Estonian Math Competitions 2017/2018

University of Tartu Youth Academy
Tartu 2018

WE THANK:

Estonian Ministry of Education and Research

University of Tartu

Problem authors: Kaarel Hänni, Joonas Jürgen Kisel, Oleg Koshik, Härmel Nestra, Ago-Erik Riet, Erik Paemurru, Kati Smotrova, Laur Tooming, Janno Veeorg, Triinu Veeorg, Indrek Zolk

Translators: Härmel Nestra, Kaur Aare Saar, Sandra Schumann, Jaan Toots

Editors: Härmel Nestra, Reimo Palm

Estonian Mathematical Olympiad

http://www.math.olympiaadid.ut.ee/

Mathematics Contests in Estonia

The Estonian Mathematical Olympiad is held annually in three rounds: at the school, town/regional and national levels. The best students of each round (except the final) are invited to participate in the next round. Every year, about 110 students altogether reach the final round.

In each round of the Olympiad, separate problem sets are given to the students of each grade. Students of grade 9 to 12 compete in all rounds, students of grade 7 to 8 participate at school and regional levels only. Some towns, regions and schools also organize olympiads for even younger students. The school round usually takes place in December, the regional round in January or February and the final round in March or April in Tartu. The problems for every grade are usually in compliance with the school curriculum of that grade but, in the final round, also problems requiring additional knowledge may be given.

The first problem solving contest in Estonia took place in 1950. The next one, which was held in 1954, is considered as the first Estonian Mathematical Olympiad.

Apart from the Olympiad, open contests take place in September and in December. In addition to students of Estonian middle and secondary schools who have never been enrolled in a university or other higher educational institution, all Estonian citizens who meet the participation criteria of the forthcoming IMO may participate in these contests. The contestants compete in two categories: Juniors and Seniors. In the former category, only students up to the 10th grade may participate. Being successful in the open contests generally assumes knowledge outside the school curriculum.

Based on the results of all competitions during the year, about 20 IMO team candidates are selected. IMO team selection contest for them is held in April or May in two rounds. Each round is an IMO-style two-day competition with 4.5 hours to solve 3 problems on both days. Some problems in our selection contest are at the level of difficulty of the IMO but easier problems are usually also included.

The problems of previous competitions can be downloaded at the Estonian Mathematical Olympiads website.

Besides the above-mentioned contests and the quiz "Kangaroo", other regional and international competitions and matches between schools are held.

*

This booklet presents selected problems of the open contests, the final round of national olympiad and the team selection contest. Selection has been made to include only problems that have not been taken from other competitions or problem sources without significant modification and seem to be interesting enough.

Selected Problems from Open Contests

O1. *(Juniors.)* A slider on the scrolling bar of Paul's mail client shows the proportion of emails which are preceding the email which is currently open. Paul noticed that before deleting some emails the slider was on 10%. Paul deleted some consecutive emails, starting from the one which was currently open. After that the slider was on 50%. What is the percentage of the emails that Paul deleted?

Answer: 80%.

Solution. Let n be the number of emails before deletion. As the slider showed 10% before deletion there were $0.1n$ emails preceding the email that was open when Paul started the deletion. After the end of the deletion those emails accounted for 50% from the remaining emails which means that $0.2n$ emails remained in the inbox. Thus Paul deleted $0.8n$ emails which makes up 80% of the emails present in the inbox before deletion.

O2. *(Juniors.)* Find all triplets of consecutive integers, such that one of these numbers is the sum of the two others.

Answer: $(1, 2, 3)$, $(-1, 0, 1)$, $(-3, -2, -1)$.

Solution 1. Let the consecutive numbers be x, $x+1$, and $x+2$. There are three cases based on which of the numbers is the sum of the other two. If $x+2 = x+(x+1)$, then $x = 1$, which gives $(1, 2, 3)$. If $x+1 = x+(x+2)$, then $x = -1$, which gives $(-1, 0, 1)$. If $x = (x+1)+(x+2)$, then $x = -3$, which gives $(-3, -2, -1)$.

Solution 2. If all three numbers are positive, then only the greatest of them can be the sum of the two others. As the sum has to be greater than one of the summands by 1, the other summand has to be 1. This results in the triplet $(1, 2, 3)$. If all the three numbers are negative, then only the smallest of the three numbers can be the sum of the other two. As the sum has to be smaller than one of the summands by 1, the other summand has to be -1. This results in the triplet $(-3, -2, -1)$. If the numbers are neither all positive nor all negative, then they have to include 0. Number 0 cannot be a summand, otherwise the sum would equal the other summand. Hence 0 has to be the sum of the other numbers. In that case the summands must be -1 and 1. This results in the final triplet $(-1, 0, 1)$.

Solution 3. One of the summands has to be either greater than the sum by 1 or less than the sum by 1. Hence the second summand is equal to the difference of the sum and the first summand, so it is either 1 or -1. It cannot be the middle number in the triplet (otherwise the difference of the other two numbers cannot be 1 or -1). Hence, we only need to investigate the triplets in which either the smallest or the largest number is either 1 or -1. The only such triplets are $(1, 2, 3)$, $(-1, 0, 1)$, and $(-3, -2, -1)$. All of them result in solutions, as $1+2 = 3$, $-1+1 = 0$, and $-2+(-1) = -3$.

O3. *(Juniors.)* Mari writes 8 prime numbers (not necessarily different) to her notebook, all smaller than 200. She then adds 1 to the first number, 2 to the second number and so on until adding 8 to the eighth number. She then finds the product of the eight sums. Find the largest power of two which can divide the product found.

Answer: 2^{31}.

Solution. To have the product divisible by as great power of 2 as possible, each of the factors has to be divisible by the greatest power of 2 possible. By adding an even number to a prime number, the sum is divisible by 2 only if the original prime is even. As the only even prime number is 2, it has to be chosen to all even positions. Their respective sums are 4, 6, 8, and 10, and the respective maximal powers of 2 which they are divisible with are 2^2, 2^1, 2^3, and 2^1. Let's investigate the odd positions one by one.

As 127 is a prime and $127 + 1$ is the maximal power of 2 less than $200 + 1$, the first position is 127. As $125 = 5^3$, the maximal power of 2 dividing the third sum cannot be 2^7. However, 61 is a prime and $61 + 3 = 2^6$, hence 61 is suitable for the third position. As $123 = 3 \cdot 41$, the maximal power of 2 dividing the fifth sum cannot be 2^7. However, 59 is a prime and $59 + 5 = 2^6$, hence 59 is suitable for the fifth position. As $121 = 11^2$, the maximal power of 2 dividing the seventh sum cannot be 2^7. The natural numbers smaller than 200 which are divisible by 2^6 are 64 and 192. The respective numbers which would result in these numbers are $57 = 3 \cdot 19$ and $185 = 5 \cdot 37$, neither of which is a prime. However, 89 is a prime and $89 + 7 = 3 \cdot 2^5$, hence 89 is suitable for the seventh position. In summary, the maximal power of 2 which divides the product of the eight numbers is $2^{7+2+6+1+6+3+5+1} = 2^{31}$.

Remark. The set of prime numbers found is the only set which satisfies the problem condition, as $189 = 3 \cdot 61$, $187 = 11 \cdot 17$, $25 = 5^2$, and $153 = 3^2 \cdot 17$, hence the third position cannot be 189, fifth position cannot be 187 and seventh position cannot be 25 or 153, which could result in factors which are divisible by the same power of 2.

O4. *(Juniors.)* Let M be the intersection of the diagonals of a cyclic quadrilateral $ABCD$. Find the length of AD, if it is known that $AB = 2\,\text{mm}$, $BC = 5\,\text{mm}$, $AM = 4\,\text{mm}$, and $\frac{CD}{CM} = 0.6$.

Answer: 6 mm.

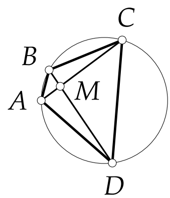
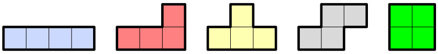
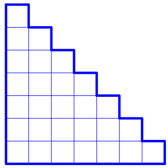


Fig. 1

Solution. Opposite angles AMB and DMC equal. Also notice that $\angle ABM = \angle ABD = \angle ACD = \angle MCD$, as ABD and ACD are subtended to the same arc. Therefore the triangles AMB and DMC are similar. Hence $\frac{BA}{BM} = \frac{CD}{CM}$, from which $BM = BA \cdot \frac{CM}{CD}$. Analogously the opposite angles AMD and BMC are equal. By the property of inscribed angles $\angle ADM = \angle ADB = \angle ACB = \angle MCB$. Hence, the triangles AMD and BMC are similar and $\frac{AD}{AM} = \frac{BC}{BM}$. In summary, $AD = \frac{AM \cdot BC}{BM} = \frac{AM \cdot BC}{BA} \cdot \frac{CD}{CM} = \frac{4\,\text{mm} \cdot 5\,\text{mm}}{2\,\text{mm}} \cdot 0.6 = 6\,\text{mm}$.

O5. *(Juniors.)* There are five different types of puzzle pieces in the figure.





Kati wants to cover (without overlapping) the figure of 7 stairs (composed of equal squares) in the figure on the right. The supply for each type of puzzle piece is unlimited and it is possible to rotate and reflect them. a) Which type of puzzle piece does Kati have to use in all cases? b) Is there a type of puzzle piece which cannot be used in any case?

Answer: a) The middle one; b) No.

Solution. Colour the stairs in a checkerboard pattern in black and white. Without loss of generality, assume that the corner square is black. Then there are 16 black squares and 12 white squares (Fig. 2). Each type of puzzle piece except the middle one covers an equal number of black and white squares, hence to cover an unequal number of black and white squares, the middle puzzle piece must be used. Fig. 3 and Fig. 4 display two of the possible layouts which show that the other puzzle pieces can be used.

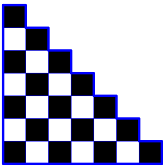


Fig. 2

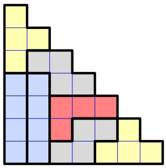


Fig. 3

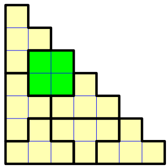


Fig. 4

O6. *(Juniors.)* Teacher drew a pentagon on the blackboard. The following conditions hold for the pentagon.

a) Two of the pentagon's interior angles are equal.
b) There exist three interior angles such that the first one equals the sum of the other two.
c) There exist four interior angles such that one of them equals the sum of the other three.
d) There exists an interior angle that equals the sum of the other four.

Find the interior angles of the pentagon.

Answer: $270^\circ, 135^\circ, 67.5^\circ, 33.75^\circ, 33.75^\circ$.

Solution. Let the sizes of the angles of the pentagon be denoted in decreasing order as $\alpha \geqslant \beta \geqslant \gamma \geqslant \delta \geqslant \varepsilon$. The sum of all the interior angles is $(5-2) \cdot 180^\circ$, in other words $\alpha + \beta + \gamma + \delta + \varepsilon = 540^\circ$. The angle that equals the sum of the other four is greater than the other four. Therefore $\alpha = \beta + \gamma + \delta + \varepsilon = \frac{540^\circ}{2} = 270^\circ$. The angle which equals the sum of some other three cannot be equal to α, because then $\varepsilon = 0^\circ$. As it must be greater than the other three angles, $\beta = \gamma + \delta + \varepsilon = \frac{270^\circ}{2} = 135^\circ$. Analogously, the angle that is the sum of some other two angles can be equal to neither α nor β, therefore $\gamma = \delta + \varepsilon = \frac{135^\circ}{2} = 67.5^\circ$. Finally, the pentagon cannot have more angles of size 270°, 135° or 67.5°, therefore $\delta = \varepsilon = \frac{67.5^\circ}{2} = 33.75^\circ$.

Note. Figures 5–10 depict all the possible ways the angles can be arranged in the pentagon.

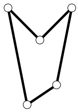


Fig. 5

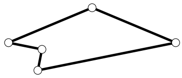


Fig. 6

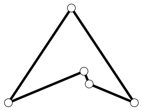


Fig. 7

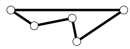


Fig. 8

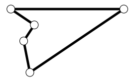


Fig. 9

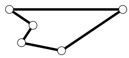


Fig. 10

O7. *(Juniors.)* A plus or a minus sign is placed between every pair of consecutive digits in the sequence 0 1 2 3 4 5 6 7 8 9.

a) Find the smallest positive odd number that cannot be equal to the value of the resulting expression.

b) Find the smallest positive even number that cannot be equal to the value of the resulting expression.

Answer: a) 47; b) 2.

Solution. Let the sum of the digits with a plus sign in front of them be x and the absolute value of the sum of the digits with a minus sign in front of them be y. Then the value v of the expression equals $x - y$. Also $x + y = 0 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45$. Therefore $v = 45 - 2y$.

a) As $45 - 2y \leqslant 45$, nothing greater than 45 can be the value of the expression. We now show that we can obtain all the positive odd integers up to 45 as the result of the expression; this shows that the least positive integer that cannot be equal to the result is 47.

We previously showed that $y = \frac{45-v}{2}$. In order to make the value of the expression be v, we need to put a minus sign in front of some digits that sum up to $\frac{45-v}{2}$. As v is a positive odd integer between 1 and 45, the number $\frac{45-v}{2}$ is a nonnegative integer between 0 and 22. Each such positive integer can be written as a sum of digits as follows: numbers from 1 to 9 are among the digits themselves, numbers from 10 to 17 can be obtained as the sum of 9 and some other digit and numbers 18 to 22 can be written as the sum of 9, 8 and some other digit in the range of 1 to 5. The case $y = 0$ corresponds to the version where every digit has a plus sign in front of it.

b) Number 2 cannot be obtained as the value of the expression, because solving $2 = 45 - 2y$ gives $y = 21.5$, which is impossible in integers. The number 2 is also the least positive even number.

O8. *(Juniors.)* Is it possible to find four distinct prime numbers for which the sum of any three of them is also a prime number?

Answer: Yes.

Solution. Suitable prime numbers are 5, 7, 17, and 19. The sums of the corresponding triplets are prime numbers 29, 31, 41, and 43.

Remark 1. It is easy to see that any four numbers that include 2 or 3 violate the given conditions. The sum of 2 with two odd numbers would be even and therefore not a prime number. If one of the four numbers would be 3, then if there were two primes incongruent modulo 3 among the remaining three primes, then the sum of those two and the number 3 would be divisible by 3, however if the other three primes are all equal modulo 3, then their sum would also be divisible by 3.

Remark 2. It is also possible to find four consecutive prime numbers that satisfy the conditions, for example 19, 23, 29, and 31. The corresponding sums of the triplets are 71, 73, 79, and 83, which are consecutive prime numbers as well.

O9. *(Juniors.)* Medians AD, BE, and CF of triangle ABC intersect at point M. Is it possible that the circles with radii MD, ME, and MF a) all have areas smaller than the area of triangle ABC; b) all have areas greater than the area of triangle ABC; c) all have areas equal to the area of triangle ABC?

Answer: a) Yes; b) Yes; c) No.

Solution. a) Let triangle ABC be equilateral (Fig. 11). As the medians of this triangle all have equal lengths, the points D, E, and F are located at equal distances from M, i.e., on a circle with centre M. It suffices to show that the area of this circle is less than the area of triangle ABC. Because the medians of an equilateral triangle are simultaneously altitudes of the triangle and perpendicular to the sides of the triangle, the sides BC, CA, and AB are tangent to the circle at points D, E, and F, respectively. Therefore the circle is the incircle of the triangle ABC. The area of the incircle of the triangle is indeed less than the area of the triangle.

b) Let $AB = AC$ and $BC = 1$ and $AD = 6$ (Fig. 12). Because the median drawn from the vertex angle of an isosceles triangle is simultaneously its

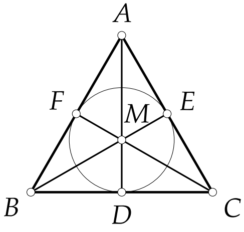


Fig. 11

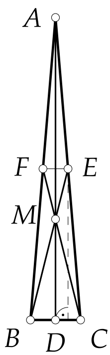


Fig. 12

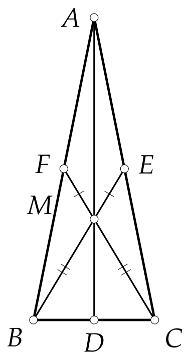


Fig. 13

altitude, the area of this triangle is $\frac{1}{2} \cdot 1 \cdot 6 = 3$. On the other hand, $BE = CF > \frac{1}{2}AD$, because EF is a midsegment of the triangle parallel to BC and $\frac{1}{2}AD$ is the distance between midsegment EF and side BC (shown on Fig. 12 with a dotted line). Therefore $ME = MF > \frac{1}{6}AD = 1$. The circles with radii ME and MF have area greater than π, which in turn is greater than the area of triangle ABC, which is 3. The circle with radius MD has area greater than the area of triangle ABC as well, because $MD = \frac{1}{3}AD = 2 > 1$.

c) If circles with radii MD, ME, and MF would all have areas equal to triangle ABC, their radii should also be equal, therefore $MD = ME = MF$. This would mean that the lengths of the parts of medians that lie on the other side of M are equal as well, in other words, $MA = MB = MC$. We show that then ABC must be equilateral. Indeed, from $ME = MF$ and $MB = MC$ we get that triangles BMF and CME must be equal (Fig. 13), therefore $BF = CE$ and also $AB = AC$. Analogously $AB = BC$. But in part a) we showed that in an equilateral triangle circles with radii MD, ME, and MF do not have the same area as the triangle.

O10. *(Juniors.)* Parents have n children, where n is a given natural number. Find all possibilities for how many children in this family can have both a brother and a sister.

Answer: 0 when $n \leqslant 2$; 0 or 2 when $n = 3$; 0, $n-1$ or n when $n \geqslant 4$.

Solution 1. If all the children have the same gender, then nobody can have a brother as well as a sister. In that case the number of children that match the condition is 0, regardless of n.

If there are children of either gender, but for at least one gender there is exactly one child of that gender, then this child does not have a brother (if he's a boy) or a sister (if she's a girl). If there is exactly one child of the other gender as well ($n = 2$), then the number of children that fulfill the condition is 0. If $n \geqslant 3$, then all other children have both a brother and a sister and there are $n-1$ children that satisfy the condition given in the problem statement. If there are at least 2 children of either gender, then all children have both a brother as well as a sister, and there are n children satisfying the condition. This can happen when $n \geqslant 4$.

Summing up, for $n \leqslant 2$ the answer is 0, for $n = 3$ there can be 0 or $n-1$ (in other words 2) such children and for $n \geqslant 4$ there are either 0, $n-1$ or n such children.

Solution 2. For some child to have both a brother and a sister, there must be at least 3 children in this family. Therefore if $n \leqslant 2$, then the answer is 0. Now assume that $n \geqslant 3$. It is clear that children of the same gender either all satisfy the condition given in problem statement or none of them do. In order for children of some gender not to satisfy the condition there must either be 1 child of such gender or 0 of the opposite gender. In the first case there are $n-1$ children satisfying the condition (since $n \geqslant 3$ and therefore there must be at least 2 children of the opposite gender), in the second case there are 0 such children. Therefore for $n \geqslant 3$ the possible answers are 0, $n-1$, and n. It suffices to note that children of both gender can only satisfy the condition if $n \geqslant 4$.

O11. *(Seniors.)* The *mediant* of two rational numbers u and v is $x = \frac{a+c}{b+d}$, where $\frac{a}{b}$ and $\frac{c}{d}$ are the reduced fractions of u and v respectively. Prove that for any two distinct positive rational numbers u and x, there exist infinitely many positive rational numbers v, such that x is the mediant of u and v.

Solution. Let $u = \frac{a}{b}$ and $x = \frac{c}{d}$ be the reduced fractions of u and x. We are looking for rational numbers v such that $v = \frac{mc-a}{md-b}$, where m is large enough integer such that $mc - a$ and $md - b$ are both positive. According to the definition, x is the mediant of u and v as soon as $\frac{mc-a}{md-b}$ is irreducible. Let's show that there are infinitely many natural numbers m such that $\frac{mc-a}{md-b}$ is irreducible. This will complete the solution.

Let us first prove a lemma: prime numbers which can be used to reduce the fractions are also the factors of $ad - bc$. Indeed, if $p \mid mc - a$ and $p \mid md - b$, then $p \mid a(md-b) - b(mc-a) = m(ad-bc)$, therefore $p \mid m$ or $p \mid ad - bc$. If $p \mid m$, then $p \mid a$ and $p \mid b$ which contradicts the irreducibility of the fraction $\frac{a}{b}$. Therefore $p \mid ad - bc$.

As u and x are different, $ad - bc \neq 0$. Therefore the number $ad - bc$ has a finite number of prime factors. Let $p_1, \ldots, p_l$ be all the different prime factors which can reduce the fraction $\frac{mc-a}{md-b}$ for at least one m and for each $i = 1, \ldots, l$ let m_i be natural number such that the fraction $\frac{m_i c-a}{m_i d-b}$ is reducible with prime p_i.

Let n be an arbitrary factor for which the fraction $\frac{nc-a}{nd-b}$ is not irreducible. This fraction must be reducible with some prime number p_i which can also reduce the fraction $\frac{m_i c-a}{m_i d-b}$. Then $p_i \mid (n - m_i)c$ and $p_i \mid (n - m_i)d$. Therefore $p_i \mid n - m_i$ as otherwise $p \mid c$ and $p \mid d$ which contradicts the irreducibility of the fraction $\frac{c}{d}$. Therefore $n \equiv m_i \pmod{p_i}$.

Therefore by choosing n such that $n \equiv m_i + 1 \pmod{p_i}$ for each $i = 1, \ldots, l$ the fraction $\frac{nc-a}{nd-b}$ must be irreducible. According to Chinese remainder theorem there are infinitely many natural numbers n which satisfy such congruence system.

O12. *(Seniors.)* Find all functions $f : \mathbb{R} \to \mathbb{R}$ which for any real numbers x and y satisfy $(f(x+y))^2 = xf(x) + 2f(xy) + (f(y))^2$.

Answer: $f(x) = 0$ and $f(x) = x$.

Solution. Substituting $x = y = 0$ to the equation, we get $(f(0))^2 = 0 + 2f(0) + (f(0))^2$. Simplifying this gives $f(0) = 0$. Substituting $y = 0$ to the original equation, we get the equation $(f(x))^2 = xf(x) + 2f(0) + (f(0))^2$ which must be satisfied for all real x. Since $f(0) = 0$, this simplifies to $f(x)(f(x) - x) = 0$. Thus, for each x, either $f(x) = 0$ or $f(x) = x$.

Assume there exists a real number $c \neq 0$ such that $f(c) = 0$. Substituting $x = c$ to the original equation, we get $(f(c+y))^2 = 2f(cy) + (f(y))^2$, or

$$2f(cy) = (f(c+y))^2 - (f(y))^2 . \qquad (1)$$

This is valid for any y. Let's analyse four cases based on whether $f(c+y) = 0$ or $f(c+y) = c+y$ and whether $f(y) = 0$ or $f(y) = y$. 1) If $f(c+y) = 0$ and $f(y) = 0$, then (1) simplifies to $f(cy) = 0$. 2) If $f(c+y) = 0$ and $f(y) = y$, then (1) gives $f(cy) = -\frac{y^2}{2}$. Assuming that $f(cy) = cy \neq 0$, we get $c = -\frac{y}{2}$ i.e. $y = -2c$. 3) If $f(c+y) = c+y$ and $f(y) = 0$, then (1) gives us $2f(cy) = (c+y)^2$. Assuming that $f(cy) = cy$, we get $c^2 + y^2 = 0$ from which $c = 0$, contradiction. 4) If $f(c+y) = c+y$ and $f(y) = y$, then (1) gives us $2f(cy) = c^2 + 2cy$. Assuming that $f(cy) = cy$, we get $c^2 = 0$, contradiction. Thus $f(cy) \neq 0$ can only be valid if $y = -2c$, i.e. $f(x) \neq 0$ can only be valid if $x = c \cdot (-2c) = -2c^2$. Thus it is possible to choose a real number d such that $d \neq 0$, $f(d) = 0$, and $|d| \neq |c|$. By replacing d by c we can analogously conclude that $f(x) \neq 0$ can only be valid when $x = -2d^2$. As $-2d^2 \neq -2c^2$, $f(x) \neq 0$ cannot be valid for any $x \neq 0$. So $f(x) = 0$ for all x. Therefore the only suitable functions are $f(x) = 0$ and $f(x) = x$.

O13. *(Seniors.)* The midpoints of the sides BC, CA, and AB of triangle ABC are D, E, and F, respectively. The reflections of centroid M of ABC around points D, E, and F are X, Y, and Z, respectively. Segments XZ and YZ intersect the side AB in points K and L, respectively. Prove that $AL = BK$.

Solution 1. As $\frac{MY}{MB} = \frac{2ME}{2ME} = 1$ and analogously $\frac{MZ}{MC} = 1$ we have $BC \parallel YZ$. Let G be the intersection of lines YZ and AD (Fig. 14). Then $\frac{MG}{MD} = \frac{MY}{MB} = 1$ from which $MG = MD = \frac{1}{3}AD$ and $AG = AD - MD - MG = \frac{1}{3}AD$. Hence $\frac{AL}{AB} = \frac{AG}{AD} = \frac{1}{3}$. By swapping the roles of A and B, the roles of D and E, roles of X and Y, and finally the roles of K and L, we get analogously $\frac{BK}{AB} = \frac{1}{3}$. Therefore $AL = BK$.

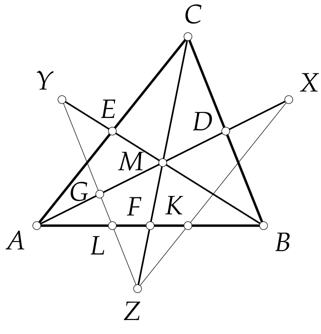


Fig. 14

Solution 2. Notice that triangle XYZ is a homothetic transformation of triangle ABC with centre M and ratio -1. Homothety preserves the directions of the lines, therefore $KL \parallel XY$ and F lies on the median drawn from vertex Z of triangle XYZ. Thus $FK = FL$ and $AL = AF - FL = BF - FK = BK$.

O14. *(Seniors.)* On a plane a finite number of points are marked of which no three are collinear. Assume that there exists a non-convex polygon with all of its vertices located at some of those points. Prove that there exists a non-convex quadrilateral such that all its vertices lie at the marked points.

Solution 1. There has to be a point inside the convex hull of the set of marked points, otherwise any subset of points would form a convex polygon. Let us partition the convex hull into triangles by drawing a necessary amount of diagonals. As no three points are on the same line, the marked point inside the convex hull must be inside one of the triangles. The vertices of that triangle along with the marked point inside the triangle form a non-convex quadrilateral.

Solution 2. We show that it is possible to find 4 points among the vertices of the non-convex n-gon such that they form a non-convex quadrilateral. Let A, B, and C be consecutive vertices of the n-gon such that the internal angle $\angle ABC > 180^\circ$. Let A' and C' be points on the extensions of AB and BC over B. Then the closed broken line corresponding to the polygon has to go through the interior of $\angle A'BC'$. If this area contains a marked point, say X, then $ABCX$ is non-convex (Fig. 15). Alternatively, if there are no vertices of the n-gon within the interior of $\angle A'BC'$, then it has to be intersected by a side of the polygon, say DE, where D is on the side of point A and E is on the side of point C (Fig. 16). Let us show that then $DBCE$ is a non-convex quadrilateral. Indeed, the opposite sides DB and CE are non-intersecting as they are located in different regions of the plane bordered by the lines AA' and CC'. The opposite sides BC and ED are also non-intersecting as they are sides of the original n-gon. In addition, $\angle DBC > 180^\circ$.

Solution 3. Let us prove by induction on n that among the n vertices of the n-gon it is possible to choose 4 vertices which form a non-convex quadrilateral. Trivially, the base case $n = 4$ is true. Assume now that $n > 4$ and that the statement is true for all non-convex polygons with less sides. A non-convex polygon has an interior angle greater than 180° and also an interior angle smaller than 180°. Let's verify that it is not possible that every pair of angles one of which is greater than 180° and the other one smaller than 180° are neighbours to each other. Indeed, as each of the vertices has 2 neighbours, there could be at most 2 angles less than 180° and at most 2 angles greater than 180°. But $n > 4$, contradiction. Therefore, it is possible to find three consecutive vertices A, B, and C and a vertex E different from the former three, such that internal angle $\angle B < 180^\circ$ and internal an-

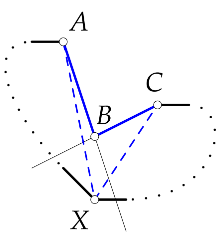


Fig. 15

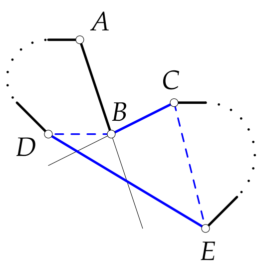


Fig. 16

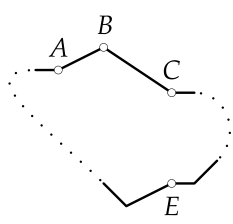


Fig. 17

gle $\angle E > 180^\circ$ (Fig. 17). If there is a vertex X in the interior of triangle ABC, then $ABCX$ is non-convex. If there are no vertices in the interior of triangle ABC, then by removing point B we get a $n-1$-gon which still has interior angle $\angle E > 180^\circ$. Hence it has 4 vertices forming a non-convex quadrilateral by the induction hypothesis.

O15. *(Seniors.)* Is it possible to find four (not necessarily distinct) real numbers b, c, p, and q and (not necessarily distinct) nonnegative integers n and m such that the functions $f(x) = x^2 + bx + c$ and $g(x) = x^2 + px + q$ have n and m distinct real-valued zeros, respectively, and the function $h(x) = (x^2 + px + q)^2 + b(x^2 + px + q) + c$ has a) less than $n + m$ distinct real-valued zeros; b) exactly $n + m$ distinct real-valued zeros; c) more than $n + m$ distinct real-valued zeros?

Answer: a) Yes; b) Yes; c) Yes.

Solution 1. a) Pick $b = 0$, $c = 1$, $p = 0$, $q = 0$. Then $n = 0$ and $m = 1$, because the function $f(x) = x^2 + 1$ does not have any zeros and function $g(x) = x^2$ has one zero 0. The function $h(x) = (x^2)^2 + 1$, however, does not have any zeros, that is it has less than $0 + 1$ zeros.

b) Pick $b = 0$, $c = 1$, $p = 0$, $q = 1$. Then $n = 0$ and $m = 0$, as the function $f(x) = g(x) = x^2 + 1$ does not have zeros. The same is true for the function $h(x) = (x^2 + 1)^2 + 1$, it has precisely $0 + 0$ real-valued zeros.

c) Pick $b = -4$, $c = 4$, $p = 0$, and $q = 1$. Then $n = 1$ and $m = 0$, because $f(x) = x^2 - 4x + 4 = (x - 2)^2$ has one zero 2 and $g(x) = x^2 + 1$ has no zeros. However, the function $h(x) = (x^2 + 1)^2 - 4(x^2 + 1) + 4 = (x^2 - 1)^2$ has zeros 1 and -1, therefore it has more than $1 + 0$ zeros.

Solution 2. Fix $p = q = 0$, then $m = 1$ is fixed as well. Now look for functions $f(x) = x^2 + bx + c$ for which $x^2 + bx + c = (x - d)^2 - 1$. Such functions have zeros $d - 1$ and $d + 1$, because of which $n = 2$ and therefore $n + m = 3$. We show that by varying d we can make the function $h(x) = (x^2 + px + q)^2 + b(x^2 + px + q) + c = x^4 + bx^2 + c = (x^2 - d)^2 - 1$ to have 2, 3 or 4 zeros.

- If $d = 0$, then the function $f(x) = (x - d)^2 - 1$ has zeros -1 and 1. Function $h(x) = (x^2 - d)^2 - 1$ then has two zeros, 1 and -1.
- If $d = 1$, then the function $f(x) = (x - d)^2 - 1$ has zeros 0 and 2. Function $h(x) = (x^2 - d)^2 - 1$ has zeros 0, $\sqrt{2}$ and $-\sqrt{2}$.
- If $d = 2$, then the function $f(x) = (x - d)^2 - 1$ has zeros 1 and 3. Function $h(x) = (x^2 - d)^2 - 1$ has zeros 1, -1, $\sqrt{3}$, and $-\sqrt{3}$.

Remark. In Solution 2 it is possible to make the number of zeros of $h(x) = (x^2 - d)^2 - 1$ be 0 or 1. If $d = -1$, then one of the zeros of the function $f(x) = (x - d)^2 - 1$ is negative and the other one is 0, and thus the function $h(x) = (x^2 - d)^2 - 1$ only has one zero, 0. However, if $d = -2$, then the function $f(x) = (x - d)^2 - 1$ has two negative zeros, because of which the function $h(x) = (x^2 - d)^2 - 1$ does not have any real-valued zeros.

O16. *(Seniors.)* Do there exist five distinct prime numbers for which the sum of any three of them is a prime number as well?

Answer: No.

Solution. Assume that there exist five such prime numbers. If there are three among them pairwise incongruent modulo 3, then by adding each of the three separately to the sum of the other two we get three distinct sums modulo 3. One of those three is 0 modulo 3 and therefore is not a prime number. However, if among the five we only get two distinct remainders when dividing by 3, then by the pigeonhole principle there must be three among them that are congruent modulo 3. The sum of those three, however, is divisible by 3 and therefore not a prime number.

O17. *(Seniors.)* Prove that for all positive real numbers x, y, z

$$\frac{y^2z}{x} + y^2 + z \geqslant \frac{9y^2z}{x+y^2+z}.$$

Solution 1. By bringing all the terms to the same side and to the common denominator, we get an equivalent inequality

$$\frac{y^2z(x+y^2+z) + xy^2(x+y^2+z) + xz(x+y^2+z) - 9xy^2z}{x(x+y^2+z)} \geqslant 0.$$

Since x, y, and z are positive, the denominator $x(x+y^2+z)$ is positive as well. Therefore the fraction on the left-hand side is nonnegative if and only if its numerator is nonnegative. By removing the parentheses, we get an equivalent inequality $x^2y^2 + xy^4 + y^2z^2 + y^4z + x^2z + xz^2 \geqslant 6xy^2z$. However, this inequality follows directly from AM-GM for terms x^2y^2, xy^4, y^2z^2, y^4z, x^2z, and xz^2.

Solution 2. When multiplying both sides of the inequality by $\frac{x+z+y^2}{y^2z}$ we get an equivalent inequality $\left(\frac{y^2z}{x} + y^2 + z\right)\left(\frac{x+z+y^2}{y^2z}\right) \geqslant 9$. This inequality can in turn be transformed into $\left(\frac{y^2z}{x} + y^2 + z\right)\left(\frac{x}{y^2z} + \frac{1}{y^2} + \frac{1}{z}\right) \geqslant 9$. The last one, however, follows from Cauchy-Schwarz, when it is applied to $\left(\frac{y\sqrt{z}}{\sqrt{x}}, y, \sqrt{z}\right)$ and $\left(\frac{\sqrt{x}}{y\sqrt{z}}, \frac{1}{y}, \frac{1}{\sqrt{z}}\right)$.

Remark. It is also possible to finish solution 2 differently, by applying AM-GM inside both parentheses, instead Cauchy-Schwarz, that is, by multiplying the corresponding sides of inequalities $\frac{y^2z}{x} + y^2 + z \geqslant 3\sqrt[3]{\frac{y^4z^2}{x}}$ and $\frac{x}{y^2z} + \frac{1}{y^2} + \frac{1}{z} \geqslant 3\sqrt[3]{\frac{x}{y^4z^2}}$.

O18. *(Seniors.)* Let A' be the result of reflection of vertex A of triangle ABC through line BC and let B' be the result of reflection of vertex B through line AC. Given that $\angle BA'C = \angle BB'C$, can the largest angle of triangle ABC be located: a) At vertex A; b) At vertex B; c) At vertex C?

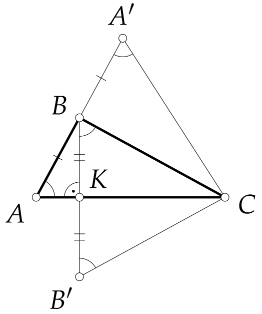


Fig. 18

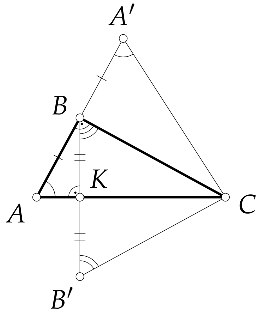


Fig. 19

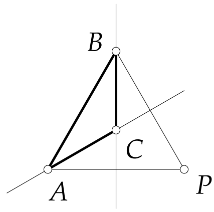


Fig. 20

Answer: a) No; b) Yes; c) Yes.

Solution 1. Let the foot of the altitude drawn from vertex B of triangle ABC be K. By symmetry, $\angle BA'C = \angle BAC$ and $\angle BB'C = \angle KB'C = \angle KBC$.

a) By assumptions, $\angle BAC = \angle KBC$ (Fig. 18). As the angle at vertex K of triangle KBC is right, $\angle KBC < 90°$. If the largest angle of triangle ABC were at vertex A, the triangle would have to be acute. This would mean that K would lie on line segment AC, whence $\angle ABC > \angle KBC = \angle BAC$. Consequently, the largest angle of triangle ABC cannot be at vertex A.

b) If the angle at vertex B of triangle ABC is right then it is the largest angle of the triangle (Fig. 19). Triangles BAC and KBC are similar, whence $\angle BAC = \angle KBC$ and $\angle BA'C = \angle BB'C$.

c) Let ABP be an equilateral triangle with midpoint C (Fig. 20). Obviously P is the reflection of A through line BC, as well as the reflection of B through line AC. Hence defining $A' = B' = P$ fulfills the conditions of the problem. The angle at vertex C of triangle ABC has size $120°$, whence it must be the largest.

Solution 2. a) By symmetry, $\angle BA'C = \angle BAC$. Hence the assumptions imply $\angle BAC = \angle BB'C$. Suppose that the largest angle of the triangle ABC is at the vertex A. Then the angle at the vertex C must be acute, implying that points A and B' lie at the same side of the line BC. Now $\angle BAC =$

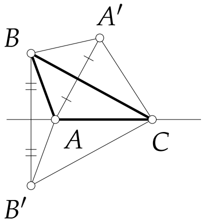


Fig. 21

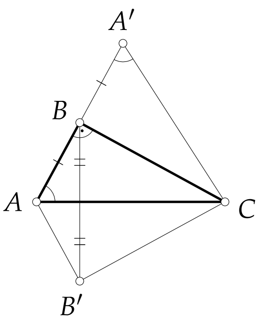


Fig. 22

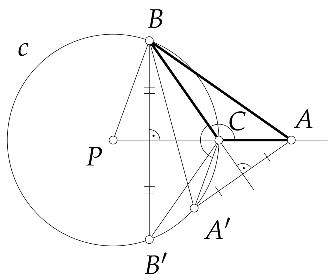


Fig. 23

$\angle BB'C$ implies that B' lies on the circumcircle of the triangle ABC (Fig. 21). Hence AC is the perpendicular bisector of the chord BB' of the circumcircle of the triangle ABC, meaning that it passes through the circumcentre of the triangle ABC. Consequently, AC is the diameter of the circumcircle of the triangle ABC. Hence the triangle ABC has right angle at the vertex B which contradicts the assumption that the largest angle is at the vertex A.

b) If the angle at vertex B of triangle ABC is right then it is the largest angle of the triangle (Fig. 22). By symmetry, $\angle AB'C = \angle ABC = 90^\circ$. Hence points A, B, C, and B' are concyclic, whereby A and B' lie on the same side from line BC. Consequently $\angle BA'C = \angle BAC = \angle BB'C$.

c) Choose points B and C on a circle c with centre P in such a way that $\angle BPC < 90^\circ$. Let B' be the reflection of B through line CP. Let A' be a point on circle c such that $\angle A'CB = 180^\circ - \angle PCB$ (Fig. 23). This choice is possible as moving with point A' along the longer arc BC of circle c gives rise to all values of $\angle A'CB$ below $180^\circ - \frac{1}{2}\angle BPC$, but $180^\circ - \angle PCB = 90^\circ + \frac{1}{2}\angle BPC < 180^\circ - \frac{1}{2}\angle BPC$. Let now A be the reflection of A' through line BC. As $\angle ACB = \angle A'CB = 180^\circ - \angle PCB$, point A lies on line CP, i.e., lines CP and AC coincide. The conditions of the problem are met, since both $BA'C$ and $BB'C$ are angles inscribed in circle c subtended by equal chords.

O19. *(Seniors.)* Let $n \geqslant 2$ be a natural number. Laura's mathematics teacher likes group exercises. In Laura's class the teacher composed new groups in every lesson of statistics. When all of the statistics curriculum had been covered, it appeared that every two different students had belonged together into exactly one group and every two different groups had contained exactly one student in common. On the day when correlation was studied there were precisely n students besides Laura in her group. How many students were in Laura's class if their number was larger than $n+2$?

Answer: $n^2 + n + 1$.

Solution. By assumptions, one group contained $n+1$ students while the whole class contains more than $n+2$ students. As every two students belonged together into one group, more than one groups with more than one members must have occurred. If some group contained only one student C, this student must have belonged to every group while every other student must have belonged to exactly one group (the one where the student was together with C). But this would mean that there was only one group with more than one student which contained the whole class. Consequently groups with only one member cannot have occurred. Consider two cases.

1) Suppose that there were two groups such that each student belonged to at least one of them. Let C be the student that belonged to both of these two groups and let these groups be $\{C, A_1, \ldots, A_k\}$ and $\{C, B_1, B_2, \ldots, B_l\}$. As every pair of different students is together in exactly one group, the other groups must have had to be of the form $\{A_i, B_j\}$ where $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, l$, whereby all such groups must have occurred. As these

groups have to pairwise share one student we have either $k = 1$ or $l = 1$. Let w.l.o.g. $k = 1$. As Laura's group on the day of correlation studies contained $n + 1$ students while $n + 1 > n \geqslant 2$, this must have been the large group with $l + 1$ students. Thus $l = n$ and the total number of students is $n + 2$. But it is given that the total number of students has to be bigger than that.

2) Suppose now that there were no two groups such that every student belonged to at least one of them. We prove that there was an equal number of students in each group. Let $\mathcal{R}$ and $\mathcal{S}$ be arbitrary groups formed and let C be an arbitrary student contained by neither of them. Let m be the number of groups that contain C. Each of these m groups has exactly one student in common with $\mathcal{R}$; all these students are distinct as C is the only common member of every two groups containing C. There can be no more students in $\mathcal{R}$ because every student in $\mathcal{R}$ must belong to some group together with C. Hence $\mathcal{R}$ contained exactly m students. Analogously we see that $\mathcal{S}$ contains exactly m students. As $\mathcal{R}$ and $\mathcal{S}$ were arbitrary, it follows that each group contains exactly m students. As on the day of studying correlation there were $n + 1$ students in Laura's group, we have $m = n + 1$. We can find the size of the class by only considering C's groups: there are $n + 1$ of them and each contains n students besides C, whence we have $1 + (n + 1)n$ students in total. This is indeed more than $n + 2$ and suits as the answer.

Remark. The conditions of the problem can be fulfilled. For $n = 2$, for instance, one can for 7 students A, B, C, D, E, F, G define 7 groups $\{A, B, C\}$, $\{A, D, E\}$, $\{A, F, G\}$, $\{B, D, F\}$, $\{B, E, G\}$, $\{C, D, G\}$, $\{C, E, F\}$. By calling each student a 'point' and each group a 'line', such a structure with $n^2 + n + 1$ students is called a finite projective plane of order n. The existence of a finite projective plane is only known and proven precisely for those n which are powers of a prime number, and in no other case. For most non-prime-powers the existence or non-existence of a projective plane of that order is unknown. For some orders, there exist several different (non-isomorphic) projective planes.

Selected Problems from the Final Round of National Olympiad

F1. *(Grade 9.)* Find all positive integers n such that $n!$ is not divisible by n^2.

Answer: All primes and 4.

Solution 1. For any prime p, $p^2 \nmid p!$ as prime p occurs only once in the prime factorization of $p!$. Additionally, $4^2 = 16$ does not divide $4! = 24$. We will show that $n^2 \mid n!$ for all other positive integers n. Let p be a prime factor of n, and k the exponent of p in the prime factorization of n. If there exists a different prime factor q of n, then both p^k and $p^k q$ are in the set of integers from 1 to n and hence $p^{2k} \mid n!$. This holds for all prime factors p of integer

n. Thus n^2 divides $n!$ for all n with at least two distinct prime factors. Now consider the remaining integers $n = p^k$ for prime p. For $k \geqslant 3$ all three integers p, p^{k-1}, and p^k are distinct factors in $n!$, implying that $p^{2k} = n^2$ divides $n!$. For $k = 2$ and $p > 2$ the integers p, $2p$, and p^2 are distinct factors in $n!$, implying that $p^4 = n^2$ divides $n!$.

Solution 2. We find all positive integers n such that $n \mid (n-1)!$; obviously this condition is equivalent to that of the problem. For prime n, no integer less than n can have a prime factor n and thus n cannot divide $(n-1)!$. For $n = 4 = 2^2$, 4 does not divide $(n-1)! = 6$. On the other hand, for $n = p^2$ where $p > 2$ is prime, the integers p and $2p$ are in the set of integers from 1 to $n-1$, implying $p^2 = n \mid (n-1)!$. If n is a cube or a higher power of some prime, or has at least two prime factors, then it obviously has a divisor d such that $1 < d < n$ and $d^2 \neq n$ (e.g. the smallest prime factor of n). Hence d and $\frac{n}{d}$ are distinct positive integers less than n and their product n divides $(n-1)!$.

F2. *(Grade 9.)* Points P and Q are chosen on the side BC of triangle ABC in such a way that P lies between B and Q, and rays AP and AQ trisect the angle BAC. The line parallel to AQ and passing through P meets the side AB of the triangle at point D, and the line parallel to AP and passing through Q meets the side AC of the triangle at point E. Can it happen that DE is a midsegment of the triangle ABC?

Answer: No.

Solution. Assume that DE is a midsegment of ABC, then D is the midpoint of AB (Fig. 24). As $DP \parallel AQ$, DP is a midsegment of triangle ABQ. Hence, P is a midpoint of BQ and AP is a median of triangle ABQ. As rays AP and AQ trisect the angle BAC, AP is a bisector of angle QAB. Thus, ABQ is an isosceles triangle with altitude AP, implying that AP is perpendicular to BC. Similarly we can see that AQ is perpendicular to BC. This leads to a contradiction as AP and AQ cannot coincide. Therefore, DE cannot be a midsegment of ABC.

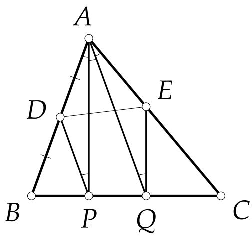


Fig. 24

F3. *(Grade 9.)* Let n and m be positive integers, $n \geqslant m$. There is a game board of size $1 \times n$ divided into n unit squares and an unlimited supply of sticky tapes of size $1 \times m$. On each move, any player adds a tape covering m consecutive unit squares on the board precisely, at least one of which is not yet covered by any tape. Two players move alternately and the one who can make no move loses.

a) Prove that if n and m are of equal parity then the first player can win regardless of how the opponent plays.
b) Is it true that, whenever n and m are of different parity, the second player can win regardless of how the first player plays?

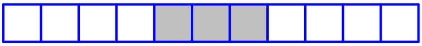


Fig. 25

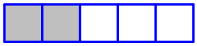


Fig. 26

Answer: b) No.

Solution. a) If n and m are of equal parity, then the first player to move can cover the central squares in such a way that an equal number of uncovered squares remain on both sides (Fig. 25 with $n = 11$ and $m = 3$). The first player can now respond to each of the opposing player's moves by a symmetric move that ensures that uncovered squares remain symmetric w.r.t. the centre of the board after their move. Indeed, the starting move ensures that any subsequent move can cover uncovered squares only to one side of the centre and the first player is free to make symmetric moves in response to the second player. The number of uncovered squares decreases on each move; thus, the second player will lose.

b) If $n = 5$ and $m = 2$, then the first player can cover two squares at the edge of the board on their first move (Fig. 26). Any move by the second player will now leave either 1 or 2 consecutive squares uncovered, and the first player can cover those on their next move, ensuring victory.

F4. *(Grade 9.)* Let $ABCDE$ be a regular pentagon and let c be the circle with diameter AB. Diagonals AC and AD intersect the circle c at points F and G, respectively. Line FG intersects the side AE at point H. Let K be the midpoint of the side DE. Prove that points F, H, E, and K are concyclic.

Solution. As AB is a diameter of c, $\angle AFB = 90^\circ$ (Fig. 27) and BF is an altitude of triangle ABC. From $AB = BC$, BF is a median and F is the midpoint of AC. By symmetry, point K lies on BF and $\angle FKE = 90^\circ$. Notice that $\angle BAC = \angle CAD = \angle DAE$, as BAC, CAD, and DAE are inscribed angles that subtend to equal arcs of the circumcircle of the regular pentagon $ABCDE$. Points A, B, F and G lie on circle c in this order, thus $\angle ABF = 180^\circ - \angle AGF = \angle AGH$. Hence, ABF and AGH are similar from two angles, implying $\angle AHG = \angle AFB = 90^\circ$. Therefore, the opposite angles at K and H of quadrilateral $FHEK$ are right angles and it is cyclic.

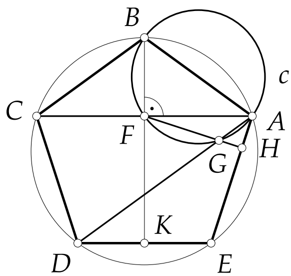


Fig. 27

F5. *(Grade 10.)* Find all integer pairs (a, b) for which $(2a^2 + b)^3 = b^3 a$.

Answer: $(27, 729)$, $(8, 128)$, $(0, 0)$, $(-1, -1)$.

If $b = 0$, then according to the equation $2a^2 + b = 0$ from which $a = 0$. Assume now that $b \neq 0$. As b^3 and $(2a^2 + b)^3$ are perfect cubes, their ratio a is the cube of a rational number; as it is an integer, it is the cube of an integer c. By taking cubic root from each side of the equation we get a relation $2c^6 + b = bc$, implying $b(c-1) = 2c^6$. As c and $c-1$ are coprime, c^6 and $c-1$ are also coprime. Therefore, $c-1$ has to divide 2 and c must be one of 3, 2, 0 or -1. If $c = 3$, then $a = 27$ and we get $2b = 2 \cdot 729$, whence $b = 729$. If $c = 2$, then $a = 8$ and we get $b = 128$. If $c = 0$, then we get $-b = 0$, which has already been analysed. If $c = -1$, then $a = -1$ and $-2b = 2$ from which $b = -1$.

F6. *(Grade 10.)* A circle with diameter AB intersects side BC of rhombus $ABCD$ at point K. A circle with diameter AD intersects side CD of rhombus $ABCD$ at point L. Find the angles of rhombus $ABCD$ if $\angle AKL = \angle ABC$.

Answer: 60° and 120°.

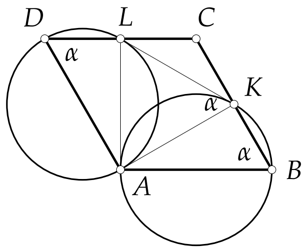


Fig. 28

Solution 1. Let $\angle ABC = \angle ADC = \alpha$; then $\angle AKL = \alpha$ (Fig. 28). According to Thales' theorem AK is perpendicular to BC and AL is perpendicular to CD. But $\angle ABK = \alpha = \angle ADL$ and $AB = AD$, so ABK and ADL are equal. Therefore, $AK = AL$ from which $\angle ALK = \alpha$. As the sum of the internal angles of a quadrilateral is 360°, we have $\angle KAL = 360^\circ - \angle KCL - \angle AKC - \angle ALC = 360^\circ - (180^\circ - \alpha) - 2 \cdot 90^\circ = \alpha$. Therefore, the triangle AKL is equilateral as all its angles are equal to α, implying that the angles of the rhombus are 60° and 120°.

Solution 2. As in Solution 1, denote $\alpha = \angle ABC = \angle ADC = \angle AKL$ and show that triangles ABK and ADL are equal. Therefore, $BK = DL$ from which $CK = CL$. We have $\angle KCL = 180^\circ - \alpha$, therefore $\angle CKL = \angle CLK = \frac{\alpha}{2}$. Thus $90^\circ = \angle AKC = \angle AKL + \angle LKC = \frac{3}{2}\alpha$ from which $\alpha = 60^\circ$. Therefore, the angles of the rhombus are 60° and 120°.

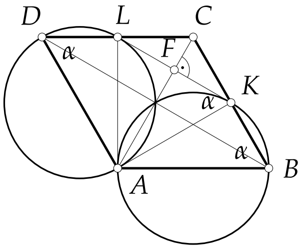


Fig. 29

Solution 3. As in Solution 1, denote $\alpha = \angle ABC = \angle ADC = \angle AKL$ and show that triangles ABK and ADL are equal. Hence, $BK = DL$ from which $\frac{BK}{BC} = \frac{DL}{CD}$. Thus $KL \parallel BD$ and $AC \perp KL$. Let AC and KL meet at F (Fig. 29). The altitude KF from the right angle of triangle CKA divides the triangle to two triangles CFK and KFA, both similar to CKA. Therefore, $\angle ACK = \angle AKF = \alpha$. The diagonal of the rhombus is also the angle bisector, therefore, $\angle BCD = 2\alpha$. Equation $180^\circ - \alpha = 2\alpha$ gives us $\alpha = 60^\circ$ and the angles of the rhombus are 60° and 120°.

F7. *(Grade 10.)* Every sound in a certain language can be either long or short. A sound is classified either as a vowel or as a consonant. Every word consists of exactly two sounds (without repetitions) and satisfies the following conditions.

1) Every word contains a short sound.
2) Words beginning with a vowel contain a long sound.
3) Words beginning with a consonant or ending with a vowel have a short sound at the end.

All sequences of two distinct sounds satisfying these conditions are words. The written language optimisation committee has decided to denote each sound with a different letter. However, they are considering two possibilities for denoting length. The first proposes denoting vowels with single letters and consonants with single or double letters based on length. The second instead proposes denoting consonants with single letters and vowels with single or double letters based on length. Is it possible to determine the lengths of sounds in all words from writing: a) in the case of the first proposal; b) in the case of the second proposal?

Answer: a) Yes; b) No.

Solution. a) If the word consists of two vowels, then based on the rule 3) the second of them is short and based on the rule 2) the first of them is long. If the word begins with a vowel and ends with a consonant, then the length of the second sound is determined uniquely by its writing and the length of first sound must be of the opposite length based on 1) and 2). If the word begins with a consonant, the length of first sound is determined uniquely by its writing and the second sound is short based on 3). Therefore, all writings of words in the language have unique pronunciation.

b) The rules allow both: a word consisting of two short consonants, and a word starting with a long consonant and ending with a short consonant. According to the proposal, the first sound of these words have identical writing and cannot be uniquely determined.

F8. *(Grade 10.)* There are n candies on the table. On every turn, a player eats a number of candies that is greater than 1 and divides the number of candies on the table at the start of the turn, but must leave at least 1 candy on the table. Two players take alternate turns and the player who is unable to make a move loses. Find all positive integers n for which the first player can always win.

Answer: All even numbers, except odd powers of 2.

Solution. Define all even numbers which are not odd powers of 2 as *good* and the rest of the positive integers as *bad*. We show that the player before whose turn the number of candies is good has a move which yields in a bad number of candies, whereas the player before whose turn the number of candies is bad either has lost or is forced to leave a good number of candies on the table. As the number of candies is reduced in each move, the starting player wins iff initially the number of candies on the table is good.

A player before whose turn the number of candies is even but not a power of 2 can eat the number of candies equal to its odd factor different from 1. After that, the number of candies on the table is odd, i.e. bad. If before the turn the number of candies is equal to an even power of 2, one can eat exactly half of the candies, leaving an odd power of 2 candies on the table, i.e. a bad number.

On the other hand, if the number of candies before the turn is odd, the player can only choose odd factors. This yields in an even number of candies left on the table. Furthermore, the number of candies left on the table is divisible by the number of candies taken from the table which is odd, hence, the number of candies cannot be a power of 2. Therefore, the number of candies left on the table is good. However, if the number of candies before the turn is equal to an odd power of 2, the player can only choose even factors, which results in an even number of candies left on the table. Furthermore, the rules do not allow eating more than half of the candies. Therefore, the number of candies left of the table can only be a power of 2 if its exponent is less by 1 than before the move. This would be an even power of 2 which is also a good number.

F9. *(Grade 11.)* Find all prime numbers p such that $2p^3 + 4p^2 - 3p + 12$ is the fifth power of an integer.

Answer: 11.

Solution. Denote $f(n) = 2n^3 + 4n^2 - 3n + 12$. The following table shows the remainders of n^2, n^3, n^5 and $f(n)$ upon division by 11:

n	0	1	2	3	4	5	6	7	8	9	10
n^2	0	1	4	9	5	3	3	5	9	4	1
n^3	0	1	8	5	9	4	7	2	6	3	10
n^5	0	1	10	1	1	1	10	10	10	1	10
$f(n)$	1	4	5	5	5	6	9	4	3	7	6

As one can see from the table, the only remainders upon division by 11 that the fifth power of an arbitrary integer n can give are 0, 1 and 10. On the other hand, integers of the form $f(n)$ give only remainders 1, 3, 4, 5, 6, 7, and 9 upon division by 11, whereby the remainder is 1 only if n is divisible by 11. Consequently, $f(p)$ can be the fifth power of an integer only if p is divisible by 11. As p is prime, the only possibility is $p = 11$. And indeed, $f(11) = 2 \cdot 11^3 + 4 \cdot 11^2 - 3 \cdot 11 + 12 = 3125 = 5^5$.

F10. *(Grade 11.)* An acute angle with vertex A and size α is given on a plane. Points B_0 and B_1 are chosen on different sides of the angle in such a way that $\angle AB_0B_1 = \beta$. Whenever points B_0, B_1, $\ldots$, B_{n-1} are defined, the next point B_n on side AB_{n-2} is allowed to be defined in such a way that $B_n \neq B_{n-2}$ and $B_{n-1}B_n = B_{n-2}B_{n-1}$. Prove that this process cannot last infinitely and determine the largest index n (depending on α and β) for which B_n can be defined.

Answer: $\frac{\beta-90^\circ}{\alpha} + 1$ if $\frac{\beta-90^\circ}{\alpha}$ is a natural number, and $\lfloor \frac{\beta}{\alpha} \rfloor + 1$ otherwise.

Solution. Let B_n for some $n > 1$ be definable (Fig. 30). By construction, A, B_{n-2}, and B_n are collinear, whereby A cannot lie between B_{n-2} and B_n. Since $B_{n-1}B_{n-2} = B_{n-1}B_n$ and $B_{n-2} \neq B_n$, the triangle $B_{n-2}B_{n-1}B_n$ is isosceles, so that $\angle AB_{n-2}B_{n-1} + \angle AB_nB_{n-1} = 180°$ and $\angle AB_{n-2}B_{n-1} \neq 90°$. Consequently, $\angle AB_{n-1}B_n = 180° - \alpha - \angle AB_nB_{n-1} = 180° - \alpha - (180° - \angle AB_{n-2}B_{n-1}) = \angle AB_{n-2}B_{n-1} - \alpha$. Since $\angle AB_0B_1 = \beta$, by induction we get $\angle AB_{n-1}B_n = \beta - (n-1)\alpha$. Hence defining of B_n assumes that $\beta - (n-2)\alpha \neq 90°$ and $\beta - (n-1)\alpha \geqslant 0°$. These conditions are also sufficient, because if $\beta - (n-2)\alpha \neq 90°$, meaning that $\angle AB_{n-2}B_{n-1} \neq 90°$, then point B_n can be chosen different from B_{n-2} on the line AB_{n-2}, and if $\beta - (n-1)\alpha \geqslant 0°$, meaning that $\angle AB_{n-2}B_{n-1} \geqslant \alpha$, then this point is located on the side AB_{n-2}.

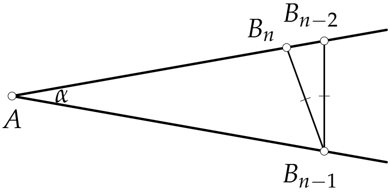


Fig. 30

Summing up, the largest n for which B_n is definable equals $\frac{\beta-90°}{\alpha} + 1$ if $\beta - (n-2)\alpha = 90°$ for some integer $n > 1$, or equivalently, $\frac{\beta-90°}{\alpha}$ is a non-negative integer, and $\lfloor \frac{\beta}{\alpha} \rfloor + 1$ otherwise. Figures 31 and 32 depict the situation in the first and second case, respectively.

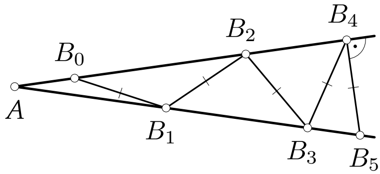


Fig. 31

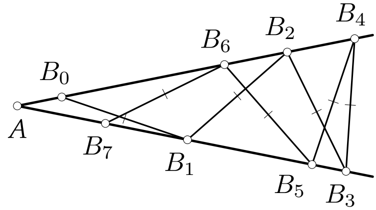


Fig. 32

F11. *(Grade 11.)* A rectangular grid whose side lengths are integers greater than 1 is given. Smaller rectangles with area equal to an odd integer and length of each side equal to an integer greater than 1 are cut out one by one. Finally one single unit square is left. Find the least possible area of the initial grid before the cuttings.

Answer: 121.

Solution. Denote by X the unit square left. Then X cannot lie in a corner of the initial rectangle, as the strip between the neighbouring rectangle of X and the edge of the rectangle could not be cut out (painted gray in Fig. 33). Similarly, X cannot lie at a side of the initial rectangle, because the strip between two neighbouring rectangles of X could not be cut out (Fig. 34).

Consider all possibilities of how the unit squares around X can be distributed between rectangles that are cut out from the initial figure; for simplicity, we identify the 8 unit squares by principal winds. Suppose that the eastern neighbour of X belongs to rectangle A. Then either the northeastern

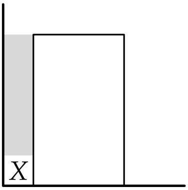


Fig. 33

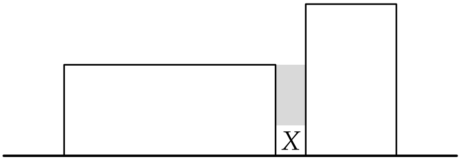


Fig. 34

or southeastern neighbour of X belongs to A, too; w.l.o.g., let the northeastern neighbour of X belong to A (Fig. 35). The northern neighbour of X must belong to a rectangle B distinct from A. Then the northwestern neighbour of X must belong to B. Similarly, the western and southwestern neighbours of X must belong to a third rectangle C and the southern and southeastern neighbour of X must belong to a fourth rectangle D (Fig. 36).

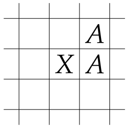


Fig. 35

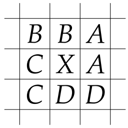


Fig. 36

The unit squares of the strip starting from the eastern neighbour of X and continuing eastward until the edge of the figure distribute between rectangles, as well as the unit squares of the strip starting from the southern neighbour of X and continuing eastward until the edge of the figure distribute between rectangles. As the area of each rectangle is odd, their side lengths are odd. Thus the lengths of both strips must be representable as sums of 1 or more odd integers greater than 1. These lengths themselves are two consecutive positive integers. The least two consecutive positive integers representable as sums of odd integers greater than 1 are 5 (which is odd itself) and 6 (which is $3 + 3$). Hence at least 5 unit squares of the initial figure must be located to the east of X. Similarly, at least 5 unit squares of the initial figure must be located in each cardinal direction from X. Thus each side length of the initial rectangle is at least 11 and the area is at least 121. As Fig. 37 shows, this limit can be achieved.

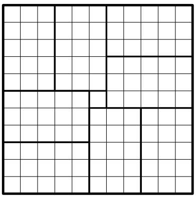


Fig. 37

F12. *(Grade 11.)* A beetle is creeping on the coordinate plane, starting from point $(0; -1)$, along a straight line until reaching the x-axis at point $(-x; 0)$ where x is a positive real number. After that it turns 90° to the right and creeps again along a straight line until reaching the y-axis. Then it again turns right by 90° and creeps along a straight line until reaching the x-axis, where it once more turns right by 90° and creeps along a straight line until reaching the y-axis.

a) Can it happen that both the length of the beetle's journey and the distance between its initial and final point are rational numbers?

b) Can it happen that both the length of the beetle's journey and the distance between its initial and final point are integers?

Answer: a) Yes; b) No.

Solution. Let O be the origin of coordinates. Let A_0 be the starting point of the beetle's journey, A_1 the first turning point, A_2 the second turning point, A_3 the third turning point and A_4 the endpoint (Fig. 38). The right triangles OA_0A_1 and OA_1A_2 are similar with ratio x because $\angle OA_1A_2 = 90^\circ - \angle OA_1A_0 = \angle OA_0A_1$ and $\frac{OA_1}{OA_0} = x$. So are the triangles OA_1A_2 and OA_2A_3 similar with ratio x since $\frac{OA_2}{OA_1} = x$ by similarity of the triangles OA_0A_1 and OA_1A_2. Analogously, the triangles OA_2A_3 and OA_3A_4 are similar with ratio x. Consequently, $OA_2 = x^2$, $OA_3 = x^3$ and $OA_4 = x^4$. By the Pythagorean theorem, the length of the beetle's journey is $\sqrt{1+x^2} + \sqrt{x^2+x^4} + \sqrt{x^4+x^6} + \sqrt{x^6+x^8}$, or equivalently, $(1+x+x^2+x^3)\sqrt{1+x^2}$ which equals $(x^4-1)\cdot\frac{\sqrt{x^2+1}}{x-1}$ if $x \neq 1$. On the other hand, the distance between A_0 and A_4 is $|x^4-1|$.

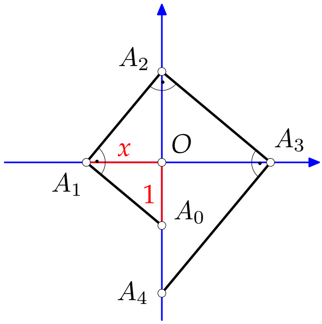


Fig. 38

a) Taking $x = \frac{4}{3}$, the number $x^4 - 1$ is rational, since x is rational. As $\sqrt{1+x^2} = \sqrt{1+\frac{16}{9}} = \frac{5}{3}$, the number $(x^4-1)\cdot\frac{\sqrt{x^2+1}}{x-1}$ is also rational.

b) Assume that the distance between A_0 and A_4 is an integer; then x^4 is an integer. Suppose that $(x^4-1)\cdot\frac{\sqrt{x^2+1}}{x-1}$, the length of the beetle's journey, is also an integer (one can assume $x \neq 1$ since otherwise the length of the journey is $4\sqrt{2}$ that is not an integer). As x^4-1 is an integer, the number $\frac{\sqrt{x^2+1}}{x-1}$ must be rational. Consider three cases.

1) If x is an integer then $x-1$ is integer whence $\sqrt{x^2+1}$ must be rational and x^2+1 must be a perfect square. This is impossible as two consecutive positive integers cannot be perfect squares.

2) Suppose that x is irrational and x^2 is an integer. As $\frac{x^2+1}{(x-1)^2}$ is the square of a rational number, so is also $\frac{(x-1)^2}{x^2+1}$. Hence $\frac{2x}{x^2+1}$ must be rational. But this is impossible, since $2x$ is irrational and x^2+1 is an integer.

3) Let x^2 be irrational. Similarly to the previous case we see that $\frac{2x}{x^2+1}$ is rational. Hence $\frac{4x^2}{(x^2+1)^2}$ is the square of a rational number, implying that $\frac{(x^2+1)^2}{4x^2}$ is rational. The latter in turn implies that $\frac{x^4+1}{4x^2}$ must be rational. This is impossible as $4x^2$ is irrational and x^4+1 is a positive integer.

F13. *(Grade 12.)* Find all triples (p,q,r) of primes such that $2018(p^2+q^2) = r^2+1$.

Answer: No such triples exist.

Solution. Suppose that both p and q are odd. Then p^2+q^2 is even and the l.h.s. of the equation is divisible by 4. Squares of integers are congruent to 0 or 1 modulo 4 whence the r.h.s. is congruent to 1 or 2 modulo 4. The contradiction shows that one of p and q equals 2; let w.l.o.g. $p = 2$. Squares of integers are congruent to 0 or 1 modulo 3. Obviously $r > 3$ as the l.h.s. is greater than 10. As r is prime, r is not divisible by 3. Thus $r^2 \equiv 1 \pmod 3$, whence $r^2 + 1 \equiv 2 \pmod 3$. Now $2018 \equiv 2 \pmod 3$ implies $p^2 + q^2 \equiv 1 \pmod 3$ and $p^2 = 4 \equiv 1 \pmod 3$ in turn implies $q^2 \equiv 0 \pmod 3$. Hence q is divisible by 3, i.e., $q = 3$. Therefore the l.h.s. is $2018 \cdot 13$. As $2018 \cdot 13 \equiv 3 \cdot 3 = 9 \equiv 4 \pmod 5$, we must have $r^2 \equiv 3 \pmod 5$. But 3 is not a quadratic residue modulo 5.

F14. *(Grade 12.)* Prove that every positive real number satisfies

$$(x+1)(x+2)(x+5) \geqslant 36x.$$

Solution 1. The given inequality is equivalent to $x^3 + 8x^2 - 19x + 10 \geqslant 0$. Note that $x^3 + 8x^2 - 19x + 10 = (x-1)^2(x+10)$. As $(x-1)^2 \geqslant 0$ and $x + 10 > 0$ for positive x, this inequality holds indeed.

Solution 2. Let $f(x) = x^3 + 8x^2 - 19x + 10$. The given inequality is equivalent to $f(x) \geqslant 0$. Note that $f'(x) = 3x^2 + 16x - 19$. As $D = 16^2 - 4 \cdot 3 \cdot (-19) > 0$, the function f' has two real roots. Let the smaller and the larger root be x_1 and x_2, respectively; then $x_2 = 1$ and $x_1 < 0$. As the coefficient of the quadratic term of $f'(x)$ is positive, f' is increasing at 1 whence the values of f' are negative in the interval $(x_1; 1)$ and positive in the interval $(1; \infty)$. Hence f is decreasing in the interval $(x_1; 1)$ and increasing in the interval $(1; \infty)$ (Fig. 39). Thus $f(x) \geqslant f(1) = 0$ holds for all x in the interval $(x_1; \infty)$ that includes every positive x.

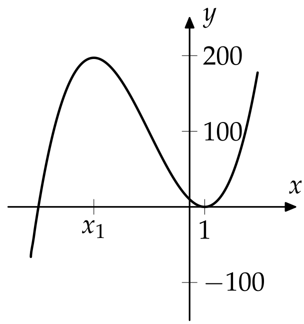


Fig. 39

Solution 3. For every positive integer k, the AM-GM inequality implies $\frac{x+1\cdot k}{k+1} \geqslant \sqrt[k+1]{x \cdot 1^k}$. Using this inequality for $k = 1$, $k = 2$, and $k = 5$, we obtain the inequalities $x + 1 \geqslant 2\sqrt{x}$, $x + 2 \geqslant 3\sqrt[3]{x}$, $x + 5 \geqslant 6\sqrt[6]{x}$, respectively. The desired result is now obtained by multiplying the corresponding sides.

F15. *(Grade 12.)* Let BE be an altitude of an acute triangle ABC and let P be the point on side AB such that $AP = AE$. Let N be the point for which $BCEN$ is a parallelogram. The areas of the triangles AEP and BNP are equal. Lines NE and AB intersect at Q.

a) Prove that the median of triangle ABC drawn from the vertex C bisects the line segment PQ.
b) Prove that in the triangle ABC, the bisector of the angle A, the altitude drawn from the vertex B and the median drawn from the vertex C meet in one point.

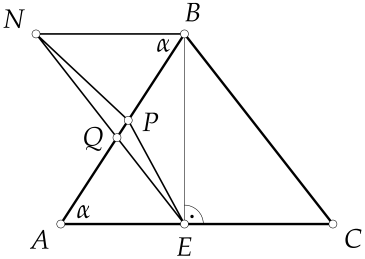


Fig. 40

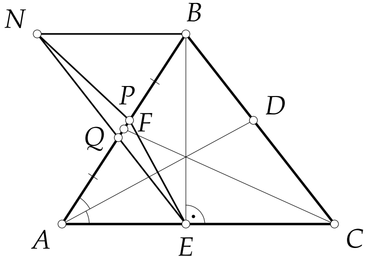


Fig. 41

Solution. Denote $AC = b$, $AB = c$, $\angle BAC = \alpha$, and $AE = AP = u$. As $BCEN$ is a parallelogram, $BN = CE = b - u$ and $BN \parallel CE$. The latter implies $\angle NBP = \alpha$ (Fig. 40). As $\angle QAE = \angle QBN$ and $\angle AQE = \angle BQN$, triangles AQE and BQN are similar.

a) The triangles AEP and BNP have areas $\frac{1}{2}u^2 \sin\alpha$ and $\frac{1}{2}(b-u)(c-u)\sin\alpha$, respectively. Thus $u^2 = (b-u)(c-u)$, whence $\frac{c-u}{u} = \frac{u}{b-u}$. Similarity of triangles AQE and BQN implies $\frac{AQ}{BQ} = \frac{AE}{BN}$ which, after defining $BQ = x$, rewrites to $\frac{c-x}{x} = \frac{u}{b-u}$. This equality reduces to a linear equation of x, meaning that it has only one root. By equality $\frac{c-u}{u} = \frac{u}{b-u}$, $x = u$ must be the only root. Thus $BQ = u = AP$, whence the midpoint of the side AB coincides with the midpoint of the line segment PQ.

b) Let F be the midpoint of the side AB and let D be the point of intersection of the bisector of angle A with side BC (Fig. 41). By angle bisector theorem, $\frac{BD}{CD} = \frac{AB}{AC}$. Since NE and BC are parallel, triangles ABC and AQE are similar, whence also triangles ABC and BQN are similar. This and part a) of the problem together imply $\frac{CE}{EA} = \frac{BN}{BQ} = \frac{AC}{AB}$. Consequently, $\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1 \cdot \frac{AB}{AC} \cdot \frac{AC}{AB} = 1$, giving the desired result by Ceva's theorem.

F16. *(Grade 12.)* Is there a positive integer n for which it is possible to write a number $-1, 0$ or 1 into each cell of an $n \times n$ table in such a way that every integer from $-n$ to n occurs at least once among the row sums, column sums and the two sums of the numbers on one long diagonal? If yes then find the least such n.

Answer: Yes, 6.

Solution. Suppose that an $n \times n$ table is filled with numbers -1, 0, and 1 in such a way that the conditions are met. The sums n and $-n$ can be obtained only from a row, column or diagonal with all 1s and all -1s, respectively, whence n and $-n$ cannot arise as sums of different kind (one as a row sum and the other as a column sum or similar) as such sums have a common summand. If both n and $-n$ arise as diagonal sums, n must be even (otherwise the middle summand would be common) and each row and each diagonal must contain one 1 and one -1. But then sums $n-1$

and $-(n-1)$ would be impossible to achieve. Hence n and $-n$ must be either both row sums or both column sums. W.l.o.g., assume that they are both row sums.

The number $n-1$ can arise only as the sum of $n-1$ numbers 1 and one number 0. As -1 occurs in each column and each long diagonal, $n-1$ can be obtained as a row sum only. Similarly, $-(n-1)$ can be obtained as a row sum only. The number $n-2$ can arise as the sum of either $n-2$ numbers 1 and two numbers 0 or $n-1$ numbers 1 and one number -1. As either two -1s or numbers 0 and -1 occur in each column and each long diagonal, $n-2$ can be obtained as a row sum only. Similarly, $-(n-2)$ can be obtained as a row sum only. Therefore, the table must contain at least 6 rows. An example of a 6×6 table that fulfils the conditions is shown in Fig. 42: the row sums from the top to the bottom are 6, 5, -5, -4, 4, and -6, the column sums from the left to the right are 0, -2, 1, 2, -1, 0, and the diagonal sums are 3 and -3.

1	1	1	1	1	1
1	1	1	1	0	1
-1	-1	0	-1	-1	-1
-1	-1	-1	1	-1	-1
1	-1	1	1	1	1
-1	-1	-1	-1	-1	-1

Fig. 42

F17. *(Grade 12.)* A quadratic equation $x^2 + px + q = 0$ is written on the blackboard, whereby p and q are real numbers such that real solutions exist to the equation on the blackboard and all the solutions are positive. Two players change in turns the coefficients in the equation according to the following rules. The first player decreases the constant term by either solution of the equation and (on the same move) increases the coefficient at the linear term by 1. The second player may replace the constant term with an arbitrary real number. Alternatively, the second player may increase the constant term by the largest solution of the equation and (on the same move) decrease the coefficient at the linear term by 1, but such move is allowed only if the solutions of the equation on the blackboard before the move differ from each other by more than 1. If either player's move results in an equation that does not have real solutions or has a non-positive real solution then the first player wins. Can the first player win regardless of how the opponent plays?

Answer: Yes.

Solution. Suppose that the first player always decreases the constant term by the smaller solution. We show that this is a winning strategy. Assume the opposite, i.e., that the play lasts infinitely. As the combined effect of the first and the second player's moves, the coefficient at the linear term either increases by 1 or (if the second player uses the alternative move) remains the same. If the second player had used the main move infinitely many times, the coefficient at the linear term would have become positive sooner or later. By Viéte's theorem, the sum of the solutions would be negative. This means that at least one solution is negative, too, whence the first player must have won already. Consequently, the second player must have used only the alternative move starting from some position on. If the equation on the blackboard before the first player's move is $x^2 + px + q = 0$ with solutions x_1, x_2, where $x_1 \leqslant x_2$, then Viéte's theorem implies $p = -(x_1 + x_2)$ and $q = x_1 x_2$, whence after the first player's move the coefficient at the linear term is $-(x_1 + x_2) + 1$ and the constant term is $x_1 x_2 - x_1$. This means that the quadratic equation after the first player's move is $x^2 - (x_1 + (x_2 - 1)) + x_1(x_2 - 1)$. By Viéte's theorem, this equation has solutions x_1 and $x_2 - 1$. Note that if before the first player's move the solutions of the equations differ by more than 1, then the difference decreases by 1 as the result of the move, but if the solutions before the move differ by at most 1 then the difference of the solutions after the move is still at most 1. Analogously, the same holds for the second player's alternative move. Hence the play must reach a position where the second player cannot make the alternative move, contradiction.

Selected Problems from the IMO Team Selection Contest

S1. There are distinct points $O, A, B, K_1, \ldots, K_n, L_1, \ldots, L_n$ on a plane such that no three points are collinear. The open line segments $K_1L_1, \ldots, K_nL_n$ are coloured red, other points on the plane are left uncoloured. An *allowed path* from point O to point X is a polygonal chain with first and last vertices at points O and X, containing no red points. For example, for $n = 1$, and $K_1 = (-1;0)$, $L_1 = (1;0)$, $O = (0;-1)$, and $X = (0;1)$, OK_1X and OL_1X are examples of allowed paths from O to X; there are no shorter allowed paths. Find the least positive integer n such that it is possible that the first vertex that is not O on any shortest possible allowed path from O to A is closer to B than to A, and the first vertex that is not O on any shortest possible allowed path from O to B is closer to A than to B.

Answer: 2.

Solution. A path $OX_1 \ldots X_{k-1}A$ is *suitable* if it is a shortest allowed path from O to A and $X_1A \leqslant X_1B$. Similarly, a path $OY_1 \ldots Y_{k-1}B$ is *suitable* if it is the shortest allowed path from O to B and $Y_1B \leqslant Y_1A$. Let us show that for $n = 2$ it is possible to choose points A, B, O, K_1, L_1, K_2, L_2 such that no shortest path from point O to point A, nor one from point O to point B is suitable. Take $A = (2;2)$, $B = (-2;-2)$, $K_1 =$

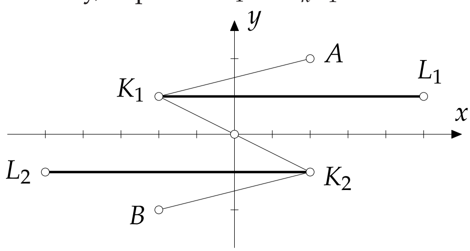
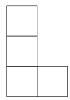


Fig. 43

$(-2;1)$, $L_1 = (5;1)$, $K_2 = (2;-1)$, and $L_2 = (-5;-1)$ (Fig. 43). If $O = (0;0)$, then the only shortest paths from point O to points A and B are respectively OK_1A and OK_2B, whereas $K_1A > K_1B$ and $K_2B > K_2A$, and hence they are not suitable. The collinearity of three points can be avoided by shifting O slightly while leaving the situation unchanged.

We will show that for $n = 1$ there exists at least one suitable path from point O to point A or point B. If the segment OA or segment OB does not contain any red points, then it is suitable. Therefore, assume that both segments OA and OB contain red points. W.l.o.g., $K_1A \leqslant K_1B$ (can change A and B). If OK_1A is a shortest path from point O to point A, then it is suitable. Otherwise, OL_1A is the only shortest path from point O to point A. It is suitable if $L_1A \leqslant L_1B$. Let us further assume that $L_1A > L_1B$. If OL_1B is a shortest path from point O to point B, then it is suitable. Otherwise, OK_1B is the only shortest path from point O to point B. Then $OL_1 + L_1A + OK_1 + K_1B < OK_1 + K_1A + OL_1 + L_1B$. This simplifies to $K_1B + L_1A < K_1A + L_1B$. However, adding the inequalities $K_1B \geqslant K_1A$ and $L_1A > L_1B$ gives $K_1B + L_1A > K_1A + L_1B$. The contradiction shows that at least one suitable path from point O to point A or B exists.

S2. Find the greatest number of depicted pieces composed of 4 unit squares that can be placed without overlapping on an $n \times n$ grid (where n is a positive integer) in such a way that it is possible to move from some corner to the opposite corner via uncovered squares (moving between squares requires a common edge). The shapes can be rotated and reflected.



Answer: $\frac{n^2-2n}{4}$ if n is even; $\frac{(n-1)^2}{4}$ if $n \equiv 1 \pmod 4$; $\frac{(n-1)^2}{4} - 1$ if $n \equiv 3 \pmod 4$.

Solution. As moving from one corner to the opposite one involves at least $n-1$ horizontal and $n-1$ vertical steps, the path goes through at least $2n-1$ unit squares. Thus the shapes can cover no more than $(n-1)^2$ unit squares, and there can be no more than $\frac{(n-1)^2}{4}$ for odd n and $\frac{(n-1)^2-1}{4} = \frac{n^2-2n}{4}$ for even n.

We show that for $n \equiv 3 \pmod 4$ it is not possible to place $\frac{(n-1)^2}{4}$ shapes. Suppose that this number of shapes has been placed and exactly $2n-1$ unit squares are not covered along the path from one corner to the opposite. Notice that the number of shapes $\frac{(n-1)^2}{4}$ is odd as $n-1$ is divisible by 2 but not by 4 and consequently 4 is the greatest power of 2 that divides $(n-1)^2$. Let us colour the rows alternatingly black and white and let m and v be the numbers of black and white squares, respectively, covered by shapes. As every shape covers exactly 3 squares of one colour and 1 of the other, m and v must be odd.

If the path of squares not covered is along the edge (Fig. 44 for $n = 7$), then the area covered by the shapes would form an $(n-1) \times (n-1)$ grid

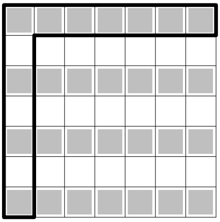


Fig. 44

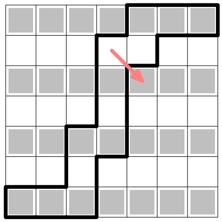


Fig. 45

containing an even number of both black and white squares (each row has an even number of squares of the same colour). Any other configuration of the path can be obtained as a transformation of this by steps of substituting a corner in the path by a square situated diagonally and hence of the opposite colour (Fig. 45). Hence the parities of the numbers of black and white squares along the path change at every step. Covering the unused area on either side of the path requires the number of squares to be divisible by 4 and every step changes the number by 1, thus 4 must divide the number of steps taken. Therefore, the parities of black and white squares under the path equal those at the initial configuration of the path, and so do the parities of black and white squares covered by the shapes equal those for the $(n-1) \times (n-1)$ grid. Thus m and v are even. The contradiction shows that it is not possible to place $\frac{(n-1)^2}{4}$ shapes according to the given conditions.

It is sufficient to place the required number of shapes on an $(n-1) \times (n-1)$ grid to show that it is possible to place $\frac{n^2-2n}{4}$ shapes for even n, $\frac{(n-1)^2}{4}$ shapes for $n \equiv 1 \pmod 4$ and $\frac{(n-1)^2}{4} - 1$ shapes for $n \equiv 3 \pmod 4$.

If $n \equiv 1 \pmod 4$, then $(n-1) \times (n-1)$ grid can be completely covered by 2×4 rectangles, each composed of 2 shapes (Fig. 46).

For $n \equiv 3 \pmod 4$ it is possible to place the shapes on an $(n-1) \times (n-1)$ grid in such a way that only a 2×2 area at the centre of the grid is not covered. This is trivial for $n = 3$. For $n > 3$, cover a strip that is 2 squares wide at the edges of the grid, as shown in Fig. 47. This yields a square with side length 4 less than the previous grid that can be covered in the same manner until reaching a square with side length 2.

For $n \equiv 2 \pmod 4$ or $n \equiv 0 \pmod 4$, it is possible to place the shapes on an $(n-1) \times (n-1)$ grid in such a way that only the middle square is not covered. This is trivial for $n = 2$ and shown for $n = 4$ on Fig. 48. For $n > 4$, cover a strip that is 2 squares wide at the edges of the grid, as shown in

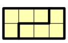


Fig. 46

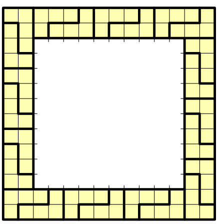


Fig. 47

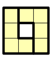


Fig. 48

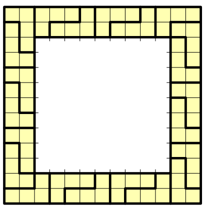


Fig. 49

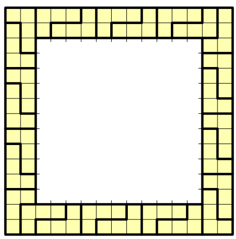


Fig. 50

Figures 49 and 50 for $n \equiv 2 \pmod 4$ and $n \equiv 0 \pmod 4$, respectively. This yields a square with side length 4 less than the previous grid that can be covered in the same manner until reaching a square with side length 1 or 3.

S3. Find all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy $f(xy + f(xy)) = 2xf(y)$ for all $x, y \in \mathbb{R}$.

Answer: $f(x) = 0$ and $f(x) = x$.

Solution 1. By choosing $(x, y) = (z, 1)$ in the original equation, where z is any real number, we get $f(z + f(z)) = 2zf(1)$. However, by choosing $(x, y) = (1, z)$ in the original equation, we get $f(z + f(z)) = 2f(z)$. From these equations together we obtain $f(z) = f(1)z$. By choosing $z = 1$ in equation $f(z + f(z)) = 2zf(1)$, we get $f(1 + f(1)) = 2f(1)$, however by choosing $z = 1 + f(1)$ in equation $f(z) = f(1)z$, we get $f(1 + f(1)) = f(1)(1 + f(1))$. Altogether we get $(1 + f(1)) f(1) = 2f(1)$ or, equivalently, $f(1)(f(1) - 1) = 0$, from which $f(1) = 0$ or $f(1) = 1$. Therefore by $f(z) = f(1)z$ either $f(z) = 0$ or $f(z) = z$ for every z. We can check that both functions indeed satisfy the original equation.

Solution 2. By choosing $y = 0$ we get $f(f(0)) = 2xf(0)$. As x can be any real number, this can hold only if $f(0) = 0$. By choosing $x = \frac{1}{y}$, we end up with $f(1 + f(1)) = \frac{2}{y} f(y)$. Hence for each $y \neq 0$ the equation $f(y) = \frac{f(1+f(1))}{2} y$ holds. In the case of $f(1 + f(1)) = 0$ this equation gives $f(y) = 0$ for every y. We get the same result if $1 + f(1) = 0$, because $f(0) = 0$. However, if $1 + f(1) \neq 0$ and $f(1 + f(1)) \neq 0$, then we can take $y = 1 + f(1)$ in the equation above and divide both sides by $f(1 + f(1))$; we obtain $\frac{1+f(1)}{2} = 1$, which implies $f(1) = 1$. Now by setting $y = 1$ in the equation above we get $\frac{f(1+f(1))}{2} = 1$, therefore $f(y) = y$ for every y.

S4. Let O be the circumcentre of an acute scalene triangle ABC. Line OA intersects the altitudes of ABC through B and C at P and Q, respectively. The altitudes meet at H. Prove that the circumcentre of triangle PQH lies on the median drawn from vertex A of triangle ABC.

Solution 1. Let AD be the altitude of triangle ABC and the midpoints of sides BC, CA, and AB be K, L, and M, respectively. Let also O_1 be the circumcentre of triangle HPQ (Fig. 51). As O is the circumcentre of ABC, we have $\angle AOL = \frac{1}{2}\angle AOC = \angle ABC$ and $\angle AOM = \frac{1}{2}\angle AOB = \angle ACB$. Therefore $\angle QPH = \angle AOL = \angle ABC$ and $\angle PQH = \angle AOM = \angle ACB$. Thus the triangles ABC and HPQ are similar. From this $\angle O_1HQ = \angle OAC = 90^\circ - \angle AOL = 90^\circ - \angle ABC = \angle HCB$. Therefore $HO_1 \parallel BC$.

Now let R and S be the points of intersection of AO with HO_1 and BC, respectively. Because of the similarity of triangles ABC and HPQ we must have $\frac{O_1R}{HR} = \frac{OS}{AS}$. Lines OK and AD both being orthogonal to BC implies $OK \parallel AD$, from which $\frac{OS}{AS} = \frac{KS}{DS}$. In summary, $\frac{O_1R}{HR} = \frac{KS}{DS}$. As HR and DS are parallel, we conclude from this equation that A, O_1, and K are collinear.

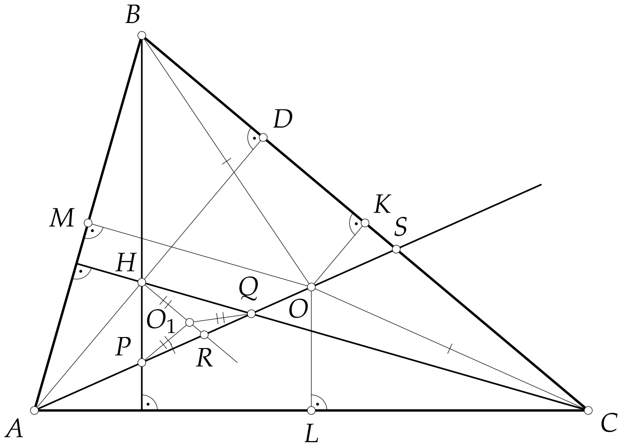


Fig. 51

Solution 2. Denote points D, K, L, and M similarly to Solution 1 and prove equations $\angle AOL = \angle ABC$ and $\angle AOM = \angle ACB$. W.l.o.g., assume that $AB < AC$. Let AO intersect the circumcircles of ABD and ACD the second time at X and Y, respectively (Fig. 52). We get $\angle DXY = 180^\circ - \angle DXA = \angle ABD = \angle AOL$ and $\angle DYX = \angle DCA = \angle MOA$, from which $DX \parallel OL$ and $DY \parallel OM$, respectively. As $OL \perp AC$ and $OM \perp AB$, also $DX \perp AC$ and $DY \perp AB$. As MK and LK are midsegments, $MK \parallel AC$ and $LK \parallel AB$, from which $DX \perp MK$ and $DY \perp LK$. Because $AD \perp BC$, points M and L are the circumcentres of triangles ABD and ACD, respectively, from which $MD = MX$ and $LD = LY$. The foot of the altitude drawn

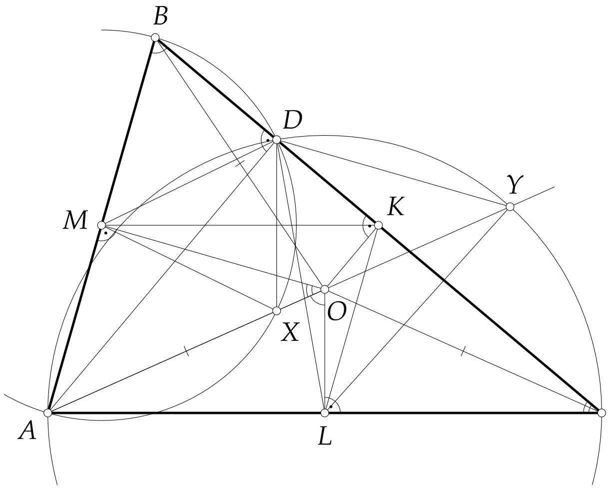


Fig. 52

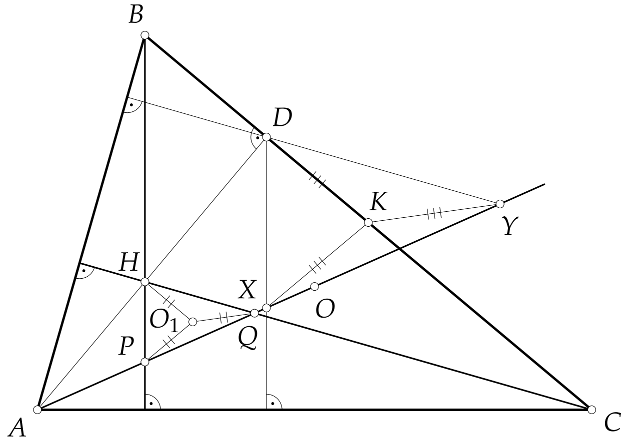


Fig. 53

from the vertex of an isosceles triangle bisects the base, from which we get that MK is the perpendicular bisector of DX and LK is the perpendicular bisector of DY. The point of intersection of those bisectors, K, is the circumcentre of triangle DXY.

From parallel lines $\frac{AH}{AD} = \frac{AP}{AX} = \frac{AQ}{AY}$ (Fig. 53). Therefore homothety centred at A with ratio $\frac{AH}{AD}$ converts triangle DXY into triangle HPQ and point K into the circumcentre of triangle HPQ. Therefore the circumcentre of triangle HPQ is located on the segment AK.

Remark. This problem was taken from the IMO-2017 Shortlist where it appeared as G3 (we slightly simplified it by saying that the median on which the circumcentre of PQH must lie is drawn from vertex A). Both solutions given here are different from that in the Shortlist.

S5. We call a positive integer n whose all digits are distinct *bright*, if either n is a one-digit number or there exists a divisor of n which can be obtained by omitting one digit of n and which is bright itself. Find the largest bright positive integer. (We assume that numbers do not start with zero.)

Answer: 146250.

Solution. First, we show by induction on the length of n that if $10n$ is bright, then n is bright as well. Assume that for one digit shorter numbers the statement holds. If after deleting 0 we obtain a bright divisor, the statement holds trivially. If the bright divisor of $10n$ is obtained after deleting some other digit, then in the end of this divisor we still have 0, i.e., it can be written as $10d$. By the induction hypothesis, d is bright. But then after deleting from n the corresponding digit, we get a bright divisor d, which means that also n is bright.

Next we show that any bright divisor of at least two-digit bright number not ending with 0 can be obtained by deleting the first or the second digit.

Assume the contrary, i.e., that a bright divisor d of a bright number n not ending with 0 is obtained by deleting the third or a further digit. This means that $n = (10a + x) \cdot 10^k + b$ and $d = a \cdot 10^k + b$, where $a \geqslant 10$, $0 \leqslant x < 10$ and $b < 10^k$. Since $9d = 9a \cdot 10^k + 9b < 9a \cdot 10^k + 9 \cdot 10^k = (9a+9) \cdot 10^k < 10a \cdot 10^k \leqslant (10a + x) \cdot 10^k + b = 10a \cdot 10^k + x \cdot 10^k + b < 10a \cdot 10^k + a \cdot 10^k + 11b = 11d$, the only possibility is $n = 10d$. Then n ends with zero, which contradicts our assumption.

Let now n be a 5-digit bright number not ending with 0 and let d be its bright divisor. We consider two cases depending on which digit is deleted to obtain d.

1) If d is obtained by deleting the first digit of n, then $d \mid n - d = 10^4 \cdot x$, where x is the deleted digit. As d does not end with 0, it is not divisible either by 2 or 5. If d is not divisible by 5, then $d \leqslant 2^4 \cdot x \leqslant 2^4 \cdot 9 < 10^3$, contradiction. Hence d is odd and $d \mid 5^4 \cdot x$. Since d is four-digit not ending with zero, it must divide one of the numbers $1875, 3125, 4375, 5625$. We may leave out 3125, because in this case the first digit of n must be $x = 5$, but the last digit is 5 as well. The four-digit divisors of the remaining numbers are 1125, 1875, 4375, 5625. The first and the last number contain equal digits, from the other two numbers we cannot obtain a divisor by deleting the first or the second digit.

2) If a bright divisor is obtained by deleting the second digit, then $d \mid n - d = 10^3 \cdot z$, where z is at most two-digit. Since d does not end with 0, it is not divisible either by 2 or 5, implying that $\frac{n-d}{d}$ is divisible either by 2^3 or by 5^3. Since n and d start with the same digit a, we have $\frac{n}{d} < \frac{(a+1) \cdot 10^4}{a \cdot 10^3} = 10 + \frac{10}{a} \leqslant 20$, implying that $\frac{n-d}{d}$ can be only 8 or 16, in both cases $5^3 \mid d$. We can write $d = 1000a + 125r$, where $r \in \{1, 3, 5, 7\}$. If $\frac{n-d}{d} = 16$, or equivalently $n = 17d$, then $n = 17000a + 2125r = 1000(17a + 2r) + 125r$. Since n starts with a and $125r < 1000$, we get $17a + 2r < 10(a+1)$ yielding $7a + 2r < 10$. This gives $a = 1$, $r = 1$, and $d = 1125$, which is not bright. If $\frac{n-d}{d} = 8$, or equivalently $n = 9d$, then $n = 9000a + 1125r = 1000(9a + r) + 125r$. Since n starts with a, we get $9a + r + \frac{r}{8} > 10a$, yielding $r > \frac{8}{9}a \geqslant a - 1$, i.e., $r \geqslant a$. Leaving out numbers with repeated digits, we get $d \in \{1375, 1625, 1875, 2375, 2875, 3625, 3875, 4625, 4875, 6875\}$. Among these numbers only 1625 is bright, deleting the first or the second digit from other candidates does not give a divisor. A check shows that $9d = 14625$ is bright as well.

Therefore 14625 is the only 5-digit bright number not ending with 0. Let now n be arbitrary 6-digit bright number. If n ends with 0, then deleting 0 we obtain a 5-digit bright number not containing 0, whence $n = 146250$. If n is not ending with 0, then after deleting the first or the second digit we obtain a bright divisor d not ending with 0. Thus $d = 14625 = 117 \cdot 125$, yielding $117 \mid n - d = 10^4 \cdot z$, where z is at most 2-digit. Since 117 and 10 are co-prime, this is not possible. Consequently, 146250 is the only 6-

digit bright number. If n was a bright 7-digit number, then by deleting its some digit we would obtain 146250. Then also n should end with 0, which means that $\frac{n}{10}$ is a bright 6-digit number not containing 0. But there are no such numbers. Since there exist no 7-digit bright numbers, there cannot be longer bright numbers either.

Remark 1. The end of the solution could be made differently by using the following observation: if a bright divisor of a bright number is divisible by 9 then the digit omitted must have been either 0 or 9. In that case, the lemma saying that if $10n$ is bright then n is bright is unnecessary.

Remark 2. Four-digit bright numbers need not be divisible by 125, the counterexamples are 2475 and 6075.

S6. A sequence of positive real numbers $a_1, a_2, a_3, \ldots$ satisfies $a_n = a_{n-1} + a_{n-2}$ for all $n \geqslant 3$. A sequence $b_1, b_2, b_3, \ldots$ is defined by equations $b_1 = a_1$, $b_n = a_n + (b_1 + b_3 + \ldots + b_{n-1})$ for even $n > 1$, $b_n = a_n + (b_2 + b_4 + \ldots + b_{n-1})$ for odd $n > 1$. Prove that if $n \geqslant 3$, then $\frac{1}{3} < \frac{b_n}{n \cdot a_n} < 1$.

Solution. The definition of sequence (b_n) indicates that $b_n - b_{n-2} = a_n - a_{n-2} + b_{n-1}$ for all $n \geqslant 3$. Therefore $b_n - b_{n-2} = a_{n-1} + b_{n-1}$, i.e., $b_n = a_{n-1} + b_{n-1} + b_{n-2}$. Notice that the left-hand inequality $\frac{1}{3} < \frac{b_n}{n \cdot a_n}$ holds for $n = 2$ and $n = 3$ as $\frac{b_2}{2a_2} = \frac{a_2+b_1}{2a_2} = \frac{a_1+a_2}{2a_2} > \frac{a_2}{2a_2} = \frac{1}{2} > \frac{1}{3}$ and $\frac{b_3}{3a_3} = \frac{a_3+b_2}{3a_3} = \frac{2(a_1+a_2)}{3(a_1+a_2)} = \frac{2}{3} > \frac{1}{3}$. Now assume that $n \geqslant 4$ and that the statement is true for $n-1$ and $n-2$. Then $b_n = a_{n-1} + b_{n-1} + b_{n-2} > a_{n-1} + \frac{1}{3}(n-1)a_{n-1} + \frac{1}{3}(n-2)a_{n-2} = \frac{1}{3}n(a_{n-1} + a_{n-2}) + \frac{2}{3}(a_{n-1} - a_{n-2}) > \frac{1}{3}na_n$, where the last inequality holds as $a_{n-1} = a_{n-2} + a_{n-3} > a_{n-2}$. By induction we get that $b_n > \frac{1}{3}na_n$ for all $n \geqslant 2$.

Now notice that the right-hand inequality $\frac{b_n}{n \cdot a_n} < 1$ holds for $n = 3$ and $n = 4$ as $\frac{b_3}{3a_3} = \frac{2}{3} < 1$ and $\frac{b_4}{4a_4} = \frac{a_4+b_3+b_1}{4a_4} = \frac{a_4+a_3+a_2+a_1}{4a_4} < 1$. Let $n \geqslant 5$ and assume that the statement is valid for $n-1$ and $n-2$. Then $b_n = a_{n-1} + b_{n-1} + b_{n-2} < a_{n-1} + (n-1)a_{n-1} + (n-2)a_{n-2} = n(a_{n-1} + a_{n-2}) - 2a_{n-2} < na_n$. By induction, $b_n < na_n$ holds for all $n \geqslant 3$.

S7. Let k be a positive integer. Find all positive integers n, such that it is possible to mark n points on the sides of a triangle (different from its vertices) and connect some of them with a line in such a way that the following conditions are satisfied: 1) there is at least 1 marked point on each side; 2) for each pair of points X and Y marked on different sides, on the third side there exist exactly k marked points which are connected to both X and Y and exactly k points which are connected to neither X nor Y.

Answer: $12k$.

Solution. Let A, B, and C be the sets of the points on different sides and a, b and c their cardinalities respectively. Let us count all the triplets (p, q, r), where $p \in A$, $q \in B$, and $r \in C$ and they are either pairwise connected or pairwise not connected. For all $p \in A$, $q \in B$ there exist exactly k points $r \in C$, for which the condition holds: if p and q are connected, then choose the k points which are connected to both; if p and q are not connected, then pick the k points which are connected to neither of them. Therefore the total number of such triplets is kab. Similarly, for all $q \in B$, $r \in C$ we find that the total number of those triplets is kbc, and for all $r \in C$, $p \in A$ we find that the number of those triplets is kca. Therefore, $ab = bc = ca$, implying $a = b = c = \frac{n}{3}$.

Let us now count all the other triplets (p, q, r) in which $p \in A$, $q \in B$ and $r \in C$. Similarly to the previous paragraph, we can see that for all $p \in A$, $q \in B$ there exist exactly k points $r \in C$ for which r is connected to neither of those if p and q are connected to each other, and r is connected to both if p and q are not connected to each other. The number of such triplets (p, q, r) is $kab = k(\frac{n}{3})^2$. Similarly, we then count the triplets (p, q, r) in which p is connected to neither of q or r if q and r are connected to each other and p is connected to both q and r if q and r are not connected to each other. The number of such points is also $k(\frac{n}{3})^2$. Finally, we count the triplets (p, q, r) in which q is connected to neither of r and p if r and p are connected to each other and q is connected to both r and p if r and p are not connected. The number of those triplets is also $k(\frac{n}{3})^2$. We have now counted all the possible triplets. Therefore, the total number of triplets (p, q, r) ($p \in A$, $q \in B$ and $r \in C$) is $4k(\frac{n}{3})^2$. On the other hand, the number of triplets is $(\frac{n}{3})^3$. The equality $4k(\frac{n}{3})^2 = (\frac{n}{3})^3$ gives us $n = 12k$.

It remains to prove that it is possible to mark $4k$ points on each side to satisfy the conditions. Let us first look the case $k = 1$. Let the sides of triangle be labelled as 0, 1, and 2 with each of the sides containing points labelled as 0, 1, 2, and 3. For all $i = 0, 1, 2$ we connect the even numbered points on the side i with points 0 and 1 on the side $(i + 1)$ mod 3 and the odd numbered points on the side i with points 2 and 3 on the side $(i + 1) \mod 3$ (Fig. 54). Then we have for each pair of points (which are not on the same side) exactly one point of the third side which are connected to both and exactly one point which is connected to neither of them. For case $k > 1$ we substitute each point with k different points and connect those which were generated from points that were connected before.

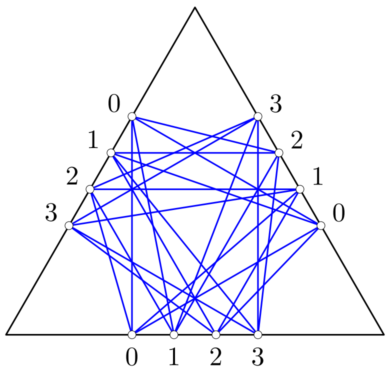


Fig. 54

Remark. This is a modified version of a problem of the Croatian Mathematical Olympiad 2017.

S8. Let a *simple polynomial function* be a polynomial function $P(x)$ whose coefficients belong to the set $\{-1, 0, 1\}$. Let n be a positive integer, $n > 1$. Find the smallest possible number of non-zero coefficients in a simple polynomial function of nth order whose values at all integral arguments are divisible by n.

Answer: 2.

Solution. A single non-zero coefficient is not sufficient for any $n > 1$ as the only simple polynomial functions with a single non-zero coefficient are $P(x) = x^n$ and $P(x) = -x^n$ but in both cases $n \nmid P(1)$. Let us show that the values of the polynomial function $P_n(x) = x^n - x^{n-\varphi(n)}$ at all integral arguments are divisible by n. (Here φ is the Euler's totient function.) This shows that having 2 non-zero coefficients is sufficient.

Let k be an integer. Let the canonical form of n be $p_1^{\alpha_1} \cdot \ldots \cdot p_m^{\alpha_m}$ and let us assume without loss of generality that k is divisible by primes $p_1, \ldots, p_l$ and is not divisible by primes $p_{l+1}, \ldots, p_m$. Define $u = p_1^{\alpha_1} \cdot \ldots \cdot p_l^{\alpha_l}$ and $v = p_{l+1}^{\alpha_{l+1}} \cdot \ldots \cdot p_m^{\alpha_m}$. Let us now show that $u \mid k^{n-\varphi(n)}$ and $v \mid k^{\varphi(n)} - 1$. Having $uv = n$, we can conclude that $n \mid P_n(k)$ as $P_n(k) = k^n - k^{n-\phi(n)} = k^{n-\varphi(n)}(k^{\varphi(n)} - 1)$.

To prove that $u \mid k^{n-\varphi(n)}$, it is sufficient to prove for all $i = 1, \ldots, l$ that $p_i^{\alpha_i} \mid k^{n-\varphi(n)}$. It is sufficient to prove that $\alpha_i \leqslant n - \varphi(n)$, as by the assumption $p_i \mid k$. Inequality $\alpha_i \leqslant n - \varphi(n)$ holds as $p_i, p_i^2, \ldots, p_i^{\alpha_i}$ are α_i positive integers which are not greater than n and not coprime with n.

To prove the statement $v \mid k^{\varphi(n)} - 1$, we derive from Euler's theorem that $v \mid k^{\varphi(v)} - 1$ as k and v are coprime. Also, u and v are coprime, therefore, $\varphi(n) = \varphi(uv) = \varphi(u)\varphi(v)$ from which $k^{\varphi(v)} - 1 \mid k^{\varphi(n)} - 1$. Consequently, $v \mid k^{\varphi(n)} - 1$.

Problems Listed by Topic

Number theory: O3, O7, O8, O11, O16, F1, F5, F9, F13, S5, S8
Algebra: O1, O2, O12, O15, O17, F12, F14, S3, S6
Geometry: O4, O6, O9, O13, O18, F2, F4, F6, F10, F15, S1, S4
Discrete mathematics: O5, O10, O14, O19, F3, F7, F8, F11, F16, F17, S2, S7