



# Estonian Math Competitions

## 2003/2004

The Gifted and Talented Development Centre

Tartu 2004

## Selected Problems from Open Contests

**1.** Diameter  $AB$  is drawn to a circle with radius 1. Two straight lines  $s$  and  $t$  touch the circle at points  $A$  and  $B$ , respectively. Points  $P$  and  $Q$  are chosen on the lines  $s$  and  $t$ , respectively, so that the line  $PQ$  touches the circle. Find the smallest possible area of the quadrangle  $APQB$ . (*Juniors.*)

*Answer:* 2.

*Solution.* As  $ABCD$  is a trapezium, its area can be found as follows:

$$S_{APQB} = \frac{|AP| + |BQ|}{2} \cdot |AB| = |AP| + |BQ|.$$

On the other hand,  $|AP| = |PK|$  and  $|BQ| = |QK|$ , thus  $S_{APQB} = |PK| + |QK| = |PQ|$ . The smallest possible length of the segment  $PQ$  is 2.

**2.** Find all pairs of positive integers  $(m, n)$ ,  $m \leq n$ , such that a rectangle of size  $m \times n$  can not be divided into rectangles of sizes  $2 \times 5$  and  $1 \times 3$ . (*Juniors.*)

*Answer:*  $(2, 2)$ ,  $(2, 4)$ ,  $(4, 4)$ ,  $(2, 7)$  and  $(1, 3k-1)$ ,  $(1, 3k-2)$ , where  $k$  is any positive integer.

*Solution.* The answer is obvious for rectangles with one side of length 1. Hence, we need to consider the rectangles with both of the sides longer than 1.

All the rectangles  $3 \times m$  can be divided into  $1 \times 3$  pieces. All the rectangles  $5 \times (2k)$  can be divided into  $2 \times 5$  pieces. All the rectangles  $5 \times (2k + 1)$  can be divided into  $k - 1$  pieces of  $2 \times 5$  and 5 pieces of  $1 \times 3$ . Thus, all the rectangles with one side having the form  $3s + 5t$ , where  $s, t \geq 0$ , can be divided in the required way. Since  $6 = 3 \cdot 2$ ,  $8 = 3 \cdot 1 + 5 \cdot 1$ ,  $9 = 3 \cdot 3$  and  $10 = 5 \cdot 2$ , every positive integer  $n > 7$  can be represented as  $8 + 3k$ ,  $9 + 3k$  or  $10 + 3k$  for some integer  $k \geq 0$ . Hence we are done, if one of the sides of the original rectangle is 3, 5, 6 or  $n > 7$  and the length of the other side is at least 2.

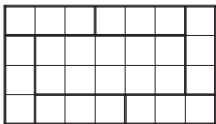


Figure 1

We still need to consider the rectangles with both sides from the set  $\{2, 4, 7\}$ . The rectangles  $2 \times 2$ ,  $2 \times 4$  and  $4 \times 4$  cannot be divided, as they cannot contain a  $2 \times 5$  piece, but their area is not divisible by 3. A similar argument holds for  $2 \times 7$  rectangle — cutting away either one or zero  $2 \times 5$  pieces leaves a part with area not divisible by 3.

It is easy to construct a required division for  $4 \times 7$  rectangle (see Figure 1), hence  $7 \times 7$  rectangle can be divided as well.

**3.** On Liarians planet, a year consists of 2004 days and for each inhabitant on the planet, each day of the year is either a truth-day or a lie-day: on truth-days a Liarian speaks only the truth and on lie-days the Liarian always lies (the number of a Liarian's truth-days or lie-days can also be 0). On each day of a given year, three Liarians were asked, "How many days a year do you lie?" On the first day of the year, the first Liarian answered that he lied on *exactly* 1 day a year, the second Liarian said that he lied on *at least* 1 day a year, and the third replied that he lied on *at most* 1 day a year. On the

second day, the first Liarian answered that he lied on exactly 2 days a year, the second, that he lied on at least 2 days a year, and the third, that he lied on at most 2 days a year, and so on, until the last, 2004th day of the year, when the first answered that he lied on exactly 2004 days, the second said that he lied on at least 2004 days, and the third replied that he lied on at most 2004 days a year. How many days a year does each of the three Liarians actually lie? (*Juniors.*)

*Answer:* the first Liarian lies on 2003 days, the second on 1002 days and the third on 0 days a year.

*Solution.* Since the answers of the first Liarian are all mutually exclusive, he could tell the truth on at most one day. If he had lied on all days, the answer he gave on the 2004th day would have been true — a contradiction. Thus, the first Liarian tells the truth on one day (the 2003th day of the year) and lies on the remaining 2003 days of the year.

Suppose that the second Liarian lies on  $n$  days of a year, then his answers are true on the first  $n$  days and false on the remaining  $2004 - n$  days. Hence  $n = 2004 - n$ , and  $n = 1002$  (the second Liarian lies on the last 1002 days of the year).

Suppose now that the third Liarian lies on  $n$  days of a year, where  $n \geq 1$ . Then the answers he gave on the first  $n - 1$  days are false and the remaining answers (starting from the  $n$ th day) are true. Thus he lies on  $n - 1$  days a year — a contradiction. However, if he lies on 0 days a year, all his answers are true and there is no contradiction. Hence the third Liarian lies on 0 days a year.

**4.** Circles  $c_1$  and  $c_2$  with centres  $O_1$  and  $O_2$ , respectively, intersect at points  $A$  and  $B$  so that the centre of each circle lies outside the other circle. Line  $O_1A$  intersects circle  $c_2$  again at point  $P_2$  and line  $O_2A$  intersects circle  $c_1$  again at point  $P_1$ . Prove that the points  $O_1, O_2, P_1, P_2$  and  $B$  are concyclic. (*Juniors.*)

*Solution 1.* Triangles  $AO_1P_1$  and  $AO_2P_2$  are isosceles and have equal base angles, thus also their vertex angles are equal. Now

$$\angle P_1O_1P_2 = \angle P_1O_1A = \angle AO_2P_2 = \angle P_1O_2P_2$$

(See figures 2 and 3 for the possible positions of points  $P_1$  and  $P_2$ , according to the angle  $\angle O_1P_1O_2 = \angle O_1P_2O_2$  being smaller or larger than a right angle). Also,

$$\angle P_1BP_2 = \angle P_1BA + \angle ABP_2 = \frac{1}{2}\angle P_1O_1A + \frac{1}{2}\angle AO_2P_2 = \angle P_1O_2P_2.$$

Hence the angles  $\angle P_1O_1P_2$ ,  $\angle P_1O_2P_2$  and  $\angle P_1BP_2$  lying on segment  $P_1P_2$  are equal, and the points  $O_1, O_2, P_1, P_2$  and  $B$  are concyclic.

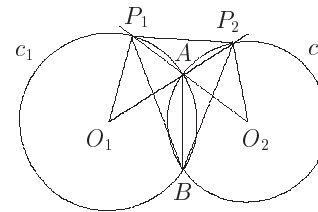


Figure 2

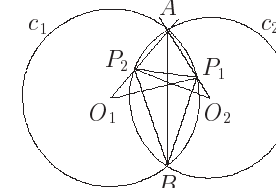


Figure 3

**Solution 2.** Since

$$\angle O_1P_1A = \angle O_1AP_1 = \angle O_2AP_2 = \angle O_2P_2A,$$

points  $O_1, P_1, P_2$  and  $O_2$  lie on the same circle. On the other hand, since

$$\angle O_1P_1O_2 + \angle O_1BO_2 = \angle O_1AP_1 + \angle O_1AO_2 = 180^\circ$$

(for the case shown on figure 2) or

$$\begin{aligned} \angle O_1P_1O_2 + \angle O_1BO_2 &= \angle 180^\circ - \angle O_1P_1A + \angle O_1AO_2 = \\ &= \angle 180^\circ - \angle O_1AO_2 + \angle O_1AO_2 = 180^\circ \end{aligned}$$

(for the case shown on figure 3), also points  $O_1, P_1, O_2$  and  $B$  lie on the same circle. These two circles coincide, since both of them are the circumcircle of triangle  $O_1P_1O_2$ .

**Solution 3.** We note that  $|O_1B| = |O_1P_1|$ ,  $|O_2B| = |O_2P_2|$ ,  $\angle AO_1O_2 = \angle BO_1O_2$  ja  $\angle AO_2O_1 = \angle BO_2O_1$ . Letting  $R_0, R_1$  and  $R_2$  be the radii of the circumcircles of triangles  $O_1O_2B, O_1O_2P_1$  and  $O_1O_2P_2$ , and applying the Sine law, we get

$$\begin{aligned} 2R_1 &= \frac{|O_1P_1|}{\sin \angle P_1O_2O_1} = \frac{|O_1P_1|}{\sin \angle AO_2O_1} = \frac{|O_1B|}{\sin \angle BO_2O_1} = 2R_0 = \\ &= \frac{|O_2B|}{\sin \angle BO_1O_2} = \frac{|O_2P_2|}{\sin \angle AO_1O_2} = \frac{|O_2P_2|}{\sin \angle P_2O_1O_2} = 2R_2. \end{aligned}$$

Hence triangles  $O_1O_2B, O_1O_2P_1$  and  $O_1O_2P_2$  share a common side  $O_1O_2$  and have equal radii of circumcircles.

If  $\angle O_1P_1O_2$  is larger than a right angle, then all the midperpendiculars of the triangles intersect at the same side of line  $O_1O_2$  as point  $B$ . If  $\angle O_1P_1O_2$  is smaller than a right angle, then all the centres of the circumcircles lie on the other side of line  $O_1O_2$  from point  $B$ . Thus, in both cases the triangles share a common circumcircle.

*Remark.* The claim of the problem is actually true without assuming that the centres of the circles lie outside the other circle.

**5.** Let  $\mathbb{R}^+$  be the set of all positive real numbers. Find all functions  $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ , such that for every  $x, y \in \mathbb{R}^+$  the equality

$$y^2 \cdot f(x) = f\left(\frac{x}{y}\right)$$

holds. (*Seniors.*)

*Answer:* All functions of the form  $f(x) = \frac{a}{x^2}$ , where  $a \in \mathbb{R}^+$ .

*Solution.* Taking  $x = 1$ , we obtain  $y^2 f(1) = f\left(\frac{1}{y}\right)$ . Substitution  $z = \frac{1}{y}$  gives  $f(z) = \frac{f(1)}{z^2}$ . Substituting this to the original equation we see that  $f(1)$  can be any positive real number.

**6.** a) Does there exist a convex quadrangle  $ABCD$  satisfying the following conditions

- (1)  $ABCD$  is not cyclic;
- (2) the sides  $AB, BC, CD$  and  $DA$  have pairwise different lengths;
- (3) the circumradii of the triangles  $ABC, ADC, BAD$  and  $BCD$  are equal?

b) Does there exist such a non-convex quadrangle? (*Seniors.*)

*Answer:* a) no; b) yes.

*Solution.* a) Assume the quadrangle  $ABCD$  satisfies the conditions of the problem. Let  $R$  be the common circumradius of triangles  $ABC, ADC, BAD$  and  $BCD$ . The Sine law for triangles  $ABC$  and  $BAD$  gives

$$\frac{|AB|}{\sin \angle ACB} = 2R = \frac{|AB|}{\sin \angle ADB},$$

or  $\sin \angle ACB = \sin \angle ADB$ . If  $\angle ACB = \angle ADB$ , the quadrangle  $ABCD$  would be cyclic, hence  $\angle ACB + \angle ADB = \pi$  (see Figure 4). Similarly we find  $\angle DAC + \angle DBC = \pi$ , but then the sum of interior angles of  $ABCD$  would be greater than  $2\pi$ , a contradiction.

b) Let  $ABD$  be an isosceles triangle with  $|AB| = a$  and  $|BD| = |DA| > 2a$ . Let  $C$  be the orthocenter of the triangle  $ABD$  and let  $B_1$  and  $D_1$  be the feet of the perpendiculars drawn from the vertices  $B$  and  $D$  to the opposing sides (see Figure 5).

Denote  $\angle ADC = \alpha$ . As the triangles  $ADD_1$  and  $ABB_1$  are right with a common acute angle, they are similar and  $\angle ABC = \angle ABB_1 = \angle ADD_1 = \angle ADC = \alpha$ . Due to symmetry we also have  $\angle BAC = \angle BDC = \alpha$ . As the triangles  $ADC$  and  $BDC$  are congruent, their circumradii are equal — let it be  $R$ . The Sine law for triangles  $ABC$  and  $BDC$  gives

$$R_{ABC} = \frac{|CB|}{2 \sin \angle CAB} = \frac{|CB|}{2 \sin \alpha} = \frac{|CB|}{2 \sin \angle CDB} = R,$$

and the Sine law for triangles  $BAD$  and  $ABC$  gives

$$R_{BAD} = \frac{|AB|}{2 \sin \angle ADB} = \frac{|AB|}{2 \sin 2\alpha} = \frac{|AB|}{2 \sin(\pi - 2\alpha)} = \frac{|AB|}{2 \sin \angle ACB} = R.$$

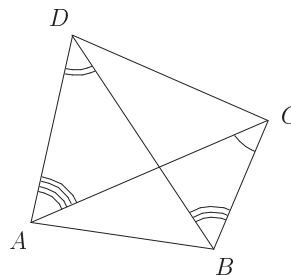


Figure 4

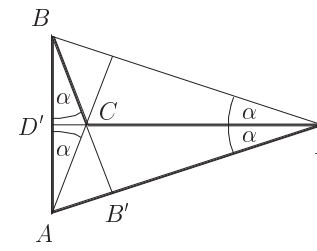


Figure 5

Thus we have shown that the circumradii of the triangles  $ABC, ADC, BAD$  and  $BCD$  are equal. We still have to prove that the lengths of the sides of  $ABCD$  are all different. Indeed, as  $AB$  is the hypotenuse of  $ABB_1$ , we have  $|AB| > |BB_1| > |BC|$ . Similarly,

in triangle  $ADD_1$  we have  $|AD| > |DD_1| > |CD|$ . Since  $|BD| > 2a$ ,  $|BC| < a$  and  $|BD| < |BC| + |CD|$ , we get

$$|CD| > |BD| - |BC| > 2a - a = a = |AB|.$$

Thus  $|DA| > |CD| > |AB| > |BC|$  as required. As a non-convex quadrangle can not be cyclic, all the conditions for the quadrangle  $ABCD$  are satisfied.

**7.** Find all positive integers  $n$  such that the number

$$\frac{n^1}{1!} + \frac{n^2}{2!} + \dots + \frac{n^{n-1}}{(n-1)!} + \frac{n^n}{n!}$$

is an integer. (*Seniors.*)

*Answer:*  $n = 1, 2, 3$ .

*Solution.* Suitability of the solutions 1, 2 and 3 can be verified directly. We will prove that there are no other solutions.

Let  $n \geq 2$ , then the given expression can be written as:

$$\begin{aligned} & \frac{n^1}{1!} + \frac{n^2}{2!} + \dots + \frac{n^{n-2}}{(n-2)!} + \frac{n^{n-1}}{(n-1)!} + \frac{n^n}{n!} = \\ & = \frac{n \cdot (n-1)!}{(n-1)!} + \frac{n^2 \cdot 3 \cdot 4 \cdot \dots \cdot (n-1)}{(n-1)!} + \dots + \\ & \quad + \frac{n^{n-2} \cdot (n-1)}{(n-1)!} + \frac{n^{n-1}}{(n-1)!} + \frac{n^{n-1}}{(n-1)!} = \\ & = \frac{n \cdot (n-1)! + n^2 \cdot 3 \cdot 4 \cdot \dots \cdot (n-1) + \dots + n^{n-2} \cdot (n-1) + 2 \cdot n^{n-1}}{(n-1)!}. \end{aligned}$$

As the first  $n-1$  terms in the numerator and the denominator are divisible by  $n-1$ , the last summand  $2 \cdot n^{n-1}$  has to be divisible by  $n-1$  as well. As the numbers  $n-1$  and  $n^{n-1}$  are coprime,  $n-1$  must divide 2, hence  $n = 2$  or  $n = 3$ .

**8.** Find the smallest real number  $x$  for which there exist two non-congruent triangles with integral side lengths having area  $x$ . (*Seniors.*)

*Answer:*  $\frac{3\sqrt{7}}{4}$ .

*Solution.* Denote

$$s(a, b, c) = (a+b+c)(b+c-a)(c+a-b)(a+b-c).$$

The area of a triangle with sides  $a, b$  and  $c$  can be found from the Heron's formula and is equal to  $\frac{\sqrt{s(a, b, c)}}{4}$ . W.L.O.G. we may assume that  $b = c+x$  and  $a = b+y = c+x+y$ , where  $x, y \geq 0$ . Then

$$\begin{aligned} s(a, b, c) &= s(c+x+y, c+x, c) = \\ &= (3c+2x+y)(c-y)(c+y)(c+2x+y), \end{aligned}$$

hence  $y < c$ . On the other hand, if  $c, x, y$  are real numbers satisfying  $x, y \geq 0$  and  $y < c$ , there exists a triangle with sides  $c+x+y, c+x$  and  $c$  (since all the triangle inequalities are satisfied). We will show that numbers less than 63 can not be represented in two different ways in the form  $s(c+x+y, c+x, c)$ , where  $x, y, c$  are integers and  $x, y \geq 0, y < c$  (disregarding the order of the numbers). Note that when  $x$  increases, the value of the expression  $s(c+x+y, c+x, c)$  increases as well. Taking this into account, we will consider all the possibilities for  $c$ .

(1) If  $c = 1$ , we have  $y = 0$  and for  $x = 0, 1, 2, 3$ , we have 3, 15, 35, 63 as the values for the expression  $s(c+x+y, c+x, c)$ , respectively.

(2) Let  $c = 2$ . If  $y = 0$ , then for  $x = 0, 1$  the expression  $s(c+x+y, c+x, c)$  takes the values 48 and 128, respectively. If  $y = 1$ , then for  $x = 0$  we have  $s(c+x+y, c+x, c) = 63$ .

(3) Let  $c \geq 3$ . As  $x, y \geq 0$  and  $y < c$ , we obtain  $s(c+x+y, c+x, c) \geq 3c \cdot 1 \cdot c \cdot c = 3c^3 \geq 81 > 63$ .

Hence we have covered all the cases when  $s(c+x+y, c+x, c)$  can be less than 63. We have also seen that

$$s(4, 4, 1) = s(3, 2, 2) = 63$$

and the respective area is  $\frac{\sqrt{63}}{4} = \frac{3\sqrt{7}}{4}$ .

**9.** Find all functions  $f$  that are defined on the set of positive real numbers, have real values and satisfy for all positive real numbers  $x$  and  $y$  the equation

$$f(x)f(y) = f(xy) + \frac{1}{x} + \frac{1}{y}.$$

(*Seniors.*)

*Answer:*  $f(x) = 1 + \frac{1}{x}$ .

*Solution.* Taking  $y = 1$ , we get for each  $x$

$$f(x)f(1) = f(x) + \frac{1}{x} + 1. \tag{1}$$

Taking also  $x = 1$ , we obtain a quadratic equation  $f(1)^2 - f(1) - 2 = 0$ , that has solutions  $f(1) = -1$  and  $f(1) = 2$ . Thus  $f(1) - 1 \neq 0$  and from the equation (1) we get

$$f(x) = \left( \frac{1}{f(1)-1} \right) \cdot \left( 1 + \frac{1}{x} \right).$$

Now either  $f(x) = -\frac{1}{2} \cdot \left( 1 + \frac{1}{x} \right)$  or  $f(x) = 1 + \frac{1}{x}$ . The second solution satisfies the initial condition, whereas the first one does not for e.g.  $x = y = \frac{1}{2}$ .

**10.** There are  $N$  lightbulbs on a circle, labelled clockwise with numbers 1 to  $N$ . Initially, none of the lightbulbs are lit. Then the following operation is performed for each positive divisor  $d$  of the number  $N$  (1 and  $N$  included): starting from bulb number 1

# Selected Problems from the Final Round of National Olympiad

and moving clockwise, the state of each  $d$ -th bulb is changed i.e. it is lit, if it is off, and switched off, if it is lit, and this is repeated exactly  $N$  times. (E.g. for  $N = 6$  and  $d = 3$ , the bulbs are lit or switched off in the following order: 3, 6, 3, 6, 3, 6.)

For which values of  $N$  will all bulbs be lit, after the procedure has been completed for all divisors of  $N$ ? (*Seniors.*)

*Answer:*  $N = 2^k$ , where  $k$  is any non-negative integer.

*Solution.* We note that for a given divisor  $d$  of the number  $N$ , all states of bulbs having number divisible by  $d$  are changed, and only these. Since there are  $\frac{N}{d}$  such bulbs and exactly  $N$  changes are made, the state of each bulb is changed exactly  $d$  times. In particular, for  $d = 1$  the state of each of the  $N$  bulbs is changed once.

To have all bulbs lit by the end of the procedure, the state of each bulb must be changed an odd number of times. As we showed, the state of bulb  $m$  is changed once for divisor 1, and additionally, for each such divisor  $d > 1$ , that is also a divisor of  $m$ , the state is changed  $d$  times. Obviously, the condition is satisfied when all divisors  $d > 1$  of  $N$  are even, i.e.  $N = 2^k$ , where  $k \geq 0$ . Suppose now that the number  $N$  has odd divisors greater than 1, and let  $d$  be the smallest of them. Then the state of bulb number  $d$  is changed once for divisor 1,  $d$  times for divisor  $d$ , and 0 times for all other divisors of  $N$ , since  $d$  is not divisible by them. Hence the state of bulb  $d$  is changed an even number of times and the bulb is not lit by the end of the procedure.

**11.** On the circumcircle of triangle  $ABC$ , point  $P$  is chosen, such that the perpendicular drawn from point  $P$  to line  $AC$  intersects the circle again at a point  $Q$ , the perpendicular drawn from point  $Q$  to line  $AB$  intersects the circle again at a point  $R$  and the perpendicular drawn from point  $R$  to line  $BC$  intersects the circle again at the initial point  $P$ . Let  $O$  be the centre of this circle. Prove that  $\angle POC = 90^\circ$ . (*Seniors.*)

*Solution 1.* Rotate triangle  $PQR$  clockwise  $90^\circ$  and denote the triangle obtained  $P'Q'R'$ . Since  $PQ \perp AC$ ,  $QR \perp AB$  and  $RP \perp BC$  (see figure 6),  $P'Q' \parallel AC$ ,  $Q'R' \parallel AB$  and  $R'P' \parallel BC$ . Hence triangle  $Q'R'P'$  is similar to triangle  $ABC$ . Since the triangles  $ABC$  and  $Q'R'P'$  share a common circumcircle, they are actually congruent and either coincide or are rotated  $180^\circ$  from the midpoint of the circle. In the first case,  $C = P'$ , and in the second case, points  $C$  and  $P'$  are endpoints of the same diameter of the triangle. Rotating triangle  $P'Q'R'$  back to triangle  $PQR$ , we obtain in both cases that  $\angle POC = 90^\circ$ .

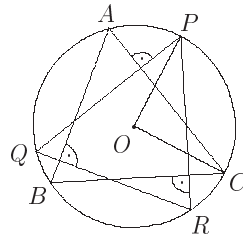


Figure 6

*Solution 2.* Since chords  $AC$  and  $PQ$  intersect at a right angle,  $\frac{\widehat{PC} + \widehat{AQ}}{2} = 90^\circ$ . Analogously,  $\frac{\widehat{AQ} + \widehat{RB}}{2} = 90^\circ$  and  $\frac{\widehat{RB} + \widehat{PC}}{2} = 90^\circ$ . These equations give  $\widehat{PC} = \widehat{AQ} = \widehat{RB} = 90^\circ$ .

**1.** Find all pairs of real numbers  $(x, y)$  that satisfy the equation

$$\frac{x+6}{y} + \frac{13}{xy} = \frac{4-y}{x}.$$

(Grade 9.)

*Answer:*  $x = -3, y = 2$ .

*Solution.* Multiply the equation by  $xy$  to obtain

$$(x+6) \cdot x + 13 = (4-y) \cdot y$$

or equivalently

$$x^2 + 6x + y^2 - 4y + 13 = 0.$$

Notice that the left side can be expressed as sum of squares:

$$(x+3)^2 + (y-2)^2 = 0,$$

giving  $x+3=0$  and  $y-2=0$ , or  $x=-3$  and  $y=2$ .

**2.** The positive differences  $a_i - a_j$  of five different positive integers  $a_1, a_2, a_3, a_4, a_5$  are all different (there are altogether 10 such differences). Find the least possible value of the largest number among the  $a_i$ . (Grade 9.)

*Answer:* 12.

*Solution 1.* Say  $a_1 < a_2 < a_3 < a_4 < a_5$ . Since the 10 positive differences  $a_i - a_j$  are all different, the largest  $a_5 - a_1$  must be greater than or equal to 10, giving  $a_5 \geq 11$ . In the case  $a_5 = 11$  the differences must be exactly 1, 2, ..., 10, hence

$$\begin{aligned} 55 &= 1 + 2 + \dots + 10 = \\ &= (a_5 - a_4) + (a_5 - a_3) + (a_5 - a_2) + (a_5 - a_1) + \\ &\quad + (a_4 - a_3) + (a_4 - a_2) + (a_4 - a_1) + \\ &\quad + (a_3 - a_2) + (a_3 - a_1) + \\ &\quad + (a_2 - a_1) = \\ &= 4a_5 + 2a_4 - 2a_2 - 4a_1. \end{aligned}$$

This is impossible, because  $4a_5 + 2a_4 - 2a_2 - 4a_1$  is even.

We have obtained that the largest of the numbers must be larger than or equal to 12. For instance, we may choose the numbers 1, 3, 8, 11, 12 (the differences  $a_i - a_j$  are therefore  $1 = 12 - 11, 2 = 3 - 1, 3 = 11 - 8, 4 = 12 - 8, 5 = 8 - 3, 7 = 8 - 1, 8 = 11 - 3, 9 = 12 - 3, 10 = 11 - 1$  and  $11 = 12 - 1$ ).

*Solution 2.* Construct the example for the largest number 12 similarly to the previous solution and notice that if the largest of numbers  $a_i$  equals 11, the differences in question

equal 1, 2, ..., 10. Therefore exactly 5 of these differences are odd. Denote by  $x$  and  $y$  the number of odd, resp. even, numbers among the  $a_i$ . The difference of two integers is odd iff the numbers are of different parity. There are  $xy$  possibilities to choose a pair of integers of different parity, hence there are  $xy$  such odd differences. So  $x$  and  $y$  must satisfy the system of equations

$$\begin{cases} x + y = 5 \\ xy = 5 \end{cases},$$

which has no positive integral solutions.

**3.** Three different circles of equal radii intersect in point  $Q$ . The circle  $\mathcal{C}$  touches all of them. Prove that  $Q$  is the center of  $\mathcal{C}$ . (Grade 9.)

*Solution.* Consider an arbitrary point  $P$  inside the circle  $\mathcal{C}$  different from the center point of  $\mathcal{C}$ . Fix  $r > 0$ . We show that there exist at most two circles of radius  $r$  that touch  $\mathcal{C}$  from inside and pass through  $P$ . For this roll the circle of radius  $r$  inside  $\mathcal{C}$ ; evidently the circle being rolled passes through point  $P$  in at most two positions (or equivalently: any fixed circle that touches  $\mathcal{C}$  from inside has at most two common points with such a circle that passes through  $P$  and has the centre point in the centre point of  $\mathcal{C}$  — see figure 7). Hence no such point  $P$  can be the intersection point of three different circles of different radii, all touching the circle  $\mathcal{C}$ .

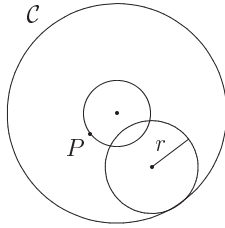


Figure 7

**4.** Find all triples of positive integers  $(x, y, z)$  satisfying  $x < y < z$ ,  $\gcd(x, y) = 6$ ,  $\gcd(y, z) = 10$ ,  $\gcd(z, x) = 8$  and  $\text{lcm}(x, y, z) = 2400$ . (Grade 10.)

*Answer:* (24, 30, 800) and (24, 150, 160).

*Solution 1.* As 6 and 8 both divide  $x$ ,  $\text{lcm}(6, 8) = 24$  divides  $x$ . Similarly  $\text{lcm}(6, 10) = 30$  divides  $y$  and  $\text{lcm}(10, 8) = 40$  divides  $z$ . Hence there exist positive integers  $x', y', z'$ , that  $x = 24x', y = 30y', z = 40z'$ , and

$$\begin{aligned} 6 &= \gcd(x, y) = \gcd(24x', 30y') = 6 \cdot \gcd(4x', 5y'), \\ 10 &= \gcd(y, z) = \gcd(30y', 40z') = 10 \cdot \gcd(3y', 4z'), \\ 8 &= \gcd(z, x) = \gcd(40z', 24x') = 8 \cdot \gcd(5z', 3x'). \end{aligned}$$

These equations imply  $\gcd(4x', 5y') = 1$ ,  $\gcd(3y', 4z') = 1$  and  $\gcd(5z', 3x') = 1$ . Hence  $x', y', z'$  are pairwise coprime and  $\gcd(x', 5) = 1$ ,  $\gcd(y', 4) = 1$  and  $\gcd(z', 3) = 1$ .

We show now that  $\text{lcm}(24x', 30y', 40z') = \text{lcm}(120x', 120y', 120z')$ . For this it suffices to prove that either side of the equation divides the other. Evidently lhs divides rhs. For the other direction, prove that  $120x', 120y', 120z'$  all divide lhs. Indeed,  $24x'$  and 5 divide lhs and, since  $\gcd(x', 5) = 1$ , we have  $\gcd(24x', 5) = 1$  giving that  $24x' \cdot 5 = 120x'$  divides lhs. Similarly prove that  $120y'$  and  $120z'$  divide lhs as well. Now

$$\begin{aligned} 2400 &= \text{lcm}(x, y, z) = \text{lcm}(24x', 30y', 40z') = \\ &= \text{lcm}(120x', 120y', 120z') = 120 \cdot \text{lcm}(x', y', z') = 120 \cdot x'y'z'. \end{aligned}$$

Therefore  $x'y'z' = 20$  and, since  $x', y'$  and  $z'$  are pairwise coprime, they are equal to (in some order) 1, 1, 20 or 1, 4, 5. Requiring  $x < y < z$  for the numbers  $x = 24x', y = 30y'$  and  $z = 40z'$ , the first case implies  $x' = 1, y' = 1$  and  $z' = 20$ , while the second case gives two possibilities:  $x' = 1, y' = 4, z' = 5$ , or  $x' = 1, y' = 5, z' = 4$ . The triples  $(x, y, z)$  are (24, 30, 800), (24, 120, 200) and (24, 150, 160), respectively. The first and the third triple satisfy the conditions of the problem, but the second triple does not, because the greatest common divisor of  $y = 120$  and  $z = 200$  is 40 rather than 10.

*Solution 2.* Let  $p \triangleright a$  denote the exponent of prime  $p$  in the canonical form of number  $a$ . First, consider the exponents of 2: the conditions of the problem imply

$$\begin{aligned} \min(2 \triangleright x, 2 \triangleright y) &= 2 \triangleright 6 = 1, & (2) \\ \min(2 \triangleright x, 2 \triangleright z) &= 2 \triangleright 8 = 3, & (3) \\ \max(2 \triangleright x, 2 \triangleright y, 2 \triangleright z) &= 2 \triangleright 2400 = 5. & (4) \end{aligned}$$

The equation (3) gives us  $2 \triangleright z \geq 3$  and  $2 \triangleright x \geq 3$ . The equation (2) now implies that  $2 \triangleright y = 1$ , and from (3) and (4) we obtain that one of the exponents  $2 \triangleright x$  and  $2 \triangleright z$  equals 3 and the other is 5. Altogether we now have two possibilities:

$$\begin{cases} 2 \triangleright x = 3 \\ 2 \triangleright y = 1 \\ 2 \triangleright z = 5 \end{cases} ; \quad \begin{cases} 2 \triangleright x = 5 \\ 2 \triangleright y = 1 \\ 2 \triangleright z = 3 \end{cases}. \quad (5)$$

Consider now the exponents of 3:

$$\begin{aligned} \min(3 \triangleright x, 3 \triangleright y) &= 3 \triangleright 6 = 1, & (6) \\ \min(3 \triangleright x, 3 \triangleright z) &= 3 \triangleright 8 = 0, & (7) \\ \max(3 \triangleright x, 3 \triangleright y, 3 \triangleright z) &= 3 \triangleright 2400 = 1. & (8) \end{aligned}$$

The equations (6) and (8) imply that  $3 \triangleright x = 3 \triangleright y = 1$ , and the equation (7) gives that  $3 \triangleright z = 0$ . Therefore

$$\begin{cases} 3 \triangleright x = 1 \\ 3 \triangleright y = 1 \\ 3 \triangleright z = 0 \end{cases}. \quad (9)$$

Finally, consider the exponents of 5:

$$\begin{aligned} \min(5 \triangleright x, 5 \triangleright y) &= 5 \triangleright 6 = 0, & (10) \\ \min(5 \triangleright y, 5 \triangleright z) &= 5 \triangleright 10 = 1, & (11) \\ \max(5 \triangleright x, 5 \triangleright y, 5 \triangleright z) &= 5 \triangleright 2400 = 2. & (12) \end{aligned}$$

The equation (11) gives us  $5 \triangleright y \geq 1$  and  $5 \triangleright z \geq 1$ . The equation (10) now implies  $5 \triangleright x = 0$  and from (11) and (12) we obtain that one of the exponents  $5 \triangleright y$  and  $5 \triangleright z$  equals 1 and the other is 2. Altogether there are again two possibilities:

$$\begin{cases} 5 \triangleright x = 0 \\ 5 \triangleright y = 1 \\ 5 \triangleright z = 2 \end{cases} ; \quad \begin{cases} 5 \triangleright x = 0 \\ 5 \triangleright y = 2 \\ 5 \triangleright z = 1 \end{cases}. \quad (13)$$

Since no other prime divides  $\text{lcm}(x, y, z) = 2400$ , no other prime divides  $x, y, z$  either. The conditions (5), (9) ja (13) now give altogether four possibilities:

$$\begin{cases} x = 2^3 \cdot 3^1 \cdot 5^0 = 24 \\ y = 2^1 \cdot 3^1 \cdot 5^1 = 30 \\ z = 2^5 \cdot 3^0 \cdot 5^2 = 800 \end{cases}, \quad \begin{cases} x = 2^3 \cdot 3^1 \cdot 5^0 = 24 \\ y = 2^1 \cdot 3^1 \cdot 5^2 = 150 \\ z = 2^5 \cdot 3^0 \cdot 5^1 = 160 \end{cases},$$

$$\begin{cases} x = 2^5 \cdot 3^1 \cdot 5^0 = 96 \\ y = 2^1 \cdot 3^1 \cdot 5^1 = 30 \\ z = 2^3 \cdot 3^0 \cdot 5^2 = 200 \end{cases}, \quad \begin{cases} x = 2^5 \cdot 3^1 \cdot 5^0 = 96 \\ y = 2^1 \cdot 3^1 \cdot 5^2 = 150 \\ z = 2^3 \cdot 3^0 \cdot 5^1 = 40 \end{cases}.$$

We may check that all the conditions of the problem, except  $x < y < z$ , are satisfied, which leaves us the two former possibilities.

**5.** In the beginning, number 1 has been written to point  $(0, 0)$  and 0 has been written to any other point of integral coordinates. After every second, all numbers are replaced with the sum of the numbers in four neighbouring points at the previous second. Find the sum of numbers in all points of integral coordinates after  $n$  seconds. (Grade 10.)

*Answer:*  $4^n$ .

*Solution.* At the initial moment, the sum of numbers in points of integral coordinates is  $1 = 4^0$ . After every second, any number  $x$  on the plane contributes to the sum of the next second as  $4x$ , hence the total sum of numbers increases 4 times at every step, giving  $4^n$  after  $n$  seconds.

**6.** Real numbers  $a, b$  and  $c$  satisfy  $a^2 + b^2 + c^2 = 1$  and  $a^3 + b^3 + c^3 = 1$ . Find  $a + b + c$ . (Grade 10.)

*Answer:* 1.

*Solution.* As  $a^2 + b^2 + c^2 = 1$ , we have  $-1 \leq a, b, c \leq 1$ , implying  $a^2 \geq a^3, b^2 \geq b^3$  and  $c^2 \geq c^3$ . Since  $a^2 + b^2 + c^2 = a^3 + b^3 + c^3$ , all three inequalities must have equality:  $a^2 = a^3, b^2 = b^3$  and  $c^2 = c^3$ . Therefore  $a, b, c$  can only be equal to 0 or 1 and exactly two of them must be 0 and the third one 1 because  $a^2 + b^2 + c^2 = 1$ . All cases imply  $a + b + c = 1$ .

**7.** Find all functions  $f$  which are defined on all non-negative real numbers, take non-negative real values only, and satisfy the condition

$$x \cdot f(y) + y \cdot f(x) = f(x) \cdot f(y) \cdot (f(x) + f(y))$$

for all non-negative real numbers  $x, y$ . (Grade 11.)

*Answer:*  $f(x) \equiv 0$  and  $f(x) \equiv \sqrt{x}$ .

*Solution.* Taking  $x = y = 1$ , one gets  $2f(1) = 2f(1)^3$ , so either  $f(1) = 1$  or  $f(1) = 0$ . If  $f(1) = 1$  then, taking  $y = 1$  in the original equality, one gets  $x + f(x) = f(x)(f(x) + 1)$  for all  $x$ . From this we get  $f(x) \equiv \sqrt{x}$  as one solution. If  $f(1) = 0$  then the same substitution leads to  $f(x) \equiv 0$ .

It remains to check that both functions satisfy the conditions of the problem.

**8.** The alphabet of language BAU consists of letters B, A, and U. Independently of the choice of the BAU word of length  $n$  from which to start, one can construct all the BAU words with length  $n$  using iteratively the following rules:

- (1) invert the order of the letters in the word;
- (2) replace two consecutive letters: BA  $\rightarrow$  UU, AU  $\rightarrow$  BB, UB  $\rightarrow$  AA, UU  $\rightarrow$  BA, BB  $\rightarrow$  AU or AA  $\rightarrow$  UB.

Given that BBAUABAUUABAUUUABAUUUABB is a BAU word, does BAU have

- a) the word BUABUABUABUABAUBAUBAUB?
- b) the word ABUABUABUABUAUBAUBAUBAUBA?

(Grade 11.)

*Answer:* a) no; b) yes.

*Solution 1.* a) Define the *value* of a word  $w$  as the number  $b - a$  where  $b$  and  $a$  are the numbers of Bs and As, respectively, in  $w$ . Note that any allowed operation preserves the value of word modulo 3. The value of the known BAU word is  $7 - 8 = -1$  and the value of the word under consideration is  $9 - 8 = 1$ , so the latter cannot belong to BAU.

b) At first, note that the allowed operations enable to interchange any two consecutive letters of a word. Indeed, if these letters are BA, AU or UB then replace them with UU, BB or AA, respectively, then invert the order of letters, then replace the letters UU, BB or AA back, and finally invert the order once more. If the letters to be interchanged are AB, UA or BU then perform all these steps in reverse order.

Iteration of the interchanging operation enables to interchange any two letters (move the second letter side by side with the first, interchange them, and finally move the first letter to the original place of the second). Note that the known BAU word contains 7 letters B, 8 letters A and 10 letters U, the word under consideration has 8 letters B, 9 letters A and 8 letters U. Thus we can replace some consecutive letters UU with BA and then reorder the letters in the word, obtaining the required word.

*Solution 2.* a) Part b) of the previous solution implies that a word  $w$  belongs to BAU if and only if there is a word  $w'$  in BAU containing any letter the same number of times as  $w$ . Suppose the right numbers of letters can be obtained from the known BAU word by a sequence of the allowed operations. Let  $x$  be the difference of the number of performed replacements BA  $\rightarrow$  UU and the number of the performed replacements UU  $\rightarrow$  BA; analogously define  $y$  and  $z$  for the rules AU  $\rightarrow$  BB and UB  $\rightarrow$  AA, respectively. The known word contains 7 letters B, 8 letters A ja 10 letters U, the word under consideration has 9 letters B, 8 letters A and 8 letters U. This leads to the system of equations

$$\begin{cases} 7 - x + 2y - z = 9 \\ 8 - x - y + 2z = 8 \\ 10 + 2x - y - z = 8 \end{cases}$$

(each equation describes the number of one letter). This system has no integral solutions, so the answer must be "no".

b) Here the word under consideration has 8 letters B, 9 letters A and 9 letters U. The corresponding system of equations is

$$\begin{cases} 7 - x + 2y - z = 8 \\ 8 - x - y + 2z = 9 \\ 10 + 2x - y - z = 8 \end{cases}.$$

One of the solutions of it is  $x = 0, y = z = 1$ . Hence one can construct the word under consideration from the known word, performing both the replacements  $AU \rightarrow BB$  and  $UB \rightarrow AA$  once and then reordering the letters.

**9.** Inside a circle, point  $K$  is taken such that the ray drawn from  $K$  through the centre  $O$  of the circle and the chord perpendicular to this ray passing through  $K$  divide the circle into three pieces with equal area. Let  $L$  be one of the endpoints of the chord mentioned. Does the inequality  $\angle KOL < 75^\circ$  hold? (Grade 12.)

*Answer:* yes.

*Solution.* Let  $AB$  be the diameter containing  $K$  (see figure 8). Moving point  $K$  together with chord  $LM$  along this diameter away from  $O$ , the angle  $KOL$  decreases, so the segment bounded by chord  $LM$  and arc  $LAM$  decreases, too. Hence it suffices to show that, if  $\angle KOL = 75^\circ$ , the area of this segment is more than one third of the area of the circle.

Let  $r$  be the radius and  $\angle KOL = 75^\circ$ ; then the area of the sector  $LOM$  is  $\frac{150}{360} \cdot \pi r^2 = \frac{5}{12} \cdot \pi r^2$  and the area of triangle  $LOM$  is

$$\begin{aligned} S_{LOM} &= 2S_{LOK} = |LK| \cdot |OK| = r^2 \cdot \sin \angle KOL \cdot \cos \angle KOL = \\ &= \frac{1}{2} r^2 \cdot \sin 2\angle KOL = \frac{1}{4} r^2. \end{aligned}$$

Hence the area of the segment bounded by chord  $LM$  and arc  $LAM$  is

$$\frac{5}{12} \cdot \pi r^2 - \frac{1}{4} r^2 = \left( \frac{5}{12} - \frac{1}{4\pi} \right) \pi r^2.$$

As  $\pi > 3$  implies  $\frac{1}{4\pi} < \frac{1}{12}$ , we have  $\frac{5}{12} - \frac{1}{4\pi} > \frac{4}{12} = \frac{1}{3}$ . Consequently, the area of the segment under consideration is indeed greater than one third of the area of the circle.

**10.** Let  $K, L, M$  be the basepoints of the altitudes drawn from the vertices  $A, B, C$  of triangle  $ABC$ , respectively. Prove that  $\vec{AK} + \vec{BL} + \vec{CM} = \vec{0}$  if and only if  $ABC$  is equilateral. (Grade 12.)

*Solution.* If triangle  $ABC$  is equilateral then vectors  $\vec{AK}, \vec{BL}$  and  $\vec{CM}$  have equal lengths and the sizes of the angles between them are equal to  $120^\circ$ . So  $\vec{AK} + \vec{BL} + \vec{CM} = \vec{0}$ .

Assume now that  $\vec{AK} + \vec{BL} + \vec{CM} = \vec{0}$ . Let  $a, b, c$  be the lengths of the sides  $BC, CA, AB$ , respectively, and let  $S$  be the area of triangle  $ABC$ . Then  $|\vec{AK}| = \frac{2S}{a}$ ,  $|\vec{BL}| = \frac{2S}{b}$  and  $|\vec{CM}| = \frac{2S}{c}$ . Rotating vectors  $\vec{AK}, \vec{BL}$  and  $\vec{CM}$  counterclockwise by  $90^\circ$ , they become parallel to the corresponding sides of the triangle and, if we draw them one by one onto the plane, every vector starting from the endpoint of the previous, we obtain a triangle similar to  $ABC$  because the sum of the vectors is zero. Let  $k$  be the similarity coefficient; then  $|\vec{AK}| = ka$ ,  $|\vec{BL}| = kb$  and  $|\vec{CM}| = kc$ . Hence  $\frac{2S}{a} = |\vec{AK}| = ka$  implying  $a = \sqrt{\frac{2S}{k}}$ .

Analogously we find  $b = c = \sqrt{\frac{2S}{k}}$ , i.e. triangle  $ABC$  is equilateral.

**11.** Let  $a, b, c$  be positive real numbers such that  $a^2 + b^2 + c^2 = 3$ . Prove that

$$\frac{1}{1+2ab} + \frac{1}{1+2bc} + \frac{1}{1+2ca} \geq 1.$$

(Grade 12.)

*Solution 1.* Applying the AM-GM inequality to each denominator, one obtains

$$\frac{1}{1+2ab} + \frac{1}{1+2bc} + \frac{1}{1+2ca} \geq \frac{1}{1+a^2+b^2} + \frac{1}{1+b^2+c^2} + \frac{1}{1+c^2+a^2}.$$

Applying now the AM-HM inequality to the whole expression, one obtains

$$\begin{aligned} \frac{1}{1+a^2+b^2} + \frac{1}{1+b^2+c^2} + \frac{1}{1+c^2+a^2} &\geq \\ &\geq 3 \cdot \frac{\frac{1}{3}}{(1+a^2+b^2) + (1+b^2+c^2) + (1+c^2+a^2)} = \\ &= \frac{9}{3+2(a^2+b^2+c^2)} = \frac{9}{3+2 \cdot 3} = 1. \end{aligned}$$

*Solution 2.* Applying the AM-HM inequality, one obtains

$$\begin{aligned} \frac{1}{1+2ab} + \frac{1}{1+2bc} + \frac{1}{1+2ca} &\geq \frac{9}{3+2ab+2bc+2ca} = \\ &= \frac{9}{a^2+b^2+c^2+2ab+2bc+2ca} = \frac{9}{(a+b+c)^2}. \end{aligned}$$

The Jensen inequality for the square function establishes

$$\frac{(a+b+c)^2}{9} \leq \frac{a^2+b^2+c^2}{3} = \frac{3}{3} = 1.$$

Thus

$$\frac{1}{1+2ab} + \frac{1}{1+2bc} + \frac{1}{1+2ca} \geq \frac{9}{(a+b+c)^2} \geq 1.$$

*Solution 3.* Multiplying both sides of the desired inequality by  $\frac{1}{8}(1+2ab)(1+2bc)(1+2ca)$ , one gets the equivalent inequality

$$\frac{1+ab+bc+ac}{4} \geq a^2b^2c^2.$$

Note that

$$1 = \frac{3}{3} = \frac{a^2+b^2+c^2}{3} \geq \sqrt[3]{a^2b^2c^2},$$

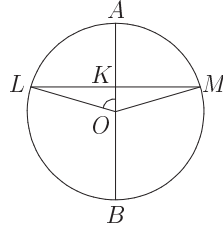


Figure 8



giving  $abc \leq 1$ ; thus

$$a^2b^2c^2 = (abc)^2 \leq \sqrt{abc}.$$

So indeed one has

$$\frac{1 + ab + bc + ac}{4} \geq \sqrt[4]{a^2b^2c^2} = \sqrt{abc} \geq a^2b^2c^2.$$

**12.** Let  $n$  and  $c$  be coprime positive integers. For any integer  $i$ , denote by  $i'$  the remainder of division of product  $ci$  by  $n$ . Let  $A_0A_1 \dots A_{n-1}$  be a regular  $n$ -gon. Prove that

- a) if  $A_iA_j \parallel A_kA_l$  then  $A_{i'}A_{j'} \parallel A_{k'}A_{l'}$ ;  
b) if  $A_iA_j \perp A_kA_l$  then  $A_{i'}A_{j'} \perp A_{k'}A_{l'}$ .

(Grade 12.)

*Solution 1.* a) Clearly  $A_iA_j \parallel A_kA_l$  iff  $i - k \equiv l - j \pmod{n}$  (see figure 9). Consequently,

$$\begin{aligned} A_iA_j \parallel A_kA_l &\Rightarrow i - k \equiv l - j \pmod{n} \Rightarrow \\ &\Rightarrow c(i - k) \equiv c(l - j) \pmod{n} \Rightarrow \\ &\Rightarrow ci - ck \equiv cl - cj \pmod{n} \Rightarrow \\ &\Rightarrow i' - k' \equiv l' - j' \pmod{n} \Rightarrow \\ &\Rightarrow A_{i'}A_{j'} \parallel A_{k'}A_{l'}. \end{aligned}$$

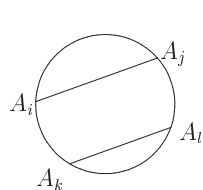


Figure 9

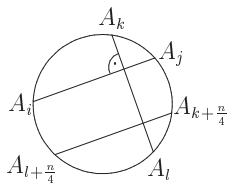


Figure 10

b) Using a well-known theorem of geometry, we obtain

$$\begin{aligned} A_iA_j \perp A_kA_l &\iff \widehat{A_iA_k} + \widehat{A_jA_l} = \widehat{A_kA_j} + \widehat{A_lA_i} \iff \\ &\iff (k - i) + (l - j) \equiv (j - k) + (i - l) \pmod{n} \iff \\ &\iff 2(k + l - i - j) \equiv 0 \pmod{n} \end{aligned}$$

where the arcs considered are taken on the circumcircle of the polygon. Thus

$$\begin{aligned} A_iA_j \perp A_kA_l &\Rightarrow 2(k + l - i - j) \equiv 0 \pmod{n} \Rightarrow \\ &\Rightarrow 2c(k + l - i - j) \equiv 0 \pmod{n} \Rightarrow \\ &\Rightarrow 2(k' + l' - i' - j') \equiv 0 \pmod{n} \Rightarrow \\ &\Rightarrow A_{i'}A_{j'} \perp A_{k'}A_{l'}. \end{aligned}$$

*Solution 2.* b) Note at first that the size of the angle between two arbitrary line segments connecting two vertices is an integral multiple of  $\frac{\pi}{n}$ , and so the angle  $\frac{\pi}{2}$  can occur in case of even  $n$  only. So  $c$  must be odd since  $c$  and  $n$  are coprime, which gives  $c \cdot \frac{n}{2} \equiv \frac{n}{2} \pmod{n}$ .

Thus  $c(i + \frac{n}{2}) \equiv ci + c \cdot \frac{n}{2} \equiv ci + \frac{n}{2} \pmod{n}$ , i.e. any diameter transforms to a diameter. Now consider two cases.

If line segment  $A_iA_j$  is a diameter then  $j \equiv i + \frac{n}{2} \pmod{n}$  and  $A_iA_j \perp A_kA_l$  if and only if  $j - k \equiv l - j \pmod{n}$  (see figure 11). Then  $cj - ck \equiv c(j - k) \equiv c(l - j) \equiv cl - cj \pmod{n}$ , i.e.  $j' - k' \equiv l' - j' \pmod{n}$  and hence  $A_{i'}A_{j'} \perp A_{k'}A_{l'}$ .

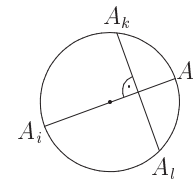


Figure 11

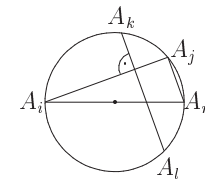


Figure 12

If line segment  $A_iA_j$  is not a diameter then consider the vertex  $A_m$  such that  $A_iA_m$  is a diameter (see figure 12). Then  $A_iA_j \perp A_jA_m$  because the angle  $A_iA_jA_m$  is supported by the diameter  $A_iA_m$ . As diameters transform to diameters, also angle  $A_{i'}A_{j'}A_{m'}$  is supported by a diameter giving  $A_{i'}A_{j'} \perp A_{j'}A_{m'}$ . Since  $A_iA_j \perp A_jA_m$  and  $A_iA_j \perp A_kA_l$ , we get  $A_jA_m \parallel A_kA_l$  and, by a), also  $A_{j'}A_{m'} \parallel A_{k'}A_{l'}$ . Hence  $A_{i'}A_{j'} \perp A_{k'}A_{l'}$ .

## IMO Team Selection Contest

### First Day

**1.** Let  $k > 1$  be a fixed natural number. Find all polynomials  $P(x)$  satisfying the condition  $P(x^k) = (P(x))^k$  for all real numbers  $x$ .

*Answer:*  $P(x) = 0$  and  $P(x) = x^n$ , where  $n$  is an arbitrary non-negative integer; in the case of odd  $k$  also  $P(x) = -x^n$ .

*Solution.* Let the degree of a polynomial  $P(x)$  be  $n > 0$ , then

$$P(x) = a_nx^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0,$$

where  $a_n \neq 0$ .

Let  $i$  be the largest index smaller than  $n$  for which  $a_i \neq 0$  (suppose that such an index  $i$  exists), then

$$P(x)^k = a_n^kx^{kn} + a_ix^{ki} + a_{i-1}x^{k(i-1)} + \dots + a_1x^k + a_0^k$$

and

$$(P(x))^k = (a_nx^n + a_ix^i + a_{i-1}x^{i-1} + \dots + a_1x + a_0)^k.$$

Find next the coefficient of the term  $x^{(k-1)n+i}$  in both polynomials. As  $i < n$ , we get  $kn > (k-1)n + i > ki$ , and the coefficient of this term in the polynomial  $P(x)^k$  is therefore 0. On the other hand, we get the term  $a_nx^n$  in  $(P(x))^k$  iff we take the term

$a_n x^n$  from  $k - 1$  factors and  $a_i x^i$  from one factor; therefore in the polynomial  $(P(x))^k$ , the coefficient of this term is  $k a_n^{k-1} a_i \neq 0$ . This contradiction shows that there is no such index  $i$  and the polynomial  $P(x)$  has the form  $P(x) = a_n x^n$ . Also notice that if the degree  $n$  of the polynomial  $P(x)$  is 0, then also  $P(x) = a_0 = a_n x^n$ .

From the equality  $P(x^k) = (P(x))^k$ , we now get  $a_n x^{nk} = a_n^k x^{nk}$  for all real  $x$ , i.e.  $a_n = a_n^k$  or  $a_n(a_n^{k-1} - 1) = 0$ . Hence  $a_n \in \{-1, 0, 1\}$  for odd  $k$  ( $a_n = 0$  is possible only if  $n = 0$ ) and  $a_n \in \{0, 1\}$  for even  $k$ .

**2.** Let  $O$  be the circumcentre of the acute triangle  $ABC$  and let lines  $AO$  and  $BC$  intersect at point  $K$ . On sides  $AB$  and  $AC$ , points  $L$  and  $M$  are chosen such that  $|KL| = |KB|$  and  $|KM| = |KC|$ . Prove that segments  $LM$  and  $BC$  are parallel.

*Solution.* Draw heights for triangles  $KBL$  and  $KCM$  from the vertex  $K$  and let their bases be  $S$  and  $T$ , respectively. Also lengthen the segment  $AK$  until it intersects the circumcircle of  $ABC$  at point  $P$  (see Figure 13). As segment  $AP$  is a diameter of the circumcircle of  $ABPC$ , the triangles  $ABP$  and  $ACP$  are right-angled. Triangle  $ASK$  is similar to triangle  $ABP$  and triangle  $ATK$  is similar to triangle  $ACP$  (their corresponding sides are parallel), so  $\frac{|AS|}{|AB|} = \frac{|AK|}{|AP|} = \frac{|AT|}{|AC|}$ . Hence triangle  $AST$  is similar to triangle  $ABC$  and therefore  $ST \parallel BC$ . As  $|LS| = |SB|$  and  $|MT| = |TC|$ , we obtain  $LM \parallel BC$ .

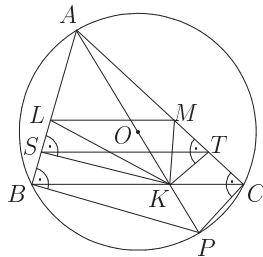


Figure 13

**3.** For which natural number  $n$  is it possible to draw  $n$  line segments between vertices of a regular  $2n$ -gon so that every vertex is an endpoint for exactly one segment and these segments have pairwise different lengths?

*Answer:*  $n = 4k$  and  $n = 4k + 1$ , where  $k$  is an arbitrary positive integer.

*Solution.* Colour the vertices of the  $2n$ -gon alternately black and white. Call the smallest number of sides needed to pass when moving from one vertex to another the *weight* of the segment with endpoints at these vertices. We see that segments with odd weights connect vertices with different colours, but segments with even weights connect vertices with the same colour.

Suppose that the required construction exists for a given  $n$ . As there is an equal number of vertices of both colours and all segments with odd weights occupy an equal number of vertices of both colours, also all segments with even weights must take occupy an equal number of vertices of both colours. Therefore the number of segments connecting two white vertices equals the number of segments connecting two black vertices, and the number of segments with even weights is an even number. Therefore there must be an even number of even numbers among numbers  $1, 2, \dots, n$ , which is possible only if  $n \equiv 0 \pmod{4}$  or  $n \equiv 1 \pmod{4}$ .

We show next that such sets exist for  $n = 4k$  and  $n = 4k + 1$ . In the following tables, the segments are grouped into blocks of parallel segments. In each row of a block,  $(x, y)$  shows that a segment is drawn between vertices  $x$  and  $y$ , next comes the weight of this segment and, after the end of a block, the number of segments in this block is shown.

Construction for  $n = 4k$ :

$(0, 4k)$	$4k$	$\} 1$			
$(2k - 1, 2k)$	$1$	$\} k$	$(6k - 1, 6k + 1)$	$2$	
$(2k - 3, 2k + 2)$	$5$		$(6k - 3, 6k + 3)$	$6$	
.....	.....		.....	.....	
$(1, 4k - 2)$	$4k - 3$	$\} k$	$(4k + 1, 8k - 1)$	$4k - 2$	
$(4k - 1, 4k + 2)$	$3$		$\} k - 1$	$(2, 8k - 2)$	$4$
$(4k - 3, 4k + 4)$	$7$			$(4, 8k - 4)$	$8$
.....	.....	.....	.....	.....	
$(2k + 1, 6k)$	$4k - 1$	$\} k - 1$	$(2k - 2, 6k + 2)$	$4k - 4$	

Construction for  $n = 4k + 1$ :

$(0, 4k + 1)$	$4k + 1$	$\} 1$	$(5k + 1, 7k + 1)$	$2k$	$\} 1$
$(k + 1, k)$	$1$	$\} k$	$(2k + 1, 8k + 1)$	$2k + 2$	$\} k$
$(k + 2, k - 1)$	$3$		$(2k + 2, 8k)$	$2k + 4$	
.....	.....		.....	.....	
$(2k, 1)$	$2k - 1$	$\} k - 1$	$(3k, 7k + 2)$	$4k$	
$(5k, 5k + 2)$	$2$		$\} k - 1$	$(4k, 6k + 1)$	$2k + 1$
$(5k - 1, 5k + 3)$	$4$			$(4k - 1, 6k + 2)$	$2k + 3$
.....	.....	.....	.....	.....	
$(4k + 2, 6k)$	$2k - 2$	$\} k - 1$	$(3k + 1, 7k)$	$4k - 1$	

(There are actually many other constructions for both cases.)

## Second Day

**4.** Denote

$$f(m) = \sum_{k=1}^m (-1)^k \cos \frac{k\pi}{2m+1}.$$

For which positive integers  $m$  is  $f(m)$  rational?

*Answer:* for all positive integers  $m$ .

*Solution* Fix a positive integer  $m$  arbitrarily and take

$$a = \cos \frac{\pi}{2(2m+1)} \neq 0.$$

Using the formula

$$\cos x \cos y = \frac{1}{2}(\cos(x - y) + \cos(x + y)),$$

we get the chain of equations

$$\begin{aligned}
f(m) \cdot a &= \left( \sum_{k=1}^m (-1)^k \cos \frac{k\pi}{2m+1} \right) \cdot a = \\
&= \sum_{k=1}^m (-1)^k \cos \frac{k\pi}{2m+1} \cos \frac{\pi}{2(2m+1)} = \\
&= \frac{1}{2} \sum_{k=1}^m (-1)^k \left( \cos \left( \frac{k\pi}{2m+1} - \frac{\pi}{2(2m+1)} \right) + \right. \\
&\quad \left. + \cos \left( \frac{k\pi}{2m+1} + \frac{\pi}{2(2m+1)} \right) \right) = \\
&= \frac{1}{2} \sum_{k=1}^m (-1)^k \left( \cos \frac{(2k-1)\pi}{2(2m+1)} + \cos \frac{(2k+1)\pi}{2(2m+1)} \right) = \\
&= \frac{1}{2} \left( -\cos \frac{\pi}{2(2m+1)} + (-1)^m \cos \frac{(2m+1)\pi}{2(2m+1)} \right) = \\
&= \frac{1}{2} (-a + (-1)^m \cdot 0) = -\frac{1}{2} a .
\end{aligned}$$

Hence  $f(m) = -\frac{1}{2}$  for all  $m$ .

**5.** Find all natural numbers  $n$  for which the number of all positive divisors of the number  $\text{lcm}(1, 2, \dots, n)$  is equal to  $2^k$  for some non-negative integer  $k$ .

*Answer:* 1, 2, 3 and 8.

*Solution.* Let  $\mathbb{P}$  be the set of all prime numbers. Let  $\delta(m)$  denote the number of positive divisors of natural number  $m$  and let  $A(n) = \delta(\text{lcm}(1, \dots, n))$ . Denote by  $p \triangleright m$  the exponent of prime number  $p$  in the canonical representation of  $m$ . Notice that

$$\begin{aligned}
A(n) &= \delta(\text{lcm}(1, \dots, n)) = \delta \left( \prod_{p \in \mathbb{P}} p^{\max(p \triangleright 1, \dots, p \triangleright n)} \right) = \\
&= \delta \left( \prod_{p \in \mathbb{P}} p^{\lfloor \log_p n \rfloor} \right) = \prod_{p \in \mathbb{P}} (\lfloor \log_p n \rfloor + 1) .
\end{aligned}$$

Therefore  $A(n)$  is a power of 2 iff all numbers in the form  $\lfloor \log_p n \rfloor + 1$ , where  $p \in \mathbb{P}$ , are powers of 2. Let  $\lfloor \log_2 n \rfloor + 1 = 2^k$  and  $\lfloor \log_3 n \rfloor + 1 = 2^l$ . As  $\log_2 n \geq \log_3 n$ , we get  $k \geq l$ . Consider two cases.

If  $k = l$  then

$$\lfloor \log_2 n \rfloor = \lfloor \log_3 n \rfloor , \quad (14)$$

this holds for  $n = 1$  and  $n = 3$ . With immediate check we see that, for  $n = 2$  and  $4 \leq n < 8$ , the equation (14) does not hold. If  $n \geq 8$  then

$$\log_2 n - 3 = \log_2 n - \log_2 8 = \log_2 3(\log_3 n - \log_3 8) > \log_3 n - \log_3 8 ,$$

hence  $\log_2 n - \log_3 n > 3 - \log_3 8 > 3 - 2 = 1$ . Therefore  $\lfloor \log_2 n \rfloor > \lfloor \log_3 n \rfloor$  for all  $n \geq 8$ . Hence (14) holds iff  $n = 1$  or  $n = 3$ .

Let now be  $k > l$ . Then  $\lfloor \log_2 n \rfloor + 1 \geq 2(\lfloor \log_3 n \rfloor + 1)$  or

$$\lfloor \log_2 n \rfloor \geq 2\lfloor \log_3 n \rfloor + 1 , \quad (15)$$

this holds for  $n = 2$  and  $n = 8$ . With immediate check we see that for  $4 \leq n \leq 7$  and  $9 \leq n < 27$  the inequality (15) does not hold. If  $n \geq 27$  then

$$\log_2 n - \log_2 27 = \log_2 3(\log_3 n - \log_3 27) < 2(\log_3 n - 3) = 2\log_3 n - 6 ,$$

hence  $2\log_3 n - \log_2 n > 6 - \log_2 27 > 6 - 5 = 1$ . Therefore  $\lfloor \log_2 n \rfloor < \lfloor 2\log_3 n \rfloor \leq 2\lfloor \log_3 n \rfloor + 1$  for all  $n \geq 27$ . Hence (15) holds iff  $n = 2$  or  $n = 8$ .

Therefore the only possible values for  $n$  are 1, 2, 3 and 8. For them, we get  $A(1) = 1 = 2^0$ ,  $A(2) = 2 = 2^1$ ,  $A(3) = 4 = 2^2$  and  $A(8) = 32 = 2^5$ . Hence all these values suit.

**6.** Call a convex polyhedron a *footballoid* if it has the following properties.

- (1) Any face is either a regular pentagon or a regular hexagon.
- (2) All neighbours of a pentagonal face are hexagonal (a *neighbour* of a face is a face that has a common edge with it).

Find all possibilities for the number of pentagonal and hexagonal faces of a footballoid.

*Answer:* there are 12 pentagonal and 20 hexagonal faces.

*Solution.* We show first that there exists a footballoid with 12 pentagonal and 20 hexagonal faces. Start with a regular icosahedron and abstract from every vertex a regular pyramid with lateral edge  $\frac{1}{3}$  of the edge of the icosahedron. In such a way, we get 12 regular pentagons instead of 12 vertices of icosahedron and we get 20 regular hexagons instead of 20 faces of icosahedron. All neighbours of any pentagonal face are hexagonal. Now show that it is the only possibility. Let  $B$  be a footballoid. Consider an arbitrary vertex of  $B$ ; let it belong to  $x$  pentagonal and  $y$  hexagonal faces. Then  $x + y \geq 3$  as every vertex of a polyhedron belongs to at least 3 faces. As the sizes of the interior angles of pentagonal and hexagonal faces are  $108^\circ$  and  $120^\circ$ , respectively, we get  $x \cdot 108^\circ + y \cdot 120^\circ < 360^\circ$ . Hence  $x + y \leq 3$  and  $x > 0$ . Therefore  $x + y = 3$ , which means that every vertex of a footballoid belongs to exactly 3 faces, at least one of which is pentagonal. As these 3 faces are pairwise neighbours and pentagonal faces cannot be neighbours, every vertex must belong to exactly one pentagonal and two hexagonal faces.

Consider an arbitrary hexagonal face. All its vertices belong to one pentagonal and one hexagonal face. Therefore the neighbours of a hexagonal face are alternately pentagonal and hexagonal, so there are exactly three of both kinds.

Now cover every pentagonal face with a regular pentagonal pyramid, whose lateral edges are continuations of the (hexagonal) neighbours of this face. In this way, hexagonal faces become equilateral triangles and pentagonal faces are replaced with vertices in which five edges meet. As any two neighbours of a pentagonal face which are neighbouring themselves meet under the same angle (two regular hexagons and one regular pentagon can meet in one vertex in principle in only one way), also the triangles meeting in a vertex of our new polyhedron meet under the same angle. Therefore the new polyhedron is a regular icosahedron. It has 12 vertices and 20 faces, so the footballoid  $B$  had to have 12 pentagonal and 20 hexagonal faces.