 2

Consider case (d). Since EFA and ADE are opposite angles of a cyclic quadrilateral $ADEF$ and $\angle ADE = \angle ADC + \angle CDE = \angle ABC + \angle CDE$, we have

$$\angle EFA = 180^\circ - \angle ADE = 180^\circ - \angle ABC - \angle CDE.$$

S2. *Answer:* $3\sqrt{3}$.

Let $a + b + c \leq K$. By the AM-GM inequality we have

$$abc \leq \left(\frac{a+b+c}{3}\right)^3 \leq \left(\frac{K}{3}\right)^3 = K \cdot \frac{K^2}{27}.$$

Hence if $\frac{K^2}{27} \leq 1$, or equivalently $K \leq 3\sqrt{3}$, the required condition is satisfied. However, if $K > 3\sqrt{3}$ and $a = b = c = \frac{K}{3}$, then $a + b + c = K$ and

$$abc = K \cdot \frac{K^2}{27} > K, \text{ so the condition is not satisfied.}$$

S3. We use induction on n .

Base: The proposition holds for $n = 0$ since 1 is divisible by $3^0 = 1$ and is not divisible by $3^1 = 3$.

Step: Observe the equality

$$\underbrace{11 \dots 1}_{3^{n+1}} = \underbrace{11 \dots 1}_{3^n} \cdot \underbrace{100 \dots 0}_{3^{n-1}} \underbrace{100 \dots 0}_{3^{n-1}} 1.$$

The first factor here is divisible by 3^n but not by 3^{n+1} by the induction hypothesis, and the second factor is divisible by 3 but not by 9. Since 3 is prime, this implies that the product is divisible by 3^{n+1} , but not by 3^{n+2} .

S4. *Answer:* -2012 .

Spring Open Contest: March 2001

Juniors (up to 10th grade)

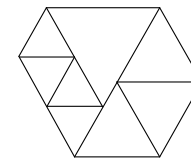
- Eight students, Anne, Mary, Cathy and Tina, Anthony, Mark, Carl and Tom have to work in four pairs, one boy and one girl in each pair. They know each other, with only these exceptions: Anthony knows neither Anne nor Mary; Mark doesn't know Mary and both Carl and Tom know neither Cathy nor Tina. How many ways are there to divide the students into pairs, so that each boy could work with a girl he knows?
- In a triangle ABC , the lengths of the sides are consecutive integers and median drawn from A is perpendicular to the bisector drawn from B . Find the lengths of the sides of triangle ABC .

- In a school locker room there are 60 lockers in three rows. The lockers in each row are labelled from left to right with numbers 1 to 20 in the top row, 21 to 40 in the middle row and 41 to 60 in the bottom row.

					K		
						L	
M							

Kate's, Lisa's and Mary's lockers are located as shown in the figure. Each of the three locker numbers is divisible by the number of Mary's house, which is not 1.

- What is the number of Mary's house?
 - What could be the numbers on the girls' lockers?
- Integers a , b , c and d satisfy $|ac + bd| = |ad + bc| = 1$. Prove that either $|a| = |b| = 1$ or $|c| = |d| = 1$.
 - A convex hexagon is constructed from n pieces, each of which is an equilateral triangle (one example is given in the figure).
 - Prove that the hexagon is equiangular.
 - Find all possible values of n .



Seniors (11th and 12th grade)

- The serial numbers of lottery tickets are 7-digit integers. It is known that the serial number of a winning ticket has seven distinct digits and is divisible by each of its digits.
 - Prove that the serial numbers of all winning tickets consist of the same digits.
 - Find the largest possible serial number of a winning ticket.

Denote $a_1 = p$ and $a_2 = q$. It is easy to see that

$$\begin{aligned} a_k &= p \text{ if } k = 1, 7, 13, \dots; \\ a_k &= q \text{ if } k = 2, 8, 14, \dots; \\ a_k &= q - p \text{ if } k = 3, 9, 15, \dots; \\ a_k &= -p \text{ if } k = 4, 10, 16, \dots; \\ a_k &= -q \text{ if } k = 5, 11, 17, \dots; \\ a_k &= p - q \text{ if } k = 6, 12, 18, \dots \end{aligned}$$

Observe that the sum of any six consecutive members of the sequence is equal to zero. Denoting $S_k = a_1 + \dots + a_k$, we get

$$\begin{aligned} S_k &= p \text{ if } k = 1, 7, 13, \dots; \\ S_k &= p + q \text{ if } k = 2, 8, 14, \dots; \\ S_k &= 2q \text{ if } k = 3, 9, 15, \dots; \\ S_k &= 2q - p \text{ if } k = 4, 10, 16, \dots; \\ S_k &= q - p \text{ if } k = 5, 11, 17, \dots; \\ S_k &= 0 \text{ if } k = 6, 12, 18, \dots \end{aligned}$$

Hence $q - p = S_{1997} = 2002$ and $2q - p = S_{2002} = 1997$, which give $q = -5$ and $p = -2007$ with $S_{2000} = p + q = -2012$.

- S5. *Answer:* $\frac{n^2}{4}$ for even n and $\frac{n^2 - 1}{4}$ for odd n .

Divide the points into two subsets with cardinalities as close to each other as possible, and draw a line segment between any two points from different subsets. Then each closed line formed by these line segments contains an even number of links and hence the line segments do not form any triangles with vertices at the given points. The number of line segments is $\left(\frac{n}{2}\right)^2 = \frac{n^2}{4}$ for even n and

$$\frac{n-1}{2} \cdot \frac{n+1}{2} = \frac{n^2-1}{4} \text{ for odd } n.$$

Now prove that there cannot be more line segments. Consider any collection of line segments satisfying the conditions of the problem. Let m be the maximal number of line segments incident to one point, and let X be any point incident to m line segments. Let A be the set of the other endpoints of these m line segments, and B be the set of the other $n - m$ points (including X). Each point of A can be joined only to points of B because any two joined points from A together with X would form a triangle. Hence each of the m points of A occurs as an endpoint for at most $n - m$ line segments. On the other hand, each of the $n - m$ points from B occurs as an endpoint for at most m line segments by the choice of m . So there is at most $m(n - m) + (n - m)m = 2m(n - m)$ segment-endpoint pairs, and since every line segment has two endpoints, we have at most $m(n - m)$ line segments. It remains to notice that this expression achieves its maximum when the difference of m and $n - m$ is as small as possible, i.e. if $m = n - m$ for even n and if $|m - (n - m)| = 1$ for odd n .

2. Let us call a convex hexagon $ABCDEF$ boring if $\angle A + \angle C + \angle E = \angle B + \angle D + \angle F$.
 - a) Is every cyclic hexagon boring?
 - b) Is every boring hexagon cyclic?
3. Find all real-valued functions $f(x)$ defined for all real numbers which satisfy the condition $f(2001x + f(0)) = 2001x^2$ for each real x .
4. For some $0 < x, y < \pi$, two of the three expressions $\sin^2 x + \sin^2 y$, $\sin^2(x + y)$ and 1 have equal values and the third one is different.
 - a) Which of the three expressions has a different value?
 - b) Give an example of x and y for which such a situation occurs.
5. There are 10 small boxes numbered from 1 to 10, and one large box. John puts some balls in some (or all) of the small boxes, and starts relocating them by the following rules:
 - during each move, John removes all balls from any small box numbered n where the number of the balls equals n ;
 - he adds these balls into boxes 1 to $n - 1$ (one ball into each box) and puts the remaining ball into the large box.

He continues this way until he cannot make another move according to these rules. Find the largest possible total number of balls in the small boxes at the beginning of the game, for which it is possible to put all balls in the large box by the end of the game.

Solutions of Spring Open Contest

J1. *Answer:* 4.

It is clear that Carl and Tom can only work with Anne and Mary: we obtain two ways to form two pairs. Now, Anthony and Mark have to work with Cathy and Tina, whom they both know: there are also two ways to form the two remaining pairs. Hence, altogether there are four ways to form the pairs.

J2. *Answer:* 2, 3 and 4.

Let D be the midpoint of BC , then the median is AD . Since the bisector of $\angle B$ is also an altitude in the triangle ABD , that triangle is equilateral, i.e. $|BD| = |BA|$ and in the original triangle ABC we have $|BC| = 2|AB|$. Since the lengths of the sides of triangle ABC are consecutive integers, the difference $|BC| - |AB|$ is either 1 or 2. In the first case $|AB| = 1$, $|BC| = 2$ and the length of the side AC must be either 0 or 3, which is impossible. In the second case we obtain $|AB| = 2$, $|BC| = 4$ and $|AC| = 3$.

J3. *Answer:* a) 7; b) 7, 28, 42 or 14, 35, 49.

From the figure we obtain $L = K + 21$ and $M = L + 14 = K + 35$. Since K , L and M are all divisible by the number of Mary's house n , the differences $L - K = 21$ and $M - L = 14$ are also divisible by n . It follows that the only possible value of n is 7. Now, since $1 \leq K \leq 20$, we obtain $K = 7$ or $K = 14$, and hence $L = 28$ and $M = 42$ or $L = 35$ and $M = 49$.

- J4. If the numbers $ac + bd$ and $ad + bc$ have the same sign, then $ac + bd = ad + bc$ and $0 = ac + bd - ad - bc = (a - b)(c - d)$. Hence $a = b$ or $c = d$. If the numbers $ac + bd$ and $ad + bc$ have distinct signs, then $0 = ac + bd + ad + bc = (a + b)(c + d)$ and $a = -b$ or $c = -d$. In both cases $|a| = |b|$ or $|c| = |d|$. If $|a| = |b|$, then $1 = |ac + bd|$ is divisible by $|a|$, therefore $|a| = |b| = 1$. In case $|c| = |d|$, we similarly obtain $|c| = |d| = 1$.
- J5. *Answer:* b) all integers $n \geq 6$.

a) Let a vertex of the hexagon be the vertex of k triangles. Then the interior angle at this vertex must be $k \cdot 60^\circ$. Since the interior angles of a convex hexagon are less than 180° , none of them can be larger than 120° . Since the sum of the angles is $720^\circ = 6 \cdot 120^\circ$, it follows that all the angles are equal to 120° .

b) There must be at least 6 pieces, because there is at least one triangle on each side of the hexagon, and since the hexagon has no angles equal to 60° , each triangle can lie on only one side of the hexagon.

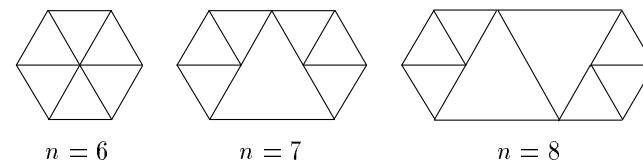


Figure 3

The constructions for $n = 6$, $n = 7$ ja $n = 8$ are given in Figure 3. We can continue the same way, adding more large triangles in the middle.

S1. *Answer:* b) 9867312.

a) Obviously, the serial number cannot contain 0 and must be even. Now, we cannot have 5 among the digits, because every even number divisible by 5 ends with a 0. If the serial number didn't have 9 as one of its digits, it would contain 3 and should be divisible by 3, but the sum of the remaining seven digits is 31, contradiction. Hence 9 is one of the digits. Now, the serial number is divisible by 9 and the sum of its digits is between 32 and 39. The only possible sum is 36 and the serial number consists of the digits 1, 2, 3, 6, 7, 8, 9.

b) Any 7-digit number consisting of these digits is divisible by 1, 3, 6 and 9. Now we must find the largest possible even number among these which is divisible by $7 \cdot 8 = 56$. This number is 9867312.

S2. *Answer:* a) yes, b) no.

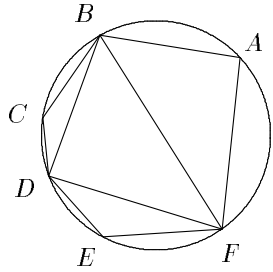


Figure 4

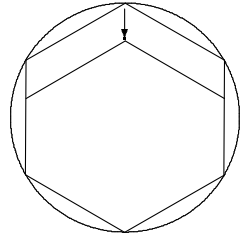


Figure 5

a) Let $ABCDEF$ be a cyclic hexagon. Since the quadrilaterals $ABDF$, $CDFB$ and $EFBD$ are also cyclic (see Fig. 4), we obtain

$$\angle BDF = \pi - \angle A, \quad \angle DFB = \pi - \angle C, \quad \angle FBD = \pi - \angle E.$$

Now, $(\pi - \angle A) + (\pi - \angle C) + (\pi - \angle E) = \pi$, and hence $\angle A + \angle C + \angle E = 2\pi$. Thus $\angle B + \angle D + \angle F = 4\pi - 2\pi = 2\pi = \angle A + \angle C + \angle E$, which proves that the hexagon $ABCDEF$ is boring.

b) Let us compress a regular hexagon along its two opposite sides (see Fig. 5). The new hexagon is boring since all its angles are equal, but it is not cyclic since three of its vertices lie on one circle and the rest on another circle.

S3. *Answer.* $f(x) = \frac{x^2}{2001}$ and $f(x) = \frac{(x - 2001)^2}{2001}$.

Since for any real number y there is an $x = \frac{y - f(0)}{2001}$ such that $y = 2001x + f(0)$,

the equality $f(y) = 2001 \cdot \left(\frac{y - f(0)}{2001}\right)^2$ holds for any real y . Taking $y = 0$ we

get $f(0) = \frac{(f(0))^2}{2001}$ and hence $f(0) = 0$ or $f(0) = 2001$. Therefore the function

must be either $f(y) = \frac{y^2}{2001}$ or $f(y) = \frac{(y - 2001)^2}{2001}$. It is easy to check that both

of these satisfy the given conditions.

S4. *Answer:* a) $\sin^2(x + y)$ can be the only expression with a different value; b) choose $0 < x < \frac{\pi}{2}$ arbitrarily and take $y = x + \frac{\pi}{2}$.

a) Let $\sin^2(x + y) = 1$. We shall prove that in this case also $\sin^2 x + \sin^2 y = 1$. From the equation $\sin^2(x + y) = 1$ we have either $x + y = \frac{\pi}{2}$ or $x + y = \frac{3\pi}{2}$.

Since $\sin\left(\frac{\pi}{2} - x\right) = \cos x$ and $\sin\left(\frac{3\pi}{2} - x\right) = \sin\left(\pi + \left(\frac{\pi}{2} - x\right)\right) = -\cos x$, then

in both cases $\sin^2 x + \sin^2 y = \sin^2 x + \cos^2 x = 1$.

Let $\sin^2 x + \sin^2 y = \sin^2(x + y)$. We shall prove that both of these are equal to 1. Applying the formula for $\sin(x + y)$ and squaring, we get

$$\sin^2 x + \sin^2 y = \sin^2 x \cos^2 y + \sin^2 y \cos^2 x + 2 \sin x \sin y \cos x \cos y,$$

$$\sin^2 x (1 - \cos^2 y) + \sin^2 y (1 - \cos^2 x) - 2 \sin x \sin y \cos x \cos y = 0,$$

$$2 \sin^2 x \sin^2 y - 2 \sin x \sin y \cos x \cos y = 0.$$

Since $0 < x, y < \pi$, neither $\sin x$ nor $\sin y$ equals 0. Hence we must have $\sin x \sin y - \cos x \cos y = 0$, i.e. $\cos(x + y) = 0$ and $\sin(x + y)$ equals to either 1 or -1 , whence $\sin^2(x + y) = 1$. So the only expression that can have a different value is $\sin^2(x + y)$.

b) Taking $0 < x < \frac{\pi}{2}$ and $y = x + \frac{\pi}{2}$, we get

$$\sin^2 x + \sin^2 y = \sin^2 x + \sin^2\left(x + \frac{\pi}{2}\right) = \sin^2 x + \cos^2 x = 1.$$

Since $\frac{\pi}{2} < y < x + y = \frac{\pi}{2} + 2x < \frac{3\pi}{2}$, we have $\sin^2(x + y) \neq 1$.

S5. *Answer:* 41.

We shall first show that the total number of balls in the small boxes cannot exceed 41. John can empty box number 10 only once, since no balls are put into it during the relocations. He can also empty box 9 only once, since at most one ball is put into it (from the tenth box). Also, boxes 8, 7 and 6 can be emptied only once. Box number 5 can be emptied at most twice (at most 5 balls will be added to it from boxes with bigger numbers). Box 4 can also be emptied at most twice, box 3 at most 4 times, box 2 at most 7 times and box 1 at most 21 times. John can therefore put no more than 41 balls in the large box.

We shall now find a way to place 41 balls in the small boxes, so that all boxes could be emptied. No balls are added to box 10, therefore it must contain 10 balls. One ball will be added to box 9, therefore it must contain 8 balls in the beginning. Similarly there must be 6, 4 and 2 balls in boxes 8, 7 and 6 respectively. Since 5 balls are added to box 5, it must contain 5 balls in the beginning in order to be emptied twice. Box 4 must contain one ball, box 3 three balls, boxes 2 and 1 must contain 1 ball. The number of balls in the small boxes is now 41. It is easy to check that if John always empties the box with the smallest possible number, all balls will finally be in the large box.

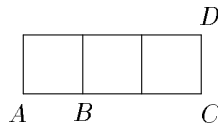
Final Round of National Olympiad: March 2001

9th grade

- John had to solve a math problem in the class. While cleaning the blackboard, he accidentally erased a part of his problem as well: the text that remained on board was $37 \cdot (72 + 3x) = 14**45$, where $*$ marks an erased digit. Show that John can still solve his problem, knowing that x is an integer.
- Dividing a three-digit number by the number obtained from it by swapping its first and last digit we get 3 as the quotient and the sum of digits of the original number as the remainder. Find all three-digit numbers with this property.
- A circle of radius 10 is tangent to two adjacent sides of a square and intersects its two remaining sides at the endpoints of a diameter of the circle. Find the side length of the square.
- It is known that the equation $|x - 1| + |x - 2| + \dots + |x - 2001| = a$ has exactly one solution. Find a .
- A table consisting of 9 rows and 2001 columns is filled with integers 1, 2, ..., 2001 in such a way that each of these integers occurs in the table exactly 9 times and the integers in any column differ by no more than 3. Find the maximum possible value of the minimal column sum (sum of the numbers in one column).

10th grade

- A convex n -gon has exactly three obtuse interior angles. Find all possible values of n .
- Find the minimum value of n such that, among any n integers, there are three whose sum is divisible by 3.
- There are three squares in the picture. Find the sum of angles ADC and BDC .



- We call a triple of positive integers (a, b, c) *harmonic* if $\frac{1}{a} + \frac{1}{b} = \frac{1}{c}$. Prove that, for any given positive integer c , the number of harmonic triples (a, b, c) is equal to the number of positive divisors of c^2 .
- A tribe called Ababab uses only letters A and B, and they create words according to the following rules:

- A is a word;
- if w is a word, then ww and $w\bar{w}$ are also words, where \bar{w} is obtained from w by replacing all letters A with B and all letters B with A (xy denotes the concatenation of x and y);
- all words are created by rules (1) and (2).

Prove that any two words with the same number of letters differ exactly in half of their letters.

11th grade

- The angles of a convex n -gon are $\alpha, 2\alpha, \dots, n\alpha$. Find all possible values of n and the corresponding values of α .
- A student wrote a correct addition operation $\frac{A}{B} + \frac{C}{D} = \frac{E}{F}$ to the blackboard, such that both summands are irreducible fractions and F is the least common multiple of B and D . After that, the student reduced the obtained sum $\frac{E}{F}$ correctly by an integer d . Prove that d is a common divisor of B and D .
- Points D, E and F are taken on the sides BC, CA, AB of a triangle ABC , respectively, so that the segments AD, BE and CF have a common point O . Prove that $\frac{|AO|}{|OD|} = \frac{|AE|}{|EC|} + \frac{|AF|}{|FB|}$.
- Let x and y be non-negative real numbers such that $x + y = 2$. Prove that $x^2 y^2 (x^2 + y^2) \leq 2$.
- Consider all trapezoids in a coordinate plane with interior angles of $90^\circ, 90^\circ, 45^\circ$ and 135° , such that their bases are parallel to one of the coordinate axes and all vertices have integer coordinates. Define the *size* of such a trapezoid as the total number of points with integer coordinates inside and on the boundary of the trapezoid.
 - How many pairwise non-congruent such trapezoids of size 2001 are there?
 - Find all positive integers not greater than 50 that do not appear as sizes of any such trapezoid.

12th grade

- Solve the system of equations

$$\begin{cases} \sin x = y \\ \sin y = x \end{cases} .$$

2. Find the maximum value of k for which one can choose k integers out of $1, 2, \dots, 2n$ so that none of the chosen integers is divisible by any other chosen integer.
3. Let I and r be the midpoint and radius of the incircle of a right-angled triangle ABC with the right angle at C . Rays AI and BI intersect the sides BC and AC at points D and E , respectively. Prove that $\frac{1}{|AE|} + \frac{1}{|BD|} = \frac{1}{r}$.
4. Prove that, for any integer $a > 1$, there is a prime p such that $1 + a + a^2 + \dots + a^{p-1}$ is composite.
5. Consider a 3×3 table, filled with real numbers in such a way that each number in the table is equal to the absolute value of the difference of the sum of numbers in its row and the sum of numbers in its column.
 - a) Prove that any number in this table can be expressed as a sum or a difference of some two numbers in the table.
 - b) Show that there exists such a table with numbers in it not all equal to 0.

Solutions of Final Round

9-1. *Answer:* $x = 1271$.

From the given equality we obtain $111(24 + x) = 14**45$. To find the number $y = 24 + x$, note that

$$111 \cdot 1000 = 111000 < 14**45 < 222000 = 111 \cdot 2000,$$

therefore y is a 4-digit number, with 1 as its first digit. Evidently y must end with 5. Let $y = \overline{1ab5}$, where $0 \leq a, b \leq 9$. Writing out the multiplication we see that $b + 5$ ends with 4, hence $b = 9$ and there is a carry of at least 1 from the third position. Since there is no carry to the first position, we have $a \leq 2$. If the carry from the third position were more than 1, we would have $a \geq 8$, a contradiction. Hence $a = 2$ and $x = 1295 - 24 = 1271$.

9-2. *Answer:* 441 and 882.

We look for a number \overline{abc} such that $\overline{abc} = 3\overline{cba} + (a + b + c)$, or $32a = 100c + 7b$. Hence $1 \leq c \leq 3$, and we have 3 cases.

1) If $c = 1$, then $100 \leq 32a = 100 + 7b \leq 163$ which implies $4 \leq a \leq 5$. If $a = 4$, then $128 = 100 + 7b$ and $b = 4$. If $a = 5$, then $160 = 100 + 7b$ and b is not an integer.

2) If $c = 2$, then $200 \leq 32a = 200 + 7b \leq 263$ which implies $7 \leq a \leq 8$. If $a = 7$, then $224 = 200 + 7b$ and b is not an integer. If $a = 8$, then $256 = 200 + 7b$, giving $b = 8$.

3) If $c = 3$, then $300 \leq 32a = 200 + 7b \leq 363$ which implies $a \geq 10$, a contradiction.

9-3. *Answer:* $10 + 5\sqrt{2}$.

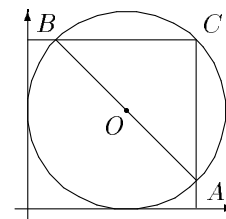


Figure 6

Introduce a coordinate system where the sides of the square tangent to the circle are on the coordinate axes—then the centre of the circle is $O(10, 10)$ (see Fig. 6). Let the side of the square be a (evidently $a > 10$) and the intersection points of the circle with its two other sides be A and B . As AB is the diameter of the circle, the common point $C(a, a)$ of these two sides lies on the circle. Since CO is a radius, we obtain $\sqrt{(a-10)^2 + (a-10)^2} = 10$, giving $a - 10 = 5\sqrt{2}$ and $a = 10 + 5\sqrt{2}$.

9-4. *Answer:* 1001000.

Note that if x is a solution of the equation, $2002 - x$ is also a solution. For uniqueness we have $x = 2002 - x$, or $x = 1001$. In this case

$$\begin{aligned} a &= 1000 + 999 + \dots + 2 + 1 + 0 + 1 + 2 + \dots + 999 + 1000 = \\ &= (1000 + 1) + (999 + 2) + \dots + (2 + 999) + (1 + 1000) = \\ &= 1000 \cdot 1001 = 1001000. \end{aligned}$$

Remark. Although this is not required in the problem, it can be verified that $x = 1001$ is indeed the only solution of the equation for $a = 1001000$.

9-5. *Answer:* 24.

The numbers 1 can be in the same column only with numbers 2, 3 and 4. As there are altogether $4 \cdot 9$ of these, the 1-s can be at most in four columns. If all 1-s are in the same column, the minimal column sum is 9. If the 1-s are in two columns, one of these must contain at least 5 of them and the sum of this column is at most $5 \cdot 1 + 4 \cdot 4 = 21$. If the 1-s are in four columns, then the sum of all numbers in these columns is $9 \cdot (1 + 2 + 3 + 4) = 90$, hence the minimal column sum is at most $\lceil \frac{90}{4} \rceil = 22$. If the 1-s are in three columns, we should have 3-s and 4-s in these

columns to obtain the largest column sum. In this case the sum of numbers in the three columns is $9 \cdot (1 + 3 + 4) = 72$ and the minimal column sum is at most 24. From the table below we see that this value is indeed attainable.

1	1	1	2	2	6	7	...	2001
1	1	1	2	2	6	7	...	2001
1	1	1	2	2	6	7	...	2001
3	3	3	2	2	6	7	...	2001
3	3	3	2	5	6	7	...	2001
3	3	3	5	5	6	7	...	2001
4	4	4	5	5	6	7	...	2001
4	4	4	5	5	6	7	...	2001
4	4	4	5	5	6	7	...	2001

10-1. *Answer:* The possible values of n are 4, 5 and 6.

The sum of the angles of a n -gon is $(n - 2) \cdot \pi$. Since three of these angles are greater than $\frac{\pi}{2}$ and less than π , and the remaining $n - 3$ angles are greater than 0 and less or equal to $\frac{\pi}{2}$, we obtain $(n - 3) \cdot 0 + 3 \cdot \frac{\pi}{2} < (n - 2) \cdot \pi < (n - 3) \cdot \frac{\pi}{2} + 3 \cdot \pi$.

Dividing by π and transforming yields $\frac{7}{2} < n < 7$. As n is an integer, we have $4 \leq n \leq 6$, and it is easy to check that all these three values are indeed possible.

10-2. *Answer:* $n = 5$.

The sum of any three integers congruent to 0, 1 and 2 modulo 3 is divisible by 3. Also, the sum of any three integers congruent to each other modulo 3 is divisible by 3. Consequently, among any five numbers there are three whose sum is divisible by 3. On the other hand, among the numbers 1, 3, 4 and 6 there are no three with a sum divisible by 3.

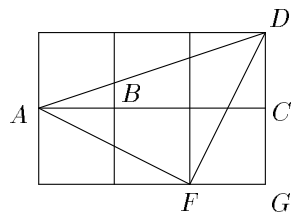


Figure 7

10-3. *Answer:* $\frac{3\pi}{4}$.

Consider points F and G as shown on Fig. 7. As BCD and DGF are congruent

right-angled triangles, we have

$$\begin{aligned} \angle ADF &= \angle ADC - \angle FDG = \angle ADC - \left(\frac{\pi}{2} - \angle DFG\right) = \\ &= \angle ADC - \left(\frac{\pi}{2} - \angle BDC\right), \end{aligned}$$

that implies $\angle ADC + \angle BDC = \frac{\pi}{2} + \angle ADF$. The segments AF and DF are transformed into each other by a 90° rotation around F . Hence AFD is an isosceles right-angled triangle with $\angle ADF = \frac{\pi}{4}$, yielding $\angle ADC + \angle BDC = \frac{3\pi}{4}$.

Remark: There are also solutions using the cosine theorem or the identity $\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$.

10-4. As a and b are non-zero integers, we have

$$\begin{aligned} \frac{1}{a} + \frac{1}{b} = \frac{1}{c} &\iff \frac{a+b}{ab} = \frac{1}{c} \iff ab = (a+b)c \iff \\ &\iff ab - ac - bc = 0 \iff ab - ac - bc + c^2 = c^2 \iff \\ &\iff (a-c)(b-c) = c^2. \end{aligned}$$

Now let $\frac{1}{a} + \frac{1}{b} = \frac{1}{c}$. If a and b are positive, then $a - c > 0$ and $b - c > 0$. On the other hand, if $a - c > 0$ and $b - c > 0$, then a and b are positive. Hence the harmonic triples (a, b, c) are in one-to-one correspondence with pairs of positive integers (r, s) , where $rs = c^2$, and there are as many such harmonic triples as there are positive divisors of c^2 .

10-5. We use induction on the length of a word. Let u_1 and u_2 be any different words of the same length, and suppose the claim holds for all shorter words. As there is only one word of length 1, u_1 and u_2 are constructed by rule (2). This implies that there exist words v_1 and v_2 so that $u_1 = v_1 v_1$ or $u_1 = v_1 \overline{v_1}$ and $u_2 = v_2 v_2$ or $u_2 = v_2 \overline{v_2}$. Note that v_1 and v_2 are of the same length. If $v_1 = v_2 = v$, then one of the words u_1 and u_2 is vv and the other $v\overline{v}$, differing exactly in half of their letters. If $v_1 \neq v_2$, then v_1 and v_2 differ exactly in half of their letters by the induction hypothesis, and it remains to show that the latter halves of u_1 and u_2 also differ exactly in half of their letters. If these halves are v_1 and v_2 or $\overline{v_1}$ and $\overline{v_2}$, this is obviously true. The words v_1 and $\overline{v_2}$, as well as v_2 and $\overline{v_1}$, differ exactly in the letters where v_1 and v_2 coincide—differing therefore also exactly in half of their letters. Hence in any case u_1 and u_2 differ in half of their letters.

11-1. *Answer:* The only possibilities are $n = 3$, $\alpha = \frac{\pi}{6}$ and $n = 4$, $\alpha = \frac{\pi}{5}$.

Obviously $n \geq 3$. As the sum of angles of the n -gon is $n \cdot \frac{(\alpha + n\alpha)}{2} = \pi(n - 2)$, we have $\alpha = \frac{2\pi(n - 2)}{n(n + 1)}$. Because of convexity, we have $n\alpha = \frac{2\pi(n - 2)}{n + 1} < \pi$ yielding $n < 5$. If $n = 3$, we obtain $\alpha = \frac{\pi}{6}$; if $n = 4$, then $\alpha = \frac{\pi}{5}$.

11-2. Let D' and B' be the multipliers of the first and the second fraction, respectively. Then $E = AD' + B'C$ and $F = BD' = DB'$, with B' and D' coprime since F is the least common multiple of the denominators. If, for a prime p , p^k divides d with $k > 0$, then p^k divides both E and F . Suppose p^k does not divide B . From $F = BD'$ we obtain that p divides D' , hence p also divides $B'C = E - AD'$. Therefore, p divides either B' or C , and as B' and D' are coprime, p divides C . From $F = DB'$ we get that D is divisible by p^k , hence p is a common factor of C and D , contradicting the irreducibility of $\frac{C}{D}$. We conclude that p^k divides B , and similarly also D . Since this is true for any prime divisor p of d , then B and D are both divisible by d .

11-3. Draw a line parallel to BC through A and denote its intersection points with rays BE and CF by L and M , respectively (see Fig. 8). From similar triangles AEL and CEB we have $\frac{|AE|}{|EC|} = \frac{|AL|}{|BC|}$. Analogously $\frac{|AF|}{|FB|} = \frac{|AM|}{|BC|}$. Moreover, from similar triangles AOL and DOB we get $\frac{|AO|}{|OD|} = \frac{|AL|}{|BD|}$, and analogously $\frac{|AO|}{|OD|} = \frac{|AM|}{|DC|}$. Hence

$$\frac{|AO|}{|OD|} = \frac{|AL| + |AM|}{|BD| + |DC|} = \frac{|AL| + |AM|}{|BC|} = \frac{|AL|}{|BC|} + \frac{|AM|}{|BC|} = \frac{|AE|}{|EC|} + \frac{|AF|}{|FB|}.$$

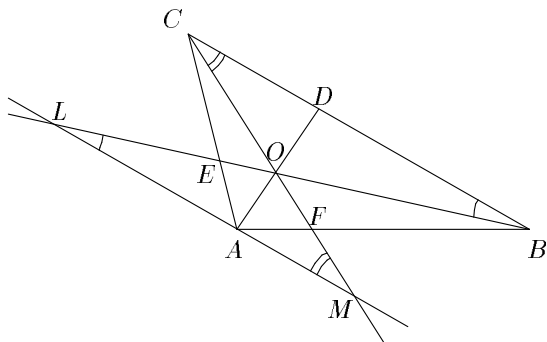


Figure 8

11-4. Denote $\alpha = 1 - x$, then $x = 1 - \alpha$ and from $x + y = 2$ we get $y = 1 + \alpha$. Now

$$\begin{aligned} x^2 y^2 (x^2 + y^2) &= (1 - \alpha)^2 (1 + \alpha)^2 \cdot ((1 - \alpha)^2 + (1 + \alpha)^2) = \\ &= ((1 - \alpha)(1 + \alpha))^2 \cdot (2 + 2\alpha^2) = \\ &= 2(1 - \alpha^2)^2 (1 + \alpha^2) = 2(1 - \alpha^4)(1 - \alpha^2). \end{aligned}$$

Since $x, y \geq 0$, we have $|\alpha| \leq 1$ that implies $0 \leq 1 - \alpha^2 \leq 1$ and $0 \leq 1 - \alpha^4 \leq 1$. Hence $2(1 - \alpha^4)(1 - \alpha^2) \leq 2$.

11-5. *Answer:* a) 7; b) 1, 2, 3, 4, 6, 8, 10, 16, 28 ja 32.

Consider a trapezoid of height h and the length of its shorter base a (see Fig. 9). The longer base of the trapezoid is of length $a + h$ and thus there is a total of

$$N(a, h) = (a+1) + (a+2) + \dots + (a+h+1) = \frac{(2a+h+2)(h+1)}{2}$$

points with integral coordinates inside and on the border of this trapezoid.

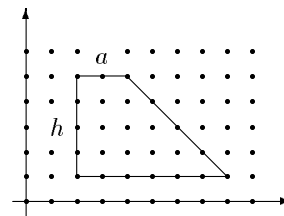


Figure 9

a) We have to find the number of distinct pairs (a, h) for which $N(a, h) = 2001$. Taking into account that $2001 = 3 \cdot 23 \cdot 29$, we consider two cases:

1) If $h = 2k$ is even, then $N(a, h) = (a + k + 1) \cdot (2k + 1)$ where $2k + 1 \geq 3$ and $a + k + 1 > k + 1 > \frac{2k + 1}{2}$. The factor $2k + 1$ can be 3, 23 or 29, yielding the pairs $(665, 2)$, $(75, 22)$ and $(54, 28)$.

2) If $h = 2k - 1$ is odd, then $N(a, h) = (2a + 2k + 1) \cdot k$, where $k \geq 1$ and $2a + 2k + 1 \geq 2k + 3$. The factor k can be 1, 3, 23 or 29, yielding the pairs $(999, 1)$, $(330, 5)$, $(20, 45)$ and $(5, 57)$.

b) For $h = 1, 2, 3, \dots, 7$ we express the size of a trapezoid in terms of a (see the table); if $h > 7$, then $N(a, h) > 50$ for any $a \geq 1$. It is easy to check that numbers 1, 2, 3, 4, 6, 8, 10, 16, 28 and 32 are the only ones that cannot be expressed by any of the formulae in the table.

h	$N(a, h)$
1	$2a + 3$
2	$3a + 6$
3	$4a + 10$
4	$5a + 15$
5	$6a + 21$
6	$7a + 28$
7	$8a + 36$

12-1. *Answer:* The only solution is $x = y = 0$.

Clearly $x = y = 0$ is a solution. We know that $|\sin x| \leq |x|$, where equality holds iff $x = 0$ (this can be easily proved using derivatives). Now

$$|x| \geq |\sin x| = |y| \geq |\sin y| = |x|,$$

and at least one of the inequalities is strict if $x \neq 0$ or $y \neq 0$.

12-2. *Answer:* n .

Let the chosen integers be a_1, \dots, a_k and, for each $i = 1, \dots, k$, let n_i be the exponent of 2 in the prime factorization of a_i , i.e. $a_i = 2^{n_i} \cdot b_i$ with b_i odd. Since $1 \leq b_i \leq 2n - 1$, there are n possibilities for the numbers b_i . If $k \geq n + 1$, then there exist indices i and j such that $b_i = b_j = b$ and $n_i > n_j$. Then $a_i = 2^{n_i} \cdot b$ is divisible by $a_j = 2^{n_j} \cdot b$.

If $k \leq n$, then choose any k numbers in the set $\{n + 1, \dots, 2n\}$. None of them is divisible by another since $2n < 2 \cdot (n + 1)$.

12-3. Let $\alpha = \angle IAE = \angle BAI$ and $\beta = \angle DBI = \angle IBA$, then $\angle EIA = \angle BID = \alpha + \beta$ (see Fig. 10). Applying the sine rule for triangle AEI and the equality $r = |AI| \sin \alpha$, we obtain

$$\frac{|AE|}{\sin(\alpha + \beta)} = \frac{|AI|}{\sin \angle AEI} = \frac{r}{\sin \alpha \sin \angle AEI}.$$

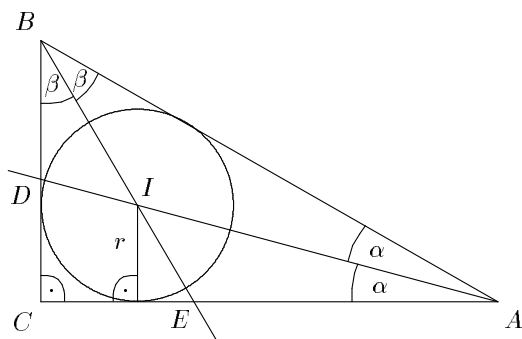


Figure 10

From triangle BDI , we similarly get

$$\frac{|BD|}{\sin(\alpha + \beta)} = \frac{|BI|}{\sin \angle IDB} = \frac{r}{\sin \beta \sin \angle IDB}.$$

Since $\sin \angle AEI = \cos \beta$ and $\sin \angle IDB = \cos \alpha$, we have

$$\frac{1}{|AE|} + \frac{1}{|BD|} = \frac{\sin \alpha \cos \beta}{r \sin(\alpha + \beta)} + \frac{\sin \beta \cos \alpha}{r \sin(\alpha + \beta)} = \frac{\sin(\alpha + \beta)}{r \sin(\alpha + \beta)} = \frac{1}{r}.$$

12-4. If $a = 2$, then $p = 11$ gives the desired result:

$$1 + 2 + 4 + \dots + 2^{10} = 2^{11} - 1 = 2047 = 23 \cdot 89.$$

If $a > 2$, then $a - 1 > 1$ and there exists a prime p that divides $a - 1$. Hence a is congruent to 1 modulo p and $M_p = 1 + a + a^2 + \dots + a^{p-1}$ is divisible by p . We also have $M_p \geq 1 + a > p$, implying that M_p is composite.

12-5. a) Let r_1, r_2, r_3 be the sums of numbers in the first, second and third row, and c_1, c_2, c_3 be the sums of numbers in the first, second and third column. Denote by a_{ij} the element in the i -th row and j -th column, and notice that all the elements of the table are non-negative.

Since $r_1 + r_2 + r_3 = c_1 + c_2 + c_3$, we have

$$\begin{aligned} a_{11} &= |r_1 - c_1| = |(r_2 + r_3) - (c_2 + c_3)| = |(r_2 - c_2) + (r_3 - c_3)| = \\ &= \pm|r_2 - c_2| \pm |r_3 - c_3| = \pm a_{22} \pm a_{33}. \end{aligned}$$

As all the elements are non-negative, a_{22} and a_{33} cannot both have minus sign here and, consequently, a_{11} is equal to the sum or difference of two numbers in the table. The proof for all other elements of the table is similar.

b) The tables below satisfy the required condition for any real $x > 0$:

$$\begin{array}{|c|c|c|} \hline 0 & x & 0 \\ \hline x & 0 & x \\ \hline 0 & x & 0 \\ \hline \end{array} ; \quad \begin{array}{|c|c|c|} \hline x & x & x \\ \hline x & x & x \\ \hline 2x & 2x & 2x \\ \hline \end{array}.$$

IMO Team Selection Test: April 2001

First Day

- Consider on the coordinate plane all rectangles whose
 - vertices have integer coordinates;
 - edges are parallel to coordinate axes;
 - area is 2^k , where $k = 0, 1, 2, \dots$

Is it possible to color all points with integer coordinates in two colors so that no such rectangle has all its vertices of the same color?

- Point X is taken inside a regular n -gon of side length a . Let h_1, h_2, \dots, h_n be the distances from X to the lines defined by the sides of the n -gon. Prove that

$$\frac{1}{h_1} + \frac{1}{h_2} + \dots + \frac{1}{h_n} > \frac{2\pi}{a}.$$

3. Let k be a fixed real number. Find all functions $f: \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(x) + (f(y))^2 = kf(x + y^2)$$

for all real numbers x and y .

Second Day

4. Consider all products by 2, 4, 6, ..., 2000 of the elements of the set $A = \left\{ \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, \frac{1}{2000}, \frac{1}{2001} \right\}$. Find the sum of all these products.

5. Find the exponent of 37 in the representation of the number $\underbrace{111\dots11}_{3 \cdot 37^{2000} \text{ digits}}$ as product of prime powers.

6. Let \mathcal{C}_1 and \mathcal{C}_2 be the incircle and the circumcircle of the triangle ABC , respectively. Prove that, for any point A' on \mathcal{C}_2 , there exist points B' and C' such that \mathcal{C}_1 and \mathcal{C}_2 are the incircle and the circumcircle of triangle $A'B'C'$, respectively.

Solutions of Selection Test

1. *Answer:* Yes.

Color the points with integer coordinates in three colors so that on each diagonal $y = x + k$ all points are of the same color and the colors change in a cyclic manner when k increases. Since $2^m \equiv 1 \pmod{3}$ for even m and $2^m \equiv 2 \pmod{3}$ for odd m , it is easy to understand that all three colors are present in vertices of each rectangle under consideration. Now recoloring the vertices of some color with one of the remaining two colors, we obtain a coloring with the required properties.

2. Let S be the area of the n -gon and r its inradius, then $S = n \cdot \frac{ar}{2}$. On the other hand, $S = \frac{1}{2} \cdot a \cdot (h_1 + h_2 + \dots + h_n)$. Using the AM-HM inequality, we get

$$\frac{n}{\frac{1}{h_1} + \frac{1}{h_2} + \dots + \frac{1}{h_n}} \leq \frac{h_1 + h_2 + \dots + h_n}{n} = \frac{2S}{na} = r.$$

Comparing the lengths of circumferences of the n -gon and its incircle, we get $na > 2\pi r$. Hence

$$\frac{1}{h_1} + \frac{1}{h_2} + \dots + \frac{1}{h_n} \geq \frac{n}{r} > \frac{2\pi}{a}.$$

3. *Answer:* If $k = 1$ then $f(x) = x$ or $f(x) = 0$; if $k \neq 1$ then $f(x) = k - 1$ or $f(x) = 0$.

Substituting $y = 0$ in the original equation we get

$$(k - 1)f(x) = f(0)^2. \tag{1}$$

If $k \neq 1$, then substituting $x = 0$ in (1) we get $f(0) = 0$ or $f(0) = k - 1$. So the solutions in the case $k \neq 1$ are the constant functions $f(x) = 0$ and $f(x) = k - 1$.

If $k = 1$, then from (1) we get $f(0) = 0$. Substituting $x = 0$ in the original equation we get $(f(y))^2 = f(y^2)$, and further substituting $y = 1$ we find that $f(1) = 1$ or $f(1) = 0$.

For any non-negative real number z there is a real number y such that $y^2 = z$, therefore from $(f(y))^2 = f(y^2)$ we get that $f(z) \geq 0$ for any $z \geq 0$. Also, substituting $x = -y^2$ in the original equation we get $f(-y^2) = -(f(y))^2$, so $f(z) \leq 0$ for any $z \leq 0$. Since $(f(y))^2 = f(y^2) = f((-y)^2) = (f(-y))^2$, then we must have $f(y) = -f(-y)$, i.e. f is an odd function.

Now let x be any real number and $z \geq 0$, then denoting $\sqrt{z} = y$ we get

$$f(x+z) = f(x+y^2) = f(x) + (f(y))^2 = f(x) + f(y^2) = f(x) + f(z). \tag{2}$$

Hence if $a \leq b$, then $f(b) = f(a + (b - a)) = f(a) + f(b - a) \geq f(a)$, i.e. f is non-decreasing.

Since f is an odd function, (2) holds also when x and z are both negative. Now we show, using induction on n , that $f(nx) = nf(x)$ for any real x and integer n . Indeed, this holds for $n = 0$ and if $f(nx) = nf(x)$ then

$$f((n+1)x) = f(nx+x) = f(nx) + f(x) = nf(x) + f(x) = (n+1)f(x).$$

Since f is odd, we also have $f(-nx) = -f(nx) = -nf(x)$, i.e. $f(nx) = nf(x)$ holds for all integers n .

Earlier we proved that $f(1) = 1$ or $f(1) = 0$. If $f(1) = 0$ then substituting $x = 1$ in $f(nx) = nf(x)$ we get that $f(n) = 0$ for all integers n , and since f is non-decreasing, we have $f(x) = 0$ for all real x . We show now that if $f(1) = 1$ then $f(x) = x$ for all x . For integers we get it from $f(nx) = nf(x)$, substituting $x = 1$. For a rational number $\frac{a}{b}$ we have

$$a = f(a) = f\left(b \cdot \frac{a}{b}\right) = b \cdot f\left(\frac{a}{b}\right),$$

so $f\left(\frac{a}{b}\right) = \frac{a}{b}$. Assume that for some real number x we have $f(x) \neq x$, then $f(x) = x + \varepsilon$ where $\varepsilon \neq 0$. If $\varepsilon > 0$, then let r be a rational number such

that $x < r < x + \varepsilon$, and if $\varepsilon < 0$, then let r be a rational number such that $x > r > x + \varepsilon$. In the first case we get $r < x + \varepsilon = f(x) \leq f(r) = r$, in the second case $r > x + \varepsilon = f(x) \geq f(r) = r$, a contradiction.

4. *Answer:* $499 \frac{1001}{2001}$.

The value of

$$\left(1 + \frac{1}{2}\right) \cdot \left(1 + \frac{1}{3}\right) \cdot \dots \cdot \left(1 + \frac{1}{2001}\right) - 1$$

is equal to the sum of all products of the elements of set A by $1, 2, 3, \dots, 2000$, and the value of

$$\left(1 - \frac{1}{2}\right) \cdot \left(1 - \frac{1}{3}\right) \cdot \dots \cdot \left(1 - \frac{1}{2001}\right) - 1$$

is equal to a similar sum where the products by $2, 4, 6, \dots, 2000$ are taken with a plus sign but the products by $1, 3, 5, \dots, 1999$ are taken with a minus sign. Denote the required sum by S , then

$$\begin{aligned} 2S &= \left(1 + \frac{1}{2}\right) \cdot \left(1 + \frac{1}{3}\right) \cdot \dots \cdot \left(1 + \frac{1}{2001}\right) + \\ &\quad + \left(1 - \frac{1}{2}\right) \cdot \left(1 - \frac{1}{3}\right) \cdot \dots \cdot \left(1 - \frac{1}{2001}\right) - 2 = \\ &= \frac{3}{2} \cdot \frac{4}{3} \cdot \dots \cdot \frac{2002}{2001} + \frac{1}{2} \cdot \frac{2}{3} \cdot \dots \cdot \frac{2000}{2001} - 2 = \\ &= \frac{2002}{2} + \frac{1}{2001} - 2 = 999 \frac{1}{2001}, \end{aligned}$$

and $S = 499 \frac{1001}{2001}$.

5. *Answer:* 2001.

As 37 and 9 are relatively prime it is sufficient to find the exponent of 37 in the representation of the number

$$\underbrace{999 \dots 99}_{3 \cdot 37^{2000} \text{ numbers}} = 10^{3 \cdot 37^{2000}} - 1 = 1000^{37^{2000}} - 1.$$

We show by induction on k that the exponent of 37 in the representation of $1000^{37^k} - 1$ is $k + 1$. In the case $k = 0$ we have

$$1000^{37^0} - 1 = 999 = 3^3 \cdot 37,$$

i.e. the exponent of 37 is 1. Suppose now that for some k our assertion holds,

and note that

$$\begin{aligned} 1000^{37^{k+1}} - 1 &= (1000^{37^k})^{37} - 1 = \\ &= \left(1000^{37^k} - 1\right) \cdot \left(1 + 1000^{37^k} + (1000^{37^k})^2 + \dots + (1000^{37^k})^{36}\right). \end{aligned}$$

The exponent of 37 in the representation of number $1000^{37^k} - 1$ is $k + 1$ by the induction hypothesis. Hence it suffices to show that the exponent of 37 in the representation of

$$1 + 1000^{37^k} + (1000^{37^k})^2 + \dots + (1000^{37^k})^{36}$$

is 1. Since $1000 \equiv 1 \pmod{37}$ then $1000^{37^k} \equiv 1 \pmod{37}$. Let $1000^{37^k} = 37q + 1$, then

$$\begin{aligned} 1 + 1000^{37^k} + (1000^{37^k})^2 + \dots + (1000^{37^k})^{36} &= \\ &= 1 + (37q + 1) + (37q + 1)^2 \dots + (37q + 1)^{36} \equiv \\ &\equiv 1 + (37q + 1) + (2 \cdot 37q + 1) + \dots + (36 \cdot 37q + 1) = \\ &= \frac{37 \cdot 36}{2} \cdot 37q + 37 = 37^2 \cdot 18 \cdot q + 37 \equiv 37 \pmod{37^2}. \end{aligned}$$

So $1 + 1000^{37^k} + (1000^{37^k})^2 + \dots + (1000^{37^k})^{36}$ is divisible by 37 but not by 37^2 , and the exponent of 37 in the representation of $1000^{37^{k+1}} - 1$ is $k + 2$.

Hence the exponent of 37 in the representation of $1000^{37^{2000}} - 1$ is 2001.

6. Let I and O be the incenter and the circumcenter of the triangle ABC , respectively. We know by Euler's formula that $|OI|^2 = R^2 - 2Rr$, where r and R are the radii of the incircle and the circumcircle, respectively.

Assume now that there exists a point A' on the circle \mathcal{C}_2 such that it is impossible to construct the points B' and C' as required in the problem. Let the tangents drawn from A' to the circle \mathcal{C}_1 touch \mathcal{C}_1 in B' and C' , hence $B'C'$ is not tangent to the circle \mathcal{C}_1 . Suppose the line $B'C'$ and the circle \mathcal{C}_1 have no points in common (the case of two common points is similar). Let the distance between the line $B'C'$ and the circle \mathcal{C}_1 be $\delta > 0$.

Now start moving the points B' and C' along the circle \mathcal{C}_2 towards A' in such a way that the distances from the circle \mathcal{C}_1 to the straight lines $A'B'$ and $A'C'$ remain equal (note that they are both equal to 0 at the beginning) — denote this distance by ϵ . The distance δ obviously decreases, whereas the distance ϵ increases, hence at some moment they must become equal. Now we can increase the radius r by $\delta = \epsilon > 0$ to make it the incircle of the triangle $A'B'C'$. Hence the triangle $A'B'C'$ has circumradius R and inradius $r + \delta$, but the distance $|OI|$ is the same as for the triangle ABC , hence Euler's formula for triangle $A'B'C'$ becomes violated.