# Estonian Math Competitions
# 2020/2021

University of Tartu Youth Academy
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Estonian Mathematical Olympiad

http://www.math.olympiaadid.ut.ee/

# Mathematics Contests in Estonia

The Estonian Mathematical Olympiad is held annually in three rounds: at the school, town/regional and national levels. The best students of each round (except the final) are invited to participate in the next round. Every year, about 110 students altogether reach the final round.

In each round of the Olympiad, separate problem sets are given to the students of each grade. Students of grade 9 to 12 compete in all rounds, students of grade 7 to 8 participate at school and regional levels only. Some towns, regions and schools also organize olympiads for even younger students. The school round usually takes place in December, the regional round in January or February and the final round in March or April in Tartu. The problems for every grade are usually in compliance with the school curriculum of that grade but, in the final round, also problems requiring additional knowledge may be given.

The first problem solving contest in Estonia took place in 1950. The next one, which was held in 1954, is considered as the first Estonian Mathematical Olympiad.

Apart from the Olympiad, open contests take place in September and in December. In addition to students of Estonian middle and secondary schools, Estonian citizens who are studying abroad may also participate in these contests. The participants must have never enrolled in a university or other higher educational institution. The contestants compete in two categories: Juniors and Seniors. In the former category, only students up to the 10th grade may participate. Being successful in the open contests generally assumes knowledge outside the school curriculum.

Based on the results of all competitions during the year, about 20 IMO team candidates are selected. IMO team selection contest for them is held in April or May in two rounds. Each round is an IMO-style two-day competition with 4.5 hours to solve 3 problems on both days. Some problems in our selection contest are at the level of difficulty of the IMO but easier problems are usually also included.

The problems of previous competitions can be downloaded at the Estonian Mathematical Olympiads website.

# Problems Listed by Topic

Number theory: O1, O7, O11, O13, O16, O18, F5, F12, S2
Algebra: O3, O8, O12, O19, F4, F8, F13, S4, S5
Geometry: O4, O9, O14, O20, F2, F6, F9, F14, F16, S3
Discrete mathematics: O2, O5, O6, O10, O21, O17, O21, F1, F3, F7, F10, F11, F15, S1

# Problems

# Selected Problems from Open Contests

**O1** *(Juniors.)* Juku claims that if the sum of the squares of all digits of a natural number is divisible by 3 then the number itself is divisible by 3. Is Juku's claim always true?

**O2** *(Juniors.)* Xavier and Olivier are playing tic-tac-toe in the rectangular grid of size $3 \times 3$ with modified rules. On every move, a player chooses an empty square and writes his token into it. Players take turns alternately, with Xavier starting. The player who is the first to occupy any three squares that either are all in the same row or column or lie in pairwise distinct rows and columns wins. Does either of the players have a winning strategy and if yes then who?

**O3** *(Juniors.)* Do there exist numbers $a, b, c$ that satisfy the equation

$$2a(c-a) - b(2a+b) + c(2b-c) = 2020?$$

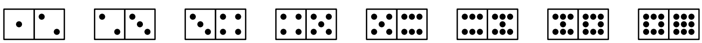
**O4** *(Juniors.)* The bisector of the angle on vertex $A$ of triangle $ABC$ intersects the circumcircle of triangle $ABC$ at point $F$ ($F \neq A$). Points $D$ and $E$ are chosen on the sides $AB$ and $AC$, respectively, in such a way that the lines $DE$ and $BC$ are parallel. Let $G$ and $H$ be the points of intersection of the rays $FD$ and $FE$, respectively, with the circumcircle of triangle $ABC$ ($G \neq F$, $H \neq F$). The circumcircles of triangles $AGD$ and $AHE$ intersect at point $P$ ($P \neq A$). Prove that point $P$ lies on the line $AF$.

**O5** *(Juniors.)* Find all integers $n \geqslant 3$ such that one can write a number (not necessarily an integer) into each vertex of a regular $n$-gon in such a way that both following conditions are met:

(1) Whenever three consecutive vertices of the $n$-gon, taken clockwise, contain numbers $x$, $y$ and $z$, respectively, the equality $x = |y - z|$ holds;

(2) The sum of the numbers in all vertices of the $n$-gon is 1.

**O6** *(Juniors.)* A set of 8 dominoes is given, each consisting of two unit squares:



Is it possible to completely cover a rectangular grid of size $4 \times 4$ with these dominoes in such a way that all rows and columns of the grid contain the same number of pips?

**O7** *(Juniors.)* Find all quadruples $(p, q, r, s)$ of primes that satisfy the following system of equations:

$$\begin{cases} 6p + 5q + 5r + 3s = 130 \\ 3p + 3q + 5r + 6s = 130 \end{cases}$$

**O8** *(Juniors.)* In a math period, the teacher asks pupils to solve quadratic equations of the form $x^2 + px + q = 0$ where $p$ and $q$ are some integers. The teacher obtains every new equation by either increasing by 1 or decreasing by 1 the value of either $p$ or $q$ in the equation just solved. In the initial equation, $p = 2020$ and $q = 2010$, whereas in the last equation, $p = 2010$ and $q = 2020$. Is it definitely true that both solutions of at least one equation solved during the period are integers?

**O9** *(Juniors.)* Let $ABC$ be a triangle such that $AB = AC$. Point $K$ lies on the altitude drawn from vertex $A$ and point $L$ is chosen on the line $BK$ in such a way that $AL \parallel BC$. Prove that if $KC \perp CL$ then point $L$ lies on the bisector of the external angle on vertex $C$ of triangle $ABC$.

**O10** *(Juniors.)* Let $n$ be a natural number, $n \geqslant 2$. There are $n$ lamps on a circle. The lamps are labeled clockwise by natural numbers from 1 kuni $n$. Each lamp can be either on or off. A switch between every two adjacent lamps enables one to change the state of both lamps simultaneously. In the beginning, all lamps are off. How many distinct configurations of states of lamps is it possible to achieve using these switches?

**O11** *(Seniors.)*

(a) Find the largest number expressible as the difference of two two-digit numbers obtained from each other by changing the order of digits.

(b) The same question with three-digit instead of two-digit numbers.

**O12** *(Seniors.)* Do there exist real numbers $x, y, z, t$ that meet the following system of equations?

$$\begin{cases} 1 + x^3 + y^2 = 0 \\ 1 + y^3 + z^2 = 0 \\ 1 + z^3 + t^2 = 0 \\ 1 + t^3 + x^2 = 0 \\ x + y + z + t = 0 \end{cases}$$

**O13** *(Seniors.)* Let $n$ be a fixed positive integer. Find all triples $(a, b, c)$ of integers satisfying the following system of equations:

$$\begin{cases} a^{n+3} + b^{n+2}c + c^{n+1}a^2 + a^n b^3 = 0 \\ b^{n+3} + c^{n+2}a + a^{n+1}b^2 + b^n c^3 = 0 \\ c^{n+3} + a^{n+2}b + b^{n+1}c^2 + c^n a^3 = 0 \end{cases}$$

**O14** *(Seniors.)* The incircle of triangle $ABC$ touches the sides $AB$ and $AC$ at points $K$ and $L$, respectively. The line $BL$ intersects the incircle of triangle $ABC$ at point $M$ ($M \neq L$). A circle passing through point $M$ touches the lines $AB$ and $BC$ at points $P$ and $Q$, respectively, and intersects the incircle of triangle $ABC$ at point $N$ ($N \neq M$). Prove that if $KM \parallel AC$ then points $P$, $N$ and $L$ are collinear.

**O15** *(Seniors.)* There is a rectangular grid with 3 rows and $n$ columns on a blackboard.

(a) Find the number of ways to write exactly one of numbers $1, 2, \ldots, 3n$ into each square in such a way that all the following conditions are met:

(1) Different squares contain different numbers.

(2) For each $i = 1, 2, \ldots, 3n - 1$, the numbers $i$ and $i + 1$ are written into adjacent squares (i.e., squares having a common side).

(3) The numbers 1 and $3n$ are written into adjacent squares.

(b) The same question, but as the 3rd condition, the number 1 must be written into the leftmost and the number $3n$ into the rightmost column.

**O16** *(Seniors.)* Find the least positive integer $n$ such that $\sqrt[5]{5n}$, $\sqrt[6]{6n}$ and $\sqrt[7]{7n}$ are integers.

**O17** *(Seniors.)* How many positive integers, where the only allowed digits are 0 and 1, are less than 11111100100?

**O18** *(Seniors.)* Prove that $a^{2020} + 10a^{1010} + 1001$ is prime for no integers $a$.

**O19** *(Seniors.)* There are $n$ lists of candidates taking part in elections. Let $h_i$ be the total number of votes given for the candidates of the $i$th list. There are $M$ seats in the representative assembly.

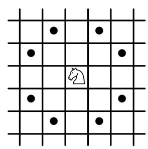
Anna proposes the following system for delivering mandates: For each list, one computes a reference number $v_i = \frac{h_i}{a_i+1}$ where $a_i$ is the number of mandates already given to the $i$th list (initially $a_i = 0$, i.e., the reference number of each list equals its number of votes). On every step ($M$ times in total), one chooses the list with the greatest reference number (if several lists share the first place, one of them is chosen randomly) and adds one mandate to this list, after which the reference number of this list is recomputed.

Bert's idea for delivering mandates is to multiply the number of votes of every list by $M/K$ where $K = h_1 + \ldots + h_n$, whereby fractional results are rounded downwards. As rounding may cause some seats to be undelivered, he proposes multiplying all numbers of votes of the lists by some suitable coefficient $\beta$, so that the number of mandates given to the $i$th list would be $m_i = \left\lfloor \frac{\beta h_i M}{K} \right\rfloor$ where $m_1 + \ldots + m_n = M$.

Prove that if such coefficient $\beta$ exists then Anna's and Bert's methods lead to the same distribution of mandates.

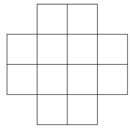
**O20** *(Seniors.)* The lines tangent to the circumcircle of triangle $ABC$ at points $B$ and $C$ intersect at point $D$. The circumcircle of triangle $BCD$ intersects the lines $AB$ and $AC$ the second time at points $K$ and $L$, respectively. Prove that the line $AD$ bisects the line segment $KL$.

**O21** *(Seniors.)* Let $n$ be a positive integer. Find the largest number of knights that can be placed on a board of size $n \times n$ in such a way that no two knights attack each other. A knight attacks precisely the squares that are located either horizontally by one square and vertically by two squares away or horizontally by two squares and vertically by one square away.



# Selected Problems from the Final Round of National Olympiad

**F1** *(Grade 9.)* Can natural numbers 1 through 12 be written into the squares of the grid shown in the figure in such a way that all following conditions are met?



(1) Each square contains exactly one number.
(2) The sums of numbers in both rows consisting of 4 squares, in both columns consisting of 4 squares, and in 4 middle squares are all equal.
(3) The numbers in any two squares with a common side or vertex differ from each other by at least 2.

**F2** *(Grade 9.)* Let $D$ be the foot of the altitude drawn to the hypotenuse $AB$ of a right triangle $ABC$. The inradii of the triangles $ABC$, $CAD$ and $CBD$ are $r$, $r_1$ and $r_2$, respectively. Prove that $CD = r + r_1 + r_2$.

**F3** *(Grade 9.)* Players A, B and C are playing the following game. Initially, the number 1 is written on a blackboard. On their move, each player replaces the number $n$ currently on the blackboard with either $n+1$, $7n+7$, or $4n^3+3n+4$ at their own choice, under the condition that the new number must not be larger than $10^9$. The player A makes the first move, then B takes turn, then C, after him A again etc., until some player cannot make a legal move. The player who makes the last move wins. Can any of the players win the game against every legal play by the opponents and if yes then who?

**F4** *(Grade 10.)* Let $a$, $b$, $c$, $d$ be positive real numbers satistfying the system of equations

$$\begin{cases} a^2 + \frac{1}{b^2} &= \frac{1}{2}, \\ b^2 + \frac{4}{c^2} &= 8, \\ c^2 + \frac{16}{d^2} &= 2, \\ d^2 + \frac{4}{a^2} &= 32. \end{cases}$$

Determine the product $abcd$.

**F5** *(Grade 10.)* Find all positive integers $k$ for which there is a right triangle with legs of integral lengths and hypotenuse of length $\sqrt{88\ldots822\ldots2}$, where the number under the root consists of exactly $k$ eights and exactly $k$ twos.

**F6** *(Grade 10.)* Let $D$ and $E$ be the midpoints of sides $AB$ and $AC$, respectively, of a triangle $ABC$. Prove that the line $AB$ is tangent to the circumcircle of the triangle $BEC$ if and only if the line $AC$ is tangent to the circumcircle of the triangle $BED$.

**F7** *(Grade 10.)* Natural numbers 1 through $n$ are written on a blackboard. On each move, one erases from the blackboard 2 or more numbers whose sum is divisible by any of the chosen numbers and writes their sum on the blackboard. Two players make moves by turns and the player who cannot move loses the game. Which player can win the game against any play by the opponent, if:

(a) $n = 6$;

(b) $n = 11$?

**F8** *(Grade 11.)* Prove that

$$\sin 10^\circ \cdot \cos 20^\circ \cdot \sin 30^\circ \cdot \cos 40^\circ \cdot \sin 50^\circ \cdot \cos 60^\circ \cdot \sin 70^\circ \cdot \cos 80^\circ = \frac{1}{256}.$$

**F9** *(Grade 11.)* The bisector of the internal angle on vertex $A$ of a triangle $ABC$ intersects the side $BC$ at point $D$. The line tangent to the circumcircle of the triangle $ABC$ at point $A$ intersects the line $BC$ at point $K$. Prove that $KA = KD$.

**F10** *(Grade 11.)* Prove that in every solved sudoku the squares marked "X" in the figure contain precisely the same digits as the squares marked "Y", counting repetitions. (For instance, if the squares marked "X" contain three digits 3 in total then the squares marked "Y" also contain three digits 3 in total.)

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| X | X | | | | | | X | X |
| X | X | | | | | | X | X |
| | | Y | Y | Y | Y | Y | | |
| | | Y | | | | Y | | |
| | | Y | | | | Y | | |
| | | Y | | | | Y | | |
| | | Y | Y | Y | Y | Y | | |
| X | X | | | | | | X | X |
| X | X | | | | | | X | X |

*Remark:* A solved sudoku is a board of size $9 \times 9$ whose every row, every column and every part of size $3 \times 3$ bounded by bold rules contains every digit from 1 to 9 exactly once.

**F11** *(Grade 11.)* Every term of the sequence $a_1, a_2, a_3, \ldots$ is either 0 or 1. It is known that both 0 and 1 occur at least 1010 times among every 2021 consecutive terms of the sequence. May one be sure that the sequence is periodic from some place on, i.e., there exist positive integers $n$ and $p$ such that $a_{n+i} = a_{n+i+p}$ for every natural number $i$?

**F12** *(Grade 12.)* Find all pairs $(a, b)$ of positive integers such that $a \geqslant b$ and

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{2021}.$$

**F13** *(Grade 12.)* Anna, Anne and Anni seek for real solutions $(x, y)$ to the system of equations

$$\begin{cases} 4x^3y - x^4 - 3x^2y^2 = 2021, \\ 4y^3x - y^4 - 3y^2x^2 = 2021. \end{cases}$$

Anna claims that the system of equations has a solution. Anne claims that the system of equations has no solution but at least one of the two equations has solutions. Anni claims that the system of equations has no solution and, even worse, neither of the two equations alone has a solution. Who is right?

**F14** *(Grade 12.)* Point $D$ inside an acute triangle $ABC$ satisfies

$$\angle ADC = \angle BDA = 180^\circ - \angle CAB.$$

Prove that the point symmetric to point $A$ w.r.t. point $D$ lies on the circumcircle of the triangle $ABC$.

**F15** *(Grade 12.)* There are 6 distinct lines chosen in the space. Find the largest possible number of points where at least 3 chosen lines intersect.

**F16** *(Grade 12.)* Two regular polygons have a common circumcircle. The sum of the areas of the incircles of these polygons equals the area of their common circumcircle. Find all possibilities of how many vertices can the two polygons have.

# Selected Problems from the IMO Team Selection Contests

**S1** Juku has the first 100 volumes of the Harrie Totter book series at his home. For every $i$ and $j$, where $1 \leqslant i < j \leqslant 100$, call the pair $(i, j)$ *reversed* if volume No $j$ is before volume No $i$ on Juku's shelf. Juku wants to arrange all volumes of the series to one row on his shelf in such a way that there does not exist numbers $i$, $j$, $k$, where $1 \leqslant i < j < k \leqslant 100$, such that pairs $(i, j)$ and $(j, k)$ are both reversed. Find the largest number of reversed pairs that can occur under this condition.

**S2** Find all polynomials $P(x)$ with integral coefficients whose values at points $x = 1, 2, \ldots, 2021$ are numbers $1, 2, \ldots, 2021$ in some order.

**S3** (a) There are $2n$ rays marked in a plane, with $n$ being a natural number. Given that no two marked rays have the same direction and no two marked rays have a common initial point, prove that there exists a line that passes through none of the initial points of the marked rays and intersects with exactly $n$ marked rays.
(b) Would the claim still hold if the assumption that no two marked rays have a common initial point was dropped?

**S4** Positive real numbers $a, b, c$ satisfy $abc = 1$. Prove that

$$\frac{a}{1+b} + \frac{b}{1+c} + \frac{c}{1+a} \geqslant \frac{3}{2}.$$

**S5** Find all polynomials $P(x, y)$ with real coefficients which for all real numbers $x$ and $y$ satisfy $P(x + y, x - y) = 2P(x, y)$.

# Problems with Solutions

# Selected Problems from Open Contests

**O1** *(Juniors.)* Juku claims that if the sum of the squares of all digits of a natural number is divisible by 3 then the number itself is divisible by 3. Is Juku's claim always true?

*Answer:* No.

*Solution:* The sum of the squares of the digits of the number 112 is 6 which is divisible by 3, while the number 112 is not divisible by 3.

**O2** *(Juniors.)* Xavier and Olivier are playing tic-tac-toe in the rectangular grid of size $3 \times 3$ with modified rules. On every move, a player chooses an empty square and writes his token into it. Players take turns alternately, with Xavier starting. The player who is the first to occupy any three squares that either are all in the same row or column or lie in pairwise distinct rows and columns wins. Does either of the players have a winning strategy and if yes then who?

*Answer:* Yes, Xavier.

*Solution:* Note that every two unit squares uniquely determine a unit square that constitutes a winning triple together with the given two squares. Label all unit squares except the middle square clockwise with numbers 1 through 8 (Fig. 1); let Xavier initially play into the middle square and if Olivier then moves into the square No. $i$ then let Xavier make his second move into the square No. $i+1$ (square No. 1 if $i = 8$; all possible positions after Xavier's second move, modulo rotation of the table, are depicted in Figures 2 and 3). In order to prevent loss of the game, Olivier must occupy the unit square that constitutes a winning triple together with the two squares occupied by Xavier. Let Xavier then move into the square that constitutes a winning triple together with the two squares previously occupied by Olivier (all possible positions after Xavier's third move are depicted in Figures 4 and 5). Then Olivier can't win on his next move. But the three squares occupied by Xavier contain three pairs, out of which only one pair has its corresponding winning third square occupied by Olivier. Thus Olivier can't block them all, whence Xavier can win on his next move.

*Remark:* The strategy described in the solution is not the only winning strategy that Xavier has.

| | | |
|---|---|---|
| 1 | 2 | 3 |
| 8 | | 4 |
| 7 | 6 | 5 |

Fig. 1

| | | |
|---|---|---|
| O | X | |
| | X | |
| | | |

Fig. 2

| | | |
|---|---|---|
| | O | X |
| | X | |
| | | |

Fig. 3

| | | |
|---|---|---|
| O | X | |
| | X | X |
| | O | |

Fig. 4

| | | |
|---|---|---|
| | O | X |
| | X | X |
| O | | |

Fig. 5

**O3** *(Juniors.)* Do there exist numbers $a$, $b$, $c$ that satisfy the equation

$$2a(c-a) - b(2a+b) + c(2b-c) = 2020?$$

*Answer:* No.
*Solution:* Transforming the l.h.s. of the equation gives

$$\begin{aligned} 2a(c-a) - b(2a+b) + c(2b-c) &= 2ac - 2a^2 - 2ab - b^2 + 2bc - c^2 \\ &= -a^2 - (a+b-c)^2. \end{aligned}$$

The equality $-a^2 - (a+b-c)^2 = 2020$ cannot be valid since all terms of its l.h.s. are non-positive whereas the r.h.s. is positive.

**O4** *(Juniors.)* The bisector of the angle on vertex $A$ of triangle $ABC$ intersects the circumcircle of triangle $ABC$ at point $F$ ($F \neq A$). Points $D$ and $E$ are chosen on the sides $AB$ and $AC$, respectively, in such a way that the lines $DE$ and $BC$ are parallel. Let $G$ and $H$ be the points of intersection of the rays $FD$ and $FE$, respectively, with the circumcircle of triangle $ABC$ ($G \neq F$, $H \neq F$). The circumcircles of triangles $AGD$ and $AHE$ intersect at point $P$ ($P \neq A$). Prove that point $P$ lies on the line $AF$.
*Solution 1:* Let $K$ and $L$ be the points of intersection of the line $AF$ with lines $BC$ and $DE$, respectively (Fig. 6). Then

$$\begin{aligned} \angle AGD &= \angle AGF = \angle AGC + \angle CGF = \angle ABC + \angle CAF \\ &= \angle ABK + \angle KAB = \angle CKA = \angle KLD = 180^\circ - \angle ALD. \end{aligned}$$

Hence the quadrilateral $AGDL$ is cyclic. Interchanging the roles of points $B$ and $C$, points $D$ and $E$ and also points $G$ and $H$, we can similarly prove that the quadrilateral $AHEL$ is cyclic. Thus the circumcircles of triangles $ADG$ and $AEH$ meet at point $L$, i.e., $P = L$. Point $L$ was chosen on the line $AF$.
*Solution 2:* Choose a point $Z$ on the same side of the line $AF$ as point $B$ such that $ZF \parallel BC \parallel DE$ (Fig. 7). Then $\angle ZFB = \angle FBC = \angle FAC = \angle FAB = \angle FCB$. Hence $ZF$ is tangent to the circumcircle of the triangle $ABC$ at $F$, implying that $\angle GFZ = \angle GHF = \angle GHE$. As $\angle GFZ = \angle DFZ = \angle FDE = 180^\circ - \angle GDE$, we altogether have $\angle GHE = 180^\circ - \angle GDE$. Consequently, the quadrilateral $DEHG$ is cyclic. Hence $|FD| \cdot |FG| = |FE| \cdot |FH|$, implying that $F$ lies on the radical axis of the circumcircles of triangles $ADG$ and $AEH$. The desired result follows.

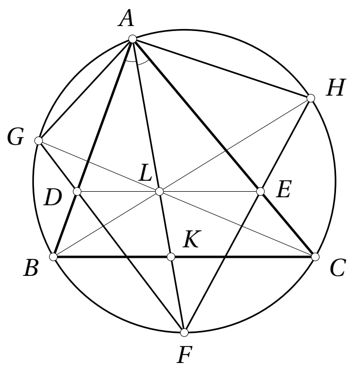



Fig. 6

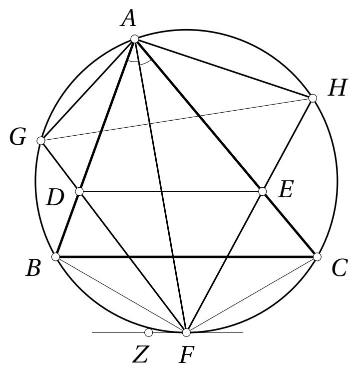



Fig. 7

**O5** *(Juniors.)* Find all integers $n \geqslant 3$ such that one can write a number (not necessarily an integer) into each vertex of a regular $n$-gon in such a way that both following conditions are met:

(1) Whenever three consecutive vertices of the $n$-gon, taken clockwise, contain numbers $x$, $y$ and $z$, respectively, the equality $x = |y - z|$ holds;

(2) The sum of the numbers in all vertices of the $n$-gon is 1.

*Answer:* All positive multiples of 3.

*Solution:* Let the vertices of an $n$-gon $A_0A_1 \ldots A_{n-1}$ be labeled with numbers satisfying the conditions. Let $a$ be the least among these numbers. W.l.o.g., assume that $A_0$ contains $a$ and the indices of vertices are increasing counterclockwise. Let $b$ and $c$ be the numbers at vertices $A_{n-1}$ and $A_{n-2}$, respectively (Fig. 8). By condition (1), $a = |b - c| \geqslant 0$. By the choice of $a$, we must have $b \geqslant a$, whence by condition (1), $A_1$ contains $b - a$. As $b - a \geqslant a$ by the choice of $a$, the condition (1) also implies that $A_2$ contains $b - 2a$. Hence $A_3$ contains $(b - a) - (b - 2a) = a$ by condition (1) and non-negativity of $a$. Since the vertices can be renumerated without changing the direction in such a way that $A_3$ becomes $A_0$, we can conclude that $A_6$ also contains $a$. Similarly, every third vertex contains $a$. If the number $n$ of vertices is not divisible by 3 then either $A_{n-1}$ or $A_1$ must contain $a$ and, as we can repeat this argument, also $A_{n-2}$ or $A_2$, respectively, contains $a$. Thus three consecutive vertices contain $a$. Applying the condition (1) to these three vertices, we obtain $a = |a - a| = 0$. But 0 being in two consecutive vertices implies that 0 is in all vertices. Then the sum of all labels is 0, contradicting the condition (2).

This shows that $n$ must be divisible by 3. Let $n = 3k$ where $k$ is a positive integer. For every $3k$-gon, the conditions of the problem can be satisfied by writing 0 into every third vertex and $\frac{1}{2k}$ into all other vertices (Fig. 9 depicts the situation for $k = 4$).

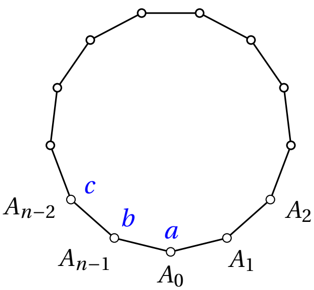



Fig. 8

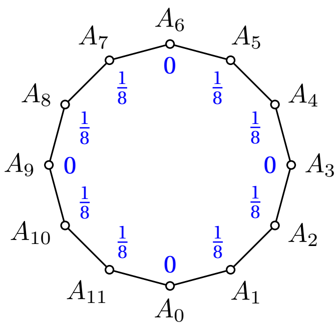
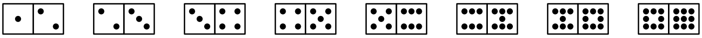



Fig. 9

**O6** *(Juniors.)* A set of 8 dominoes is given, each consisting of two unit squares:



Is it possible to completely cover a rectangular grid of size $4 \times 4$ with these dominoes in such a way that all rows and columns of the grid contain the same number of pips?
*Answer:* Yes.
*Solution:* Figure 10 shows one possibility.

| | | | |
|---|---|---|---|
| 1 | 2 | 8 | 9 |
| 4 | 5 | 5 | 6 |
| 8 | 7 | 3 | 2 |
| 7 | 6 | 4 | 3 |

Fig. 10

**O7** *(Juniors.)* Find all quadruples $(p, q, r, s)$ of primes that satisfy the following system of equations:

$$\begin{cases} 6p + 5q + 5r + 3s = 130 \\ 3p + 3q + 5r + 6s = 130 \end{cases}$$

*Answer:* $(11, 3, 2, 13)$.
*Solution 1:* Subtracting the second equation from the first one gives $3p + 2q - 3s = 0$ which implies $2q = 3(s - p)$. Thus $2q$ is divisible by 3. As both 2 and $q$ are primes, this implies $q = 3$. Substituting $q = 3$ into the initial system of equation and simplifying gives

$$\begin{cases} 6p + 5r + 3s = 115, \\ 3p + 5r + 6s = 121. \end{cases} \tag{1}$$

If $r$ and $s$ were both odd then also $5r$ and $3s$ would be odd, whence $5r + 3s$ would be even. As $6p$ is even, too, the l.h.s. of the first equation of system (1) would be even and could not equal the r.h.s. 115. Thus one of $r$ and $s$ is even, i.e., $r = 2$ or $s = 2$. Analogously if $p$ and $r$ were both odd then the second equation of system (1) would give a contradiction; hence $p = 2$ or $r = 2$. Consequently, if $r \neq 2$ then $p = s = 2$, but substituting $p = s$ into system (1) and subtracting the second equation from the first one gives $0 = -6$. The contradiction shows that $r = 2$. Substituting $r = 2$ into system (1) and simplifying gives

$$\begin{cases} 6p + 3s = 105, \\ 3p + 6s = 111. \end{cases}$$

Solving this equation gives $p = 11$ and $s = 13$.
*Solution 2:* As in Solution 1 we find $q = 3$ and obtain system (1). Subtracting the first equation from the second one in system (1) gives $3s - 3p = 6$, implying $s = p + 2$. The least pairs of prime numbers with difference 2 are $(3, 5)$, $(5, 7)$ and $(11, 13)$. The next such pair $(17, 19)$ gives $6p + 3s > 115$,

implying that no more pairs are suitable. Substituting the three pairs one by one into system (1) and simpligying each time gives $5r = 82$, $5r = 64$ and $5r = 10$, respectively. Only the last alternative leads to an integral solution $r = 2$, which is a prime, too. Consequently, the only possibility is $(p, q, r, s) = (11, 3, 2, 13)$.

**O8** *(Juniors.)* In a math period, the teacher asks pupils to solve quadratic equations of the form $x^2 + px + q = 0$ where $p$ and $q$ are some integers. The teacher obtains every new equation by either increasing by 1 or decreasing by 1 the value of either $p$ or $q$ in the equation just solved. In the initial equation, $p = 2020$ and $q = 2010$, whereas in the last equation, $p = 2010$ and $q = 2020$. Is it definitely true that both solutions of at least one equation solved during the period are integers?
*Answer:* Yes.
*Solution:* In the first equation, one has $p - q = 10$, while in the last equation, one has $p - q = -10$. At each step, either $p$ or $q$ changes exactly by 1, whence also $p - q$ changes exactly by 1. Thus at some step one must have an equation where $p - q = 1$, or equivalently, $p = q + 1$. The equation $x^2 + (q + 1)x + q = 0$ has integral solutions $-1$ and $-q$.

**O9** *(Juniors.)* Let $ABC$ be a triangle such that $AB = AC$. Point $K$ lies on the altitude drawn from vertex $A$ and point $L$ is chosen on the line $BK$ in such a way that $AL \parallel BC$. Prove that if $KC \perp CL$ then point $L$ lies on the bisector of the external angle on vertex $C$ of triangle $ABC$.
*Solution 1:* As $AL \parallel BC$ and $AK \perp BC$ (Fig. 11), we have $\angle KAL = 90°$. As $KAL$ and $KCL$ are both right angles, points $A$ and $C$ lie on circle with diameter $KL$. By inscribed angles, $\angle KCA = \angle KLA$. On the other hand, $\angle KLA = \angle KBC$ and, by symmetry of isosceles triangle, $\angle KBC = \angle KCB$. Consequently, $\angle KCA = \angle KCB$, implying that $KC$ bisects the internal angle on vertex $C$ of the triangle $ABC$. As $KCL$ is right angle, $CL$ bisects the corresponding external angle.
*Solution 2:* The altituded drawn from the vertex angle of the isosceles triangle $ABC$ is the perpendicular bisector of its base. All points of the perpendicular bisector of a line segment lie at equal distance from the endpoints of the line segment, implying that $|KB| = |KC|$. As $AL \parallel BC$ and $AK \perp BC$, we conclude $AK \perp AL$.

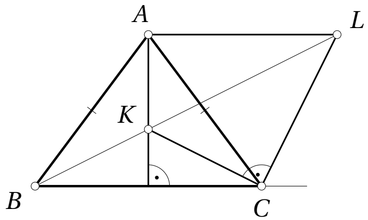



Fig. 11

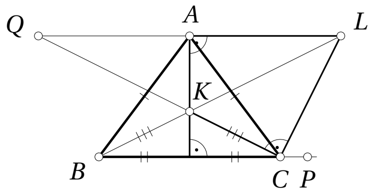



Fig. 12

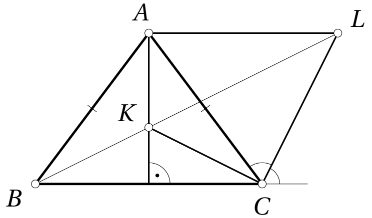



Fig. 13

Let $P$ be a point on line $BC$ such that $C$ is between $B$ and $P$, and let lines $CK$ and $AL$ intersect at $Q$ (Fig. 12). As $\angle BKC = \angle LKQ$ and $\angle KBC = \angle KLQ$, triangles $KBC$ and $KLQ$ are similar. Together with the equality $|KB| = |KC|$, it implies $|KL| = |KQ|$. Thus $KA$ is the altitude drawn from the vertex angle of the isosceles triangle $KLQ$, implying also $|AL| = |AQ|$. Hence $A$ is the midpoint of the hypotenuse of the right triangle $CLQ$, implying that $A$ is the center of the circumcircle of the triangle $CLQ$. Consequently, $|AC| = |AL|$, implying that $\angle ACL = \angle ALC = \angle LCP$, i.e., $CL$ bisects the external angle on vertex $C$ of the triangle $ABC$.

*Solution 3:* When point $K$ moves along the altitude drawn from vertex $A$ of the triangle $ABC$ farther from point $A$, the angle $KCB$ decreases, while point $L$ also moves farther from point $A$, causing the angle $LCB$ to increase. Thus the difference $\angle LCB - \angle KCB$ also increases in this process and can equal $90^\circ$ in the case of exactly one location of point $K$. Hence it suffices to show that if $L$ lies at the bisector of the external angle on vertex $C$ of the triangle $ABC$ then $\angle LCK = 90^\circ$.

So let $L$ lie at the bisector of the external angle on vertex $C$ of the triangle $ABC$ (Fig. 13). As $AL \parallel BC$ and $AK \perp BC$, we have $AK \perp AL$; as $AK$ is the altitude drawn from the vertex angle of the triangle $ABC$, it bisects the vertex angle, whence also $L$ lies on the bisector of the external angle on vertex $A$ of the triangle $ABC$. Thus $L$ is the center of the excircle tanglent to side $AC$ of the triangle $ABC$ and lies also on the bisector of the internal angle on vertex $B$ of the triangle $ABC$. Hence both $AK$ and $BK$ are angle bisectors, implying that $K$ is the intersection point of angle bisectors of the triangle $ABC$ and $CK$ is the bisector of the internal angle on vertex $C$ of the triangle $ABC$. Consequently, $CK$ and $CL$ are perpendicular.

**O10** *(Juniors.)* Let $n$ be a natural number, $n \geqslant 2$. There are $n$ lamps on a circle. The lamps are labeled clockwise by natural numbers from 1 kuni $n$. Each lamp can be either on or off. A switch between every two adjacient lamps enables one to change the state of both lamps simultaneously. In the beginning, all lamps are off. How many distinct configurations of states of lamps is it possible to achieve using these switches?

*Answer:* $2^{n-1}$.

*Solution 1:* As the state of each lamp is determined by the parity of the number of switchings that influence this lamp, the result of every sequence of switchings is determined by the set of switches that have been touched an odd number of times. Hence all possible states can be achieved by sequences of switchings that touch some switches once and do not touch other switches at all, whereby the order of switchings is irrelevant.

Note that every two set of switches, one of which contains precisely the switches that the other one does not, lead to equal final states, because touching these two sets of switches in a row one touches each switch exactly once and thus changes the state of each lamp exactly twice. As there are $2^n$ sets of switches and these sets can be divided into pairs that give equal final states, there can be at most $2^{n-1}$ final states.

If two sets of switches are neither equal nor complementary (in the sense of previous paragraph) then touching these two sets of switches in a row does not give back the initial state, because this sequence of switchings is equivalent to touching a set of switches that contains some switch and does not contain some other switch, but then there must exist a lamp, the switch in only one side of which is touched. Hence every pair of sets of switches defined in the previous paragraph gives a unique final state, i.e., there are $2^{n-1}$ possible distinct final states.

*Solution 2:* After every switching, either two more lamps are burning, two less lamps are burning, or the number of burning lamps does not change. As the initial number of burning lamps is even, the number of burning lamps stays even.

We show that every state with an even number of burning lamps can be achieved. As the switchings are invertible (repeating the same touch reverts the previous change), it suffices to show that, starting from any state with an even number of burning lamps, one can achieve the state where no lamp is burning. Indeed, while there exist two consecutive burning lamps, we can switch off both these lamps. If there do not exist such lamps, but some lamps are still burning, one can shift a lonely burning lamp clockwise by repeatedly touching the clockwise nearest switch until two consecutive lamps are burning. Switching these two lamps off decreases the total number of burning lamps by 2. This way, we can decrease the number of burning lamps until all lamps are switched off.

Thus the number of all achievable states is $C_n^0 + C_n^2 + \ldots + C_n^{2\lfloor \frac{n}{2} \rfloor}$ which equals $2^{n-1}$.

**O11** *(Seniors.)*

(a) Find the largest number expressible as the difference of two two-digit numbers obtained from each other by changing the order of digits.

(b) The same question with three-digit instead of two-digit numbers.

*Answer:* (a) 72; (b) 801.

*Solution:*

(a) Let the given two-digit number be $\overline{ab}$. The only number that can be obtained by changing the order of digits is $\overline{ba}$. The difference of these numbers is $(10a + b) - (10b + a) = 9(a - b)$. To obtain the largest difference, $a$ must be as large as possible and $b$ as small as possible. Since $b = 0$ is impossible, we must have $a = 9$ and $b = 1$ giving $a - b = 8$. Hence the desired largest difference is 72.

(b) Let the given three-digit number be $\overline{abc}$. Interchanging the last two digits can change it by less than 100. Interchanging the first two digits can change the number by at most 720 by part a) of the problem. It remains to study cases where changing the order of digits results in $\overline{cab}$, $\overline{bca}$ or $\overline{cba}$.

- The difference of numbers $\overline{abc}$ and $\overline{cab}$ is $(100a + 10b + c) - (100c + 10a + b) = 9(10a + b - 11c)$. To obtain the largest difference, $a$ and $b$ must be as large as possible and $c$ as small as possible. Since $c$ is the first digit of the number, $c = 0$ is impossible, whence the largest difference is obtained if $a = b = 9$ and $c = 1$. This difference is $991 - 199 = 792$.
- The difference of numbers $\overline{abc}$ and $\overline{bca}$ is $(100a + 10b + c) - (100b + 10c + a) = 9(11a - 10b - c)$. To obtain the largest difference, $a$ must be as large as possible and both $b$ and $c$ as small as possible, i.e., $a = 9$, $b = 1$ and $c = 0$. This difference is $910 - 109 = 801$.
- The difference of numbers $\overline{abc}$ and $\overline{cba}$ is $(100a + 10b + c) - (100c + 10b + a) = 99(a - c)$. To obtain the largest difference, $a$ must be as large as possible and $c$ as small as possible. Since $c \neq 0$, the largest difference is obtained if $a = 9$ and $c = 1$. Then the difference of the three-digit numbers is $99 \cdot 8 = 792$.

Consequently, the desired largest difference is 801.

**O12** *(Seniors.)* Do there exist real numbers $x, y, z, t$ that meet the following system of equations?

$$\begin{cases} 1 + x^3 + y^2 &= 0 \\ 1 + y^3 + z^2 &= 0 \\ 1 + z^3 + t^2 &= 0 \\ 1 + t^3 + x^2 &= 0 \\ x + y + z + t &= 0 \end{cases}$$

*Answer:* No.

*Solution 1:* The first equation implies $x^3 = -y^2 - 1$. Thus $x^3 < 0$, implying also $x < 0$. Similarly from the second, third and fourth equations we obtain $y < 0$, $z < 0$ and $t < 0$, respectively. The sum of negative numbers $x, y, z, t$ is negative, contradicting the fifth equation.

*Solution 2:* Suppose that the system has a solution. W.l.o.g., let $x$ be variable with the largest value. Then $4x \geqslant x + y + z + t$, which by the last equation implies $x \geqslant 0$. Consequently, also $x^3 \geqslant 0$. As $y^2 \geqslant 0$, this implies $1 + x^3 + y^2 \geqslant 1$, contradicting the first equation. Hence no solution can exist.

**O13** *(Seniors.)* Let $n$ be a fixed positive integer. Find all triples $(a, b, c)$ of integers satisfying the following system of equations:

$$\begin{cases} a^{n+3} + b^{n+2}c + c^{n+1}a^2 + a^n b^3 = 0 \\ b^{n+3} + c^{n+2}a + a^{n+1}b^2 + b^n c^3 = 0 \\ c^{n+3} + a^{n+2}b + b^{n+1}c^2 + c^n a^3 = 0 \end{cases}$$

*Answer:* $(0, 0, 0)$.

*Solution:* If $a = 0$ then the first equation implies $b^{n+2}c = 0$. Hence $b = 0$ or $c = 0$; w.l.o.g., $b = 0$. Then the last equation reduces to $c^{n+3} = 0$ which implies $c = 0$. Thus if $a = 0$ then $a = b = c = 0$. Analogously we can prove that if $b = 0$ or $c = 0$ then $a = b = c = 0$. The triple $(0, 0, 0)$ satisfies the equation.

It remains to consider triples $(a, b, c)$ whose all terms are different from 0. Let $p$ be an arbitrary prime number. If $p \mid a$ then the first equation implies $p \mid b^{n+2}c$. Thus $p \mid b$ or $p \mid c$; w.l.o.g., $p \mid b$. Then the last equation gives $p \mid c^{n+1}$ which implies $p \mid c$. Thus if $p \mid a$ then $p \mid a$, $p \mid b$, $p \mid c$. Analogously we can prove that if $p \mid b$ or $p \mid c$ then $p \mid a$, $p \mid b$, $p \mid c$.

But rewriting $a = pa'$, $b = pb'$, $c = pc'$ enables us to divide both sides of all equations by $p^{n+3}$, giving a similar system of equations having $a'$, $b'$, $c'$ in the role of $a$, $b$, $c$. Hence the triple $(a', b', c')$ also satisfies the system of equations. As the numbers other than 0 cannot be infinitely divided in integers, a finite number of divisions should give us a solution whose terms have no common prime factors. By the previous paragraph, this is possible only if the values of variables have no prime factors, i.e., $a = \pm 1$, $b = \pm 1$, $c = \pm 1$. We show now that such solutions do not exist. Indeed, if $n$ is even then the system of equations reduces to

$$\begin{cases} a + c + c + b = 0, \\ b + a + a + c = 0, \\ c + b + b + a = 0. \end{cases}$$

Adding all equations results in $4(a + b + c) = 0$, implying $a + b + c = 0$; but the sum of three odd numbers cannot equal the even number 0. If $n$ is odd then the system of equations reduces to

$$\begin{cases} 1 + bc + 1 + ab = 0, \\ 1 + ca + 1 + bc = 0, \\ 1 + ab + 1 + ca = 0. \end{cases}$$

Adding all equations results in $2(ab + bc + ca) + 6 = 0$, implying $ab + bc + ca = -3$. As $|ab| = |bc| = |ca| = 1$, the only possibility is $ab = bc = ca = -1$. This demands that $a$, $b$, $c$ have pairwise opposite signs which is impossible. Consequently, no solutions except $(0, 0, 0)$ exist.

**O14** *(Seniors.)* The incircle of triangle $ABC$ touches the sides $AB$ and $AC$ at points $K$ and $L$, respectively. The line $BL$ intersects the incircle of triangle $ABC$ at point $M$ ($M \neq L$). A circle passing through point $M$ touches the lines $AB$ and $BC$ at points $P$ and $Q$, respectively, and intersects the incircle of triangle $ABC$ at point $N$ ($N \neq M$). Prove that if $KM \parallel AC$ then points $P$, $N$ and $L$ are collinear.

*Solution:* Note that the circles of the problem can be obtained from each other by homothetic transformation with center $B$ since both are tangent to sides $BA$ and $BC$. Let $X$ be the other intersection point of the line $BL$ with the circumcircle of the triangle $MPQ$ (Fig. 14). The aforementioned homothety takes point $P$ to point $K$, point $M$ to point $L$, and point $X$ to point $M$. By the homothety, $\angle PXM = \angle KML$. Hence $\angle KML = \angle KLA = \angle LKM$. Finally,

$$\begin{aligned} \angle PNM + \angle LNM &= (180^\circ - \angle PXM) + \angle LKM \\ &= (180^\circ - \angle KML) + \angle KML = 180^\circ. \end{aligned}$$

This proves the desired claim that points $P$, $N$ and $L$ are collinear.

*Remark:* The converse is also true: if points $P$, $N$ and $L$ are collinear then lines $KM$ and $AC$ are parallel. The proof is analogous.

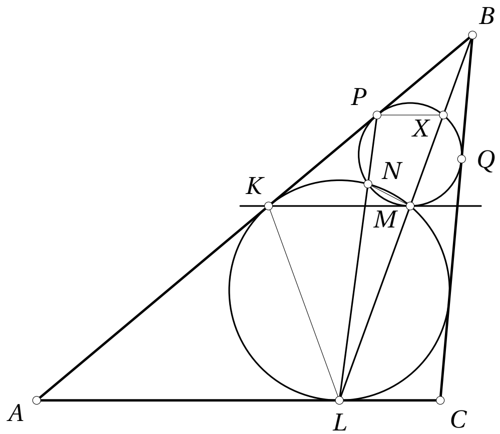



Fig. 14

**O15** *(Seniors.)* There is a rectangular grid with 3 rows and $n$ columns on a blackboard.

(a) Find the number of ways to write exactly one of numbers $1, 2, \ldots, 3n$ into each square in such a way that all the following conditions are met:

(1) Different squares contain different numbers.

(2) For each $i = 1, 2, \ldots, 3n-1$, the numbers $i$ and $i+1$ are written into adjacent squares (i.e., squares having a common side).

(3) The numbers 1 and $3n$ are written into adjacent squares.

(b) The same question, but as the 3rd condition, the number 1 must be written into the leftmost and the number $3n$ into the rightmost column.

*Answer:* (a) $3n \cdot 2^{\frac{n}{2}}$ for even $n$ and 0 for odd $n$; (b) $2^n$.

*Solution:*

(a) Note that all three squares in the leftmost and rightmost columns must be traversed consecutively because along with both corner squares of either of the columns one has to traverse the middle square of the same column and this cannot be done twice. In no other column can one traverse all three squares consecutively because such column would divide the grid into two parts between which one could not move without visiting a square twice.
Note also that one cannot traverse three or more consecutive squares of the middle row consecutively because such trajectory would also divide the grid into two isolated parts (Figures 15 and 16). For similar reasons, when visiting two consecutive squares of the middle row consecutively, the previous and the next square should be taken from one and the same row (either top or bottom).
Hence apart from the first and the last column, every legal trajectory divides the middle row into pairs of squares that are visited consecutively. Therefore the required enumeration is impossible for odd $n$. For even $n$, one can choose for every pair of squares in the middle row the row where the previous and the next square are located (either top or bottom); the number of such pairs of squares is $\frac{n-2}{2}$, whence there are $2^{\frac{n-2}{2}}$ possibilities to choose the direction of continuation for every such pair of squares. The whole trajectory is determined by these choices. For writing numbers in the case of a fixed trajectory, there are $6n$ possibilites (2 possibilities to choose the direction and $3n$ possibilities to choose the initial square). Consequently, there are $6n \cdot 2^{\frac{n-2}{2}} = 3n \cdot 2^{\frac{n}{2}}$ possibilities of the required enumeration for even $n$.

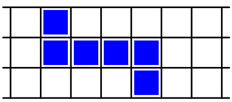


Fig. 15

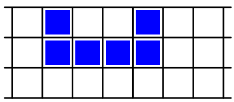


Fig. 16

(b) One cannot start from the middle square of the leftmost column since either of the corner squares would become a dead end. Hence suppose w.l.o.g. that one starts from the top left corner and visits $k$ squares in the top row (inclusive of the initial square), where $1 \leqslant k \leqslant n$ (Fig. 17). Going directly to the bottom row would generally divide the grid into two isolated parts; also turning to the right along the middle row would make it impossible to visit squares in the left and afterwards reach the rightmost column. Hence one should turn to the left along the middle row. Similarly we find that the leftmost part of size $3 \times k$ of the grid should be traversed along Z-shape (Fig. 18). After that, one should follow similar sules to traverse the remaining part of size $3 \times (n - k)$.
Repeating the argument, we can see that every legal trajectory divides the grid into Z-shapes and S-shapes, whereby Z-shapes are immediately fol-

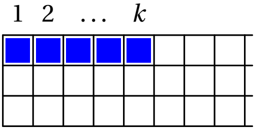



Fig. 17

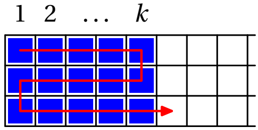



Fig. 18

lowed by S-shapes and vice versa. In every column but the last one we can choose whether to start a new letter or make the current letter wider. In the first column, we also can choose which corner to start in. Hence there are $2^n$ possibilities altogether.

**O16** *(Seniors.)* Find the least positive integer $n$ such that $\sqrt[5]{5n}$, $\sqrt[6]{6n}$ and $\sqrt[7]{7n}$ are integers.
*Answer:* $2^{35} \cdot 3^{35} \cdot 5^{84} \cdot 7^{90}$.
*Solution:* Let $n = 2^\alpha \cdot 3^\beta \cdot 5^\gamma \cdot 7^\delta \cdot s$, where $s$ is not divisible by 2, 3, 5 or 7; then $5n = 2^\alpha \cdot 3^\beta \cdot 5^{\gamma+1} \cdot 7^\delta \cdot s$, $6n = 2^{\alpha+1} \cdot 3^{\beta+1} \cdot 5^\gamma \cdot 7^\delta \cdot s$ and $7n = 2^\alpha \cdot 3^\beta \cdot 5^\gamma \cdot 7^{\delta+1} \cdot s$. Consequently:

- For $\sqrt[5]{5n}$ to be an integer, $\alpha$, $\beta$, $\gamma+1$ and $\delta$ must be divisible by 5;
- For $\sqrt[6]{6n}$ to be an integer, $\alpha+1$, $\beta+1$, $\gamma$ and $\delta$ must be divisible by 6;
- For $\sqrt[7]{7n}$ to be an integer, $\alpha$, $\beta$, $\gamma$ and $\delta+1$ must be divisible by 7.

Hence $\alpha$ and $\beta$ must be divisible by 35, $\gamma$ must be divisible by 42 and $\delta$ must be divisible by 30. The least suitable value for $\alpha$ and $\beta$ is 35 since 35 is the least positive multiple of 35 and the next integer is divisible by 6. Studying the multiples of 42 and 30 similarly shows that the least suitable value for $\gamma$ is 84 and the least suitable value for $\delta$ is 90. For the least suitable value for $n$, take $s = 1$. Hence the desired number is $2^{35} \cdot 3^{35} \cdot 5^{84} \cdot 7^{90}$.

**O17** *(Seniors.)* How many positive integers, where the only allowed digits are 0 and 1, are less than 11111100100?
*Answer:* 2019.
*Solution 1:* The given number has 11 digits. There are $2^{11} - 1 = 2047$ positive integers with at most 11 digits each of which is either 0 or 1. We solve the problem by subtracting the number of positive integers that are not less than the given number.
The 11-digit numbers larger than the given number are all of the form $\overline{111111abcde}$. There are $2^5 = 32$ numbers in this form. Among them, 4 numbers 11111100000, 11111100001, 11111100010 and 11111100011 are less than the given number. Thus the number of positive integers consisting of zeros and ones and being less than 11111100100 is $2047 - 32 + 4 = 2019$.
*Solution 2:* Ordering any two digit sequences that constitute a positional representation on different bases does not depend on the base. This means that if a number is larger than another number on base 2 then the first number is larger than the second number also on base 10. The number

11111100100 on base 2 equals $2^{10} + 2^9 + 2^8 + 2^7 + 2^6 + 2^5 + 2^2 = 2020$ on base 10. Hence, to solve the problem, it suffices to count all positive integers less than 2020. The result is obviously 2019.

**O18** *(Seniors.)* Prove that $a^{2020} + 10a^{1010} + 1001$ is prime for no integers $a$.

*Solution:* Note that $1001 = 11 \cdot 91$. If $11 \mid a$ then also $11 \mid a^{2020} + 10a^{1010} + 1001$. Otherwise, consider $a^5$ modulo 11. A case study shows that if $a$ is congruent to $1, 3, 4, 5$, or 9, then $a^5$ is congruent to 1, and in all other cases, $a^5$ is congruent to $-1$. Hence $a^{10} \equiv 1 \pmod{11}$ whenever $11 \nmid a$. This implies that $a^{1010} \equiv 1 \pmod{11}$ and $a^{2020} \equiv 1 \pmod{11}$ because $a^{1010} = \left(a^{10}\right)^{101}$ ja $a^{2020} = \left(a^{10}\right)^{202}$. Consequently, $10a^{1010} \equiv 10 \pmod{11}$, implying that $a^{2020} + 10a^{1010} + 1001 \equiv 0 \pmod{11}$. Thus $a^{2020} + 10a^{1010} + 1001$ cannot be prime since obviously $a^{2020} + 10a^{1010} + 1001 \geqslant 1001 > 11$.

*Remark:* The fact $a^{10} \equiv 1 \pmod{11}$ directly follows from Fermat's theorem.

**O19** *(Seniors.)* There are $n$ lists of candidates taking part in elections. Let $h_i$ be the total number of votes given for the candidates of the $i$th list. There are $M$ seats in the representative assembly.

Anna proposes the following system for delivering mandates: For each list, one computes a reference number $v_i = \frac{h_i}{a_i+1}$ where $a_i$ is the number of mandates already given to the $i$th list (initially $a_i = 0$, i.e., the reference number of each list equals its number of votes). On every step ($M$ times in total), one chooses the list with the greatest reference number (if several lists share the first place, one of them is chosen randomly) and adds one mandate to this list, after which the reference number of this list is recomputed.

Bert's idea for delivering mandates is to multiply the number of votes of every list by $M/K$ where $K = h_1 + \ldots + h_n$, whereby fractional results are rounded downwards. As rounding may cause some seats to be undelivered, he proposes multiplying all numbers of votes of the lists by some suitable coefficient $\beta$, so that the number of mandates given to the $i$th list would be $m_i = \left\lfloor \frac{\beta h_i M}{K} \right\rfloor$ where $m_1 + \ldots + m_n = M$.

Prove that if such coefficient $\beta$ exists then Anna's and Bert's methods lead to the same distribution of mandates.

*Solution:* Let the total number of mandates given to the $i$th list be $m_i$ in the case of Bert's method and $m'_i$ in the case of Anna's method. Suppose that these methods result in different distribution of mandates. As the sum of numbers of mandates must be the same, we must have $m'_i > m_i$ for some $i = 1, 2, \ldots, n$ and $m'_j < m_j$ for some $j = 1, 2, \ldots, n$. As the numbers of mandates are integers, we have $m'_i \geqslant m_i + 1$ and $m'_j + 1 \leqslant m_j$.

Consider the situation in the case of Anna's method immediately after the $i$th list having obtained its last mandate. Before obtaining the last mandate, the $i$th list had $m'_i - 1$ mandates and reference number $v_i = \frac{h_i}{(m'_i-1)+1} = \frac{h_i}{m'_i}$, while the $j$th list had at most $m'_j$ mandates, implying that $a_j \leqslant m'_j$ and $v_j = \frac{h_j}{a_j+1} \geqslant \frac{h_j}{m'_j+1}$. As the mandate was given to the $i$th list, $\frac{h_i}{m'_i} \geqslant \frac{h_j}{m'_j+1}$. Thus $\frac{h_i}{h_j} \geqslant \frac{m'_i}{m'_j+1}$, implying that

$$m_j \cdot \frac{h_i}{h_j} \geqslant m_j \cdot \frac{m'_i}{m'_j+1} \geqslant m_j \cdot \frac{m_i+1}{m_j} \geqslant m_i + 1.$$

On the other hand, from the specification of Bert's method we know that

$$m_j \cdot \frac{h_i}{h_j} = \frac{h_i}{h_j} \cdot \left\lfloor \frac{\beta h_j M}{K} \right\rfloor \leqslant \frac{h_i}{h_j} \cdot \frac{\beta h_j M}{K} = \frac{\beta h_i M}{K} < \left\lfloor \frac{\beta h_i M}{K} \right\rfloor + 1 = m_i + 1.$$

This inequation contradicts the previous inequation. Hence both methods indeed lead to the same distribution of mandates.

*Remark:* The first method is called D'Hondt's method after Victor D'Hondt, a Belgian lawyer. The second method is called Jefferson's method after the president of the USA Thomas Jefferson.

**O20** *(Seniors.)* The lines tangent to the circumcircle of triangle $ABC$ at points $B$ and $C$ intersect at point $D$. The circumcircle of triangle $BCD$ intersects the lines $AB$ and $AC$ the second time at points $K$ and $L$, respectively. Prove that the line $AD$ bisects the line segment $KL$.

*Solution 1:* We prove that $AKDL$ is a parallelogram; this implies the desired claim since $AD$ and $KL$ are diagonals of this quadrilateral. We prove at first that $KD \parallel AL$. If $K$ lies between $A$ and $B$ (Fig. 19) then by inscribed angles $\angle BAC = \angle BCD$ and $\angle BCD = \angle BKD$. Consequently, $\angle BAL = \angle BAC = \angle BKD$, implying $KD \parallel AL$. If $B$ lies between $A$ and $K$ (Fig. 20) then by inscribed angles $\angle BAC = \angle BCD$ and $\angle BCD = 180^\circ - \angle BKD$. Consequently, $\angle BAL + \angle BKD = \angle BAC + \angle BKD = 180^\circ$, implying $KD \parallel AL$. If $A$ lies between $K$ and $B$ (Fig. 21) then by inscribed angles $\angle BAC = 180^\circ - \angle BCD$ and $\angle BCD = \angle BKD$. Consequently, $\angle BAL = 180^\circ - \angle BAC = \angle BKD$, implying $KD \parallel AL$ again. Analogously, we can show that $LD \parallel AK$. Altogether, this establishes that $AKDL$ is a parallelogram.

*Solution 2:* By inscribed angles, $\angle ABC = \angle ALK$, implying that the triangles $ABC$ and $ALK$ are similar. Let the lines $AD$ and $KL$ intersect at point $N$

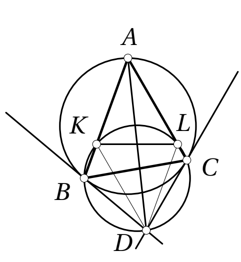


Fig. 19

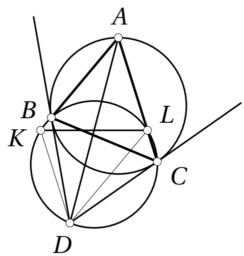


Fig. 20

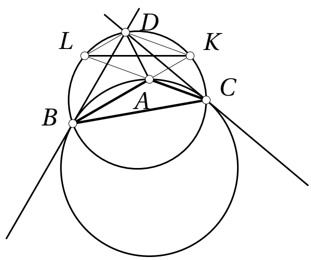
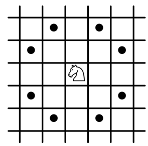


Fig. 21

and let $M$ be the midpoint of the side $BC$. It is known that the symmedian line drawn through a vertex of a triangle and the lines tangent to the circumcircle of the triangle at the other two vertices meet in one point; hence $N$ is the point of intersection of the symmedian line drawn through vertex $A$ of the triangle $ABC$ with line $KL$. By the definition of symmedian, $\angle BAM = \angle CAN = \angle LAN$, whence $M$ and $N$ are corresponding points in similar triangles $ABC$ and $ALK$. Thus $N$ bisects the line segment $KL$.

*Remark:* The claim of the problem is true if the circumcircle of the triangle $BCD$ is replaced with any circle that passes through vertices $B$ and $C$.

**O21** *(Seniors.)* Let $n$ be a positive integer. Find the largest number of knights that can be placed on a board of size $n \times n$ in such a way that no two knights attack each other. A knight attacks precisely the squares that are located either horizontally by one square and vertically by two squares away or horizontally by two squares and vertically by one square away.



*Answer:* $\left\lceil \frac{n^2}{2} \right\rceil$ if $n \neq 2$; 4 if $n = 2$.

*Solution:* Clearly one can place only 1 knight on an $1 \times 1$ board, which is $\left\lceil \frac{1^2}{2} \right\rceil$, and at most 4 knights on an $2 \times 2$ board. In the rest, assume $n \geqslant 3$.

If a set of unit squares is divided into pairs in such a way that a knight on one square of any pair attacks the other square of that pair then there cannot be more knights than half of the total number of unit squares in this set without some knights attacking each other. In Figures 22, 23 and 24, all unit squares of boards of size $2 \times 4$, $3 \times 4$ and $3 \times 6$ are divided into pairs whose members are located at one knight move from each other. In Figures 25 and 26, all unit squares of boards of size $3 \times 3$ and $5 \times 5$ except the middle square are divided into pairs whose members are located at one knight move from each other. By the above, it is impossible to place knights to more than half of unit squares of boards of size $2 \times 4$, $3 \times 4$ and $3 \times 6$. Taking into account that an additional knight may be on the middle square, it also follows that there cannot be more than $\left\lceil \frac{3^2}{2} \right\rceil$ knights on a $3 \times 3$ board or more than $\left\lceil \frac{5^2}{2} \right\rceil$ knights on a $5 \times 5$ board. As a $4 \times 4$ board can be formed from two $2 \times 4$ boards and a $6 \times 6$ board from two $3 \times 6$ boards, there cannot be more than $\frac{4^2}{2}$ knights on a $4 \times 4$ board or more than $\frac{6^2}{2}$ knights on a $6 \times 6$ board.

| | |
|---|---|
| 4 | 3 |
| 2 | 1 |
| 3 | 4 |
| 1 | 2 |

Fig. 22

| | | |
|---|---|---|
| 5 | 3 | 4 |
| 6 | 1 | 5 |
| 2 | 4 | 3 |
| 1 | 6 | 2 |

Fig. 23

| | | |
|---|---|---|
| 9 | 7 | 8 |
| 8 | 6 | 9 |
| 5 | 4 | 7 |
| 2 | 3 | 6 |
| 4 | 5 | 1 |
| 1 | 2 | 3 |

Fig. 24

| | | |
|---|---|---|
| 2 | 1 | 3 |
| 4 | | 2 |
| 1 | 3 | 4 |

Fig. 25

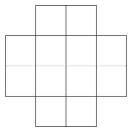
| | | | | |
|---|---|---|---|---|
| 2 | 10 | 6 | 7 | 3 |
| 6 | 9 | 2 | 10 | 11 |
| 5 | 1 | | 3 | 7 |
| 9 | 12 | 4 | 11 | 8 |
| 1 | 5 | 8 | 12 | 4 |

Fig. 26

If $n \geqslant 7$ then an $n \times n$ board can be divided into pieces of size $4 \times 4$, $r \times 4$ and $r \times r$, where $3 \leqslant r \leqslant 6$ and $4 \mid n - r$. By the above, neither $4 \times 4$ nor $r \times 4$ board can contain more knights than half of the number of unit squares ($5 \times 4$ and $6 \times 4$ are divisible into rectangles of size $2 \times 4$ and $3 \times 4$). The same holds for an $r \times r$ piece, if the middle square in the case of odd $r$ is not taken into account. Thus for no $n \geqslant 3$ can one place more than $\left\lceil \frac{n^2}{2} \right\rceil$ knights on an $n \times n$ board.
On the other hand, coloring the unit squares black and white chesswise, one can place a knight on all squares of one and the same color since a knight attacks only squares of the opposite color. The number of unit squares of a fixed color is $\frac{n^2}{2}$ if $n$ is even. In the case of odd $n$, the number of unit squares whose color coincides with the color of the middle square is $\left\lceil \frac{n^2}{2} \right\rceil$. Hence one can place $\left\lceil \frac{n^2}{2} \right\rceil$ pairwise non-attacking knights on an $n \times n$ board.

# Selected Problems from the Final Round of National Olympiad

**F1** *(Grade 9.)* Can natural numbers 1 through 12 be written into the squares of the grid shown in the figure in such a way that all following conditions are met?



(1) Each square contains exactly one number.
(2) The sums of numbers in both rows consisting of 4 squares, in both columns consisting of 4 squares, and in 4 middle squares are all equal.
(3) The numbers in any two squares with a common side or vertex differ from each other by at least 2.
*Answer:* Yes.
*Solution:* A configuration that meets all conditions is shown in Fig. 27.

| | | | |
|---|---|---|---|
| | 11 | 9 | |
| 6 | 3 | 5 | 12 |
| 1 | 8 | 10 | 7 |
| | 4 | 2 | |

Fig. 27

**F2** *(Grade 9.)* Let $D$ be the foot of the altitude drawn to the hypotenuse $AB$ of a right triangle $ABC$. The inradii of the triangles $ABC$, $CAD$ and $CBD$ are $r$, $r_1$ and $r_2$, respectively. Prove that $CD = r + r_1 + r_2$.
*Solution 1:* Let $a = |BC|$, $b = |CA|$, $c = |AB|$ and $h = |CD|$; then $ab = ch = (a + b + c)r$.

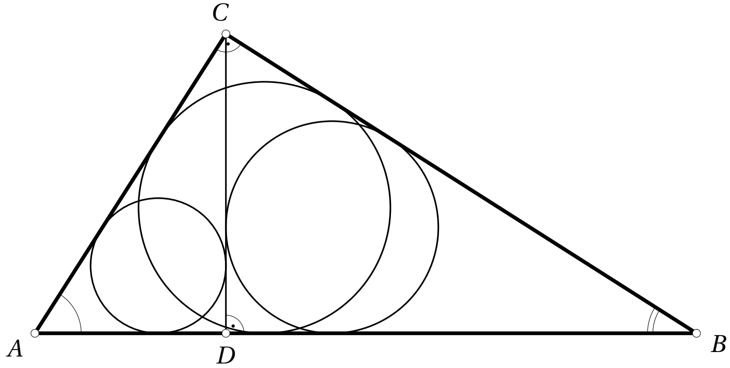



Fig. 28

Note that the triangles $ABC$, $ACD$ and $CBD$ are similar by two equal angles (Fig. 28). As the similarity ratio of triangles $ACD$ and $ABC$ is $\frac{b}{c}$ and that of triangles $CBD$ and $ABC$ is $\frac{a}{c}$, we have $r_1 = \frac{b}{c} \cdot r$ and $r_2 = \frac{a}{c} \cdot r$, whence $r + r_1 + r_2 = \frac{a+b+c}{c} \cdot r$. As the equality $ab = (a+b+c)r$ implies $r = \frac{ab}{a+b+c}$, we obtain $r + r_1 + r_2 = \frac{ab}{c}$. On the other hand, the equality $ab = ch$ implies $h = \frac{ab}{c}$. Consequently, $r + r_1 + r_2 = h$.

*Solution 2:* Let the incenters of the triangles $ABC$, $ACD$ and $BCD$ be $I$, $I_1$ and $I_2$. Let the points of tangency of the incircle of the triangle $ABC$ with sides $BC$, $CA$ and $AB$ be $K$, $L$ and $M$, respectively; let the points of tangency of the incircle of the triangle $ACD$ with sides $DC$, $CA$ and $AD$ be $K_1$, $L_1$ and $M_1$, respectively; let the points of tangency of the incircle of the triangle $BCD$ with sides $DC$, $CB$ and $BD$ be $K_2$, $L_2$ and $M_2$, respectively (Fig. 29). As a tangent line is perpendicular to the radius drawn to the point of tangency, we have $\angle IKC = 90^\circ = \angle ILC$. Thus $IKCL$ is a square by three right angles and the equality $IK = IL$. Hence $CK = CL = r$. Similarly we see that

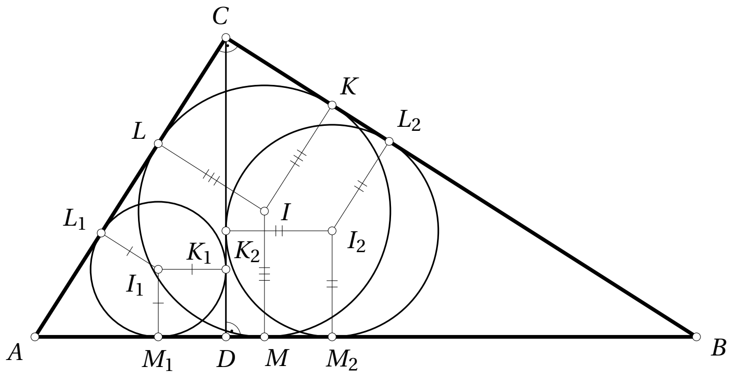



Fig. 29

$I_1K_1DM_1$ and $I_2K_2DM_2$ are squares, whereby $DK_1 = DM_1 = r_1$ ja $DK_2 = DM_2 = r_2$. By property of tangent line segments, we have $CK_1 = CL_1$, $AL = AM$ and $AL_1 = AM_1$, whence

$$\begin{aligned} CD &= CK_1 + K_1D = CL_1 + r_1 = CL + LL_1 + r_1 \\ &= r + AL - AL_1 + r_1 = r + AM - AM_1 + r_1 = r + MM_1 + r_1. \end{aligned}$$

By symmetry, we also get $CD = r + MM_2 + r_2$. After adding up these two equalities and taking into account that $MM_1 + MM_2 = DM_1 + DM_2 = r_1 + r_2$, we obtain

$$2CD = 2(r + r_1 + r_2).$$

Consequently, $CD = r + r_1 + r_2$.

**F3** *(Grade 9.)* Players A, B and C are playing the following game. Initially, the number 1 is written on a blackboard. On their move, each player replaces the number $n$ currently on the blackboard with either $n + 1$, $7n + 7$, or $4n^3 + 3n + 4$ at their own choice, under the condition that the new number must not be larger than $10^9$. The player A makes the first move, then B takes turn, then C, after him A again etc., until some player cannot make a legal move. The player who makes the last move wins. Can any of the players win the game against every legal play by the opponents and if yes then who?

*Answer:* Yes, C.

*Solution:* The game lasts while the number on the blackboard stays less than $10^9$, because it is possible to make a move of the first kind (replace $n$ with $n + 1$). As the number on the blackboard is increased by at least 1 by every move, it cannot stay less than $10^9$ infinitely. When the number on the blackboard equals $10^9$, there is no legal move. As numbers larger than $10^9$ cannot appear on the blackboard, the last move is made by the player who writes the number $10^9$ on the blackboard.

Let a number $n$ be written on the blackboard. Note that $7n + 7 \equiv n + 1 \pmod 3$ and $4n^3 + 3n + 4 \equiv n^3 + 1 \equiv n + 1 \pmod 3$. Hence all numbers that can appear on the blackboard on the next move are congruent to $n + 1$ modulo 3, whence the residues repeat in cycle $1, 2, 0, 1, 2, 0, \ldots$. As $10^9 \equiv 1 \pmod 3$, the number $10^9$ is written by the third player C.

*Remark:* The solution shows that C can win with whatever strategy.

**F4** *(Grade 10.)* Let $a$, $b$, $c$, $d$ be positive real numbers satistfying the system of equations

$$\begin{cases} a^2 + \frac{1}{b^2} &= \frac{1}{2}, \\ b^2 + \frac{4}{c^2} &= 8, \\ c^2 + \frac{16}{d^2} &= 2, \\ d^2 + \frac{4}{a^2} &= 32. \end{cases}$$

Determine the product $abcd$.

*Answer:* 4.

*Solution 1:* Multiplying all equations gives

$$\left(a^2+\frac{1}{b^2}\right)\left(b^2+\frac{4}{c^2}\right)\left(c^2+\frac{16}{d^2}\right)\left(d^2+\frac{4}{a^2}\right)=2^8.$$

By AM-GM, $a^2+\frac{1}{b^2}\geqslant 2\cdot\frac{a}{b}$, where the equality holds if and only if $a=\frac{1}{b}$. Similarly $b^2+\frac{4}{c^2}\geqslant 4\cdot\frac{b}{c}$ (equality if and only if $b=\frac{2}{c}$), $c^2+\frac{16}{d^2}\geqslant 8\cdot\frac{c}{d}$ (equality if and only if $c=\frac{4}{d}$) and $d^2+\frac{4}{a^2}\geqslant 4\cdot\frac{d}{a}$ (equality if and only if $d=\frac{2}{a}$). Multiplying the obtained four inequalities gives

$$\left(a^2+\frac{1}{b^2}\right)\left(b^2+\frac{4}{c^2}\right)\left(c^2+\frac{16}{d^2}\right)\left(d^2+\frac{4}{a^2}\right)\geqslant 2\cdot\frac{a}{b}\cdot 4\cdot\frac{b}{c}\cdot 8\cdot\frac{c}{d}\cdot 4\cdot\frac{d}{a}=2^8.$$

The resulting inequality must hold as equality by the first step of the solution. This is possible only if all four inequalities hold as equalities, whence

$$\begin{cases} a=\frac{1}{b},\\ b=\frac{2}{c},\\ c=\frac{4}{d},\\ d=\frac{2}{a}.\end{cases}$$

By multiplying the equations of this system, we get $abcd=\frac{16}{abcd}$, whence $(abcd)^2=16$. As $a$, $b$, $c$ and $d$ are positive, the only possibility is $abcd=4$.

*Solution 2:* By introducing $a=\frac{x}{2}$, $b=2y$, $c=z$, $d=4t$, rewrite the system as

$$\begin{cases} \frac{x^2}{4}+\frac{1}{4y^2}=\frac{1}{2},\\ 4y^2+\frac{4}{z^2}=8,\\ z^2+\frac{16}{16t^2}=2,\\ 16t^2+\frac{4\cdot 4}{x^2}=32.\end{cases}$$

Multiplying the first equation by 4, the second equation by $\frac{1}{4}$, and the fourth equation by $\frac{1}{16}$, we obtain an equivalent system

$$\begin{cases} x^2+\frac{1}{y^2}=2,\\ y^2+\frac{1}{z^2}=2,\\ z^2+\frac{1}{t^2}=2,\\ t^2+\frac{1}{x^2}=2.\end{cases}$$

Adding the equations of this system gives

$$x^2+\frac{1}{y^2}+y^2+\frac{1}{z^2}+z^2+\frac{1}{t^2}+t^2+\frac{1}{x^2}=8.$$

But for every real number $u$, $u+\frac{1}{u}\geqslant 2$, where equality holds only if $u=1$. By adding up inequalities $x^2+\frac{1}{x^2}\geqslant 2$, $y^2+\frac{1}{y^2}\geqslant 2$, $z^2+\frac{1}{z^2}\geqslant 2$ and

$t^2 + \frac{1}{t^2} \geqslant 2$, we get

$$x^2 + \frac{1}{x^2} + y^2 + \frac{1}{y^2} + z^2 + \frac{1}{z^2} + t^2 + \frac{1}{t^2} \geqslant 8.$$

This inequality must hold as equality by the above; hence all four previous inequalities must hold as equalities, too, i.e., $x^2 = y^2 = z^2 = t^2 = 1$. As the numbers are positive, the only possibility is $x = y = z = t = 1$. Hence $abcd = \frac{x}{2} \cdot 2y \cdot z \cdot 4t = 4xyzt = 4$.

*Remark:* This problem can also be solved by finding the values of $a$, $b$, $c$, $d$ using standard techniques (substituting variables from one equation to another) and directly computing $abcd$.

**F5** *(Grade 10.)* Find all positive integers $k$ for which there is a right triangle with legs of integral lengths and hypotenuse of length $\sqrt{88\ldots822\ldots2}$, where the number under the root consists of exactly $k$ eights and exactly $k$ twos.

*Answer:* 1.

*Solution 1:* Let the lengths of legs be $a$ and $b$. By the Pythagorean theorem,

$$a^2 + b^2 = 88\ldots822\ldots2.$$

If $k = 1$ then one can choose $a = 9$ and $b = 1$ as $9^2 + 1^2 = 82$. If $k \geqslant 2$ then $88\ldots822\ldots2 \equiv 6 \pmod 8$ since $822 \equiv 6 \pmod 8$ and $222 \equiv 6 \pmod 8$. On the other hand, the residues of $a^2$ and $b^2$ modulo 8 can be 0, 1, or 4. However, the sum of such two numbers can have residue 0, 1, 2, 4, or 5 modulo 8. Consequently, $a^2 + b^2 = 88\ldots822\ldots2$ cannot hold for $k \geqslant 2$.

*Solution 2:* Let the lengths of legs be $a$ and $b$. By the Pythagorean theorem,

$$a^2 + b^2 = 88\ldots822\ldots2.$$

If $k = 1$ then one can choose $a = 9$ and $b = 1$ as $9^2 + 1^2 = 82$. Assume in the rest that $k \geqslant 2$. It is known that a positive integer is expressible as the sum of two squares of integers if and only if its canonical representation contains primes congruent to 3 modulo 4 only with even exponents. Note that $88\ldots822\ldots2 = 2 \cdot 44\ldots411\ldots1$, where only the odd second factor can be divisible by primes congruent to 3 modulo 4. If all such primes would occur with even exponents in the canonical representation of $44\ldots411\ldots1$, this number itself would be congruent to 1 modulo 4, but actually $44\ldots411\ldots1 \equiv 3 \pmod 4$. This shows that right triangles with the required property cannot exist in the case $k \geqslant 2$.

**F6** *(Grade 10.)* Let $D$ and $E$ be the midpoints of sides $AB$ and $AC$, respectively, of a triangle $ABC$. Prove that the line $AB$ is tangent to the circumcircle of the triangle $BEC$ if and only if the line $AC$ is tangent to the circumcircle of the triangle $BED$.

*Solution 1:* The conditions of the problem imply that $DE$ is the midsegment parallel to the side $BC$ of the triangle $ABC$ (Fig. 30). By properties of inscribed angle, the line $AB$ is tangent to the circumcircle of the triangle $BEC$

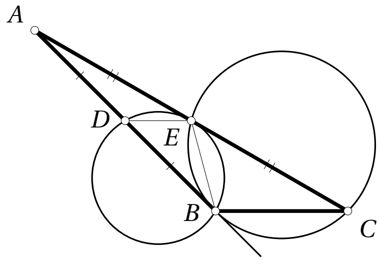



Fig. 30

if and only if $\angle ECB = \angle DBE$, and the line $AC$ is tangent to the circumcircle of the triangle $BED$ if and only if $\angle DBE = \angle AED$. But $\angle ECB = \angle AED$ since the lines $DE$ and $BC$ are parallel, whence validity of either of these two equalities implies validity of the other one.

*Solution 2:* By powers, the line $AB$ is tangent to the circumcircle of the triangle $BEC$ if and only if $AB^2 = AE \cdot AC$, and the line $AC$ is tangent to the circumcircle of the triangle $BED$ if and only if $AE^2 = AD \cdot AB$. As $AB = 2AD$ and $AC = 2AE$, the equality $AB^2 = AE \cdot AC$ is equivalent to the equality $2|AD|^2 = |AE|^2$, as well as the equality $AE^2 = AD \cdot AB$ is equivalent to the equality $AE^2 = 2AD^2$. Hence both conditions reduce to the same equality, which solves the problem.

**F7** *(Grade 10.)* Natural numbers 1 through $n$ are written on a blackboard. On each move, one erases from the blackboard 2 or more numbers whose sum is divisible by any of the chosen numbers and writes their sum on the blackboard. Two players make moves by turns and the player who cannot move loses the game. Which player can win the game against any play by the opponent, if:

(a) $n = 6$;

(b) $n = 11$?

*Answer:* (a) The first player; (b) The first player.

*Solution:*

(a) The first player can replace numbers 1, 2, 3, 6 with 12. After that, the blackboard contains numbers 4, 5, 12. In this state, the sum of no two or three numbers on the blackboard is divisible by all the added numbers. Thus the second player cannot move and the first player wins immediately.

(b) The first player can replace numbers 1, 2, 3, 4, 6, 8 with 24. After that, the blackboard contains numbers 5, 7, 9, 10, 11, 24, which sum up to 66. Among numbers 7, 9, 10, 11, 24, the l.c.m. of any two numbers is greater than 66. Thus when choosing two or more numbers from among the mentioned numbers, and perhaps also the number 5, the sum of the chosen numbers is less than their l.c.m. and cannot be divisible by all of them. Also when choosing one of the mentioned numbers together with 5, the sum of the chosen numbers is not divisible by the larger one. Hence the second player cannot move and the first player wins immediately.

**F8** *(Grade 11.)* Prove that

$$\sin 10^\circ \cdot \cos 20^\circ \cdot \sin 30^\circ \cdot \cos 40^\circ \cdot \sin 50^\circ \cdot \cos 60^\circ \cdot \sin 70^\circ \cdot \cos 80^\circ = \frac{1}{256}.$$

*Solution 1:* As $\sin 10^\circ = \cos 80^\circ$, $\sin 30^\circ = \cos 60^\circ$, $\sin 50^\circ = \cos 40^\circ$, and $\sin 70^\circ = \cos 20^\circ$, the desired equality is equivalent to

$$(\cos 20^\circ \cdot \cos 40^\circ \cdot \cos 60^\circ \cdot \cos 80^\circ)^2 = \frac{1}{256}.$$

It is known that $\cos 60^\circ = \frac{1}{2}$. Concerning the other factors, we obtain

$$\begin{aligned}\cos 20^\circ \cdot \cos 40^\circ \cdot \cos 80^\circ &= \frac{8\sin 20^\circ \cos 20^\circ \cdot \cos 40^\circ \cdot \cos 80^\circ}{8\sin 20^\circ} \\ &= \frac{4\sin 40^\circ \cos 40^\circ \cdot \cos 80^\circ}{8\sin 20^\circ} \\ &= \frac{2\sin 80^\circ \cos 80^\circ}{8\sin 20^\circ} \\ &= \frac{\sin 160^\circ}{8\sin 20^\circ} = \frac{\sin 20^\circ}{8\sin 20^\circ} = \frac{1}{8}.\end{aligned}$$

Consequently,

$$(\cos 20^\circ \cdot \cos 40^\circ \cdot \cos 60^\circ \cdot \cos 80^\circ)^2 = \left(\frac{1}{2} \cdot \frac{1}{8}\right)^2 = \left(\frac{1}{16}\right)^2 = \frac{1}{256}.$$

*Solution 2:* As $\sin 10^\circ = \cos 80^\circ$, $\sin 30^\circ = \cos 60^\circ$, $\sin 50^\circ = \cos 40^\circ$, and $\sin 70^\circ = \cos 20^\circ$, the l.h.s. of the desired equality can be rewritten as $(\cos 20^\circ \cos 40^\circ \cos 60^\circ \cos 80^\circ)^2$. Hence the desired equality is equivalent to

$$\cos 20^\circ \cos 40^\circ \cos 60^\circ \cos 80^\circ = \frac{1}{16}. \qquad (2)$$

As $\cos 60^\circ = \frac{1}{2}$ and $\sin 60^\circ = \frac{\sqrt{3}}{2}$, we have

$$\begin{aligned}&\cos 40^\circ \cos 80^\circ \\ &= \cos(60^\circ - 20^\circ)\cos(60^\circ + 20^\circ) \\ &= (\cos 60^\circ \cos 20^\circ + \sin 60^\circ \sin 20^\circ)(\cos 60^\circ \cos 20^\circ - \sin 60^\circ \sin 20^\circ) \\ &= \cos^2 60^\circ \cos^2 20^\circ - \sin^2 60^\circ \sin^2 20^\circ \\ &= \tfrac{1}{4}\cos^2 20^\circ - \tfrac{3}{4}\sin^2 20^\circ = \tfrac{1}{4}\left(\cos^2 20^\circ - 3\sin^2 20^\circ\right).\end{aligned}$$

Thus

$$\cos 20^\circ \cos 40^\circ \cos 60^\circ \cos 80^\circ = \frac{1}{8}\left(\cos^3 20^\circ - 3\sin^2 20^\circ \cos 20^\circ\right). \qquad (3)$$

By applying the formula $\cos 3x = \cos^3 x - 3\sin^2 x \cos x$ to $x = 20^\circ$, we get $\cos^3 20^\circ - 3\sin^2 20^\circ \cos 20^\circ = \cos 60^\circ = \frac{1}{2}$. Consequently, (3) reduces to (2), completing the solution of the problem.

**F9** *(Grade 11.)* The bisector of the internal angle on vertex $A$ of a triangle $ABC$ intersects the side $BC$ at point $D$. The line tangent to the circumcircle of the triangle $ABC$ at point $A$ intersects the line $BC$ at point $K$. Prove that $KA = KD$.

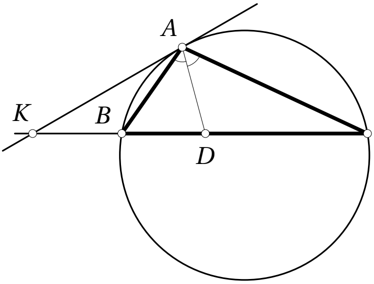



Fig. 31

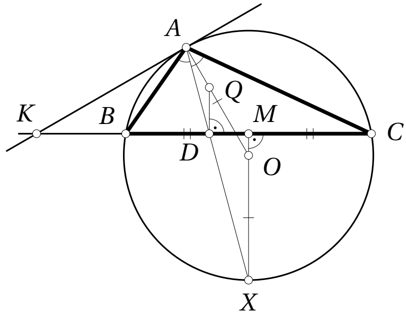



Fig. 32

*Solution 1:* Assume w.l.o.g. that $\angle ABC > \angle ACB$ (Fig. 31; otherwise change the roles of points $B$ and $C$). Note that

$$\angle ADK = 180^\circ - \angle CDA = \angle DAC + \angle ACD = \angle BAD + \angle ACB.$$

By inscribed angle property, $\angle KAB = \angle ACB$, whence

$$\angle BAD + \angle ACB = \angle BAD + \angle KAB = \angle KAD.$$

Consequently, $\angle ADK = \angle KAD$, which implies $KA = KD$.

*Solution 2:* Let $O$ be the circumcenter of the triangle $ABC$, $M$ be the midpoint of the side $BC$, and $X$ be the point of intersection of the ray $OM$ with the circumcircle of the triangle $ABC$. Moreover, let $Q$ be the point of intersection of the line perpendicular to the side $BC$ and passing through point $D$ with the line $AO$ (Fig. 32). As $OM$ is the perpendicular bisector of the side $BC$, point $X$ bisects the arc $BC$ of the circumcircle of the triangle $ABC$, whence the line $AD$ also passes through $X$. As $OA = OX$, we have $\angle XAO = \angle OXA$. On the other hand, $OX \perp BC$ and $QD \perp BC$ together imply $OX \parallel QD$, whence $\angle OXA = \angle QDA$. Thus $\angle DAQ = \angle XAO = \angle QDA$, implying $AQ = QD$. Consequently, there exists a circle with center $Q$ passing through both points $A$ and $D$. As $KA \perp AQ$ and $KD \perp DQ$, the lines $KA$ and $KD$ are tangent to this circle. By the property of tangent line segments, $KA = KD$.

**F10** *(Grade 11.)* Prove that in every solved sudoku the squares marked "X" in the figure contain precisely the same digits as the squares marked "Y", counting repetitions. (For instance, if the squares marked "X" contain three digits 3 in total then the squares marked "Y" also contain three digits 3 in total.)

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| X | X | | | | | | X | X |
| X | X | | | | | | X | X |
| | | Y | Y | Y | Y | Y | | |
| | | Y | | | | Y | | |
| | | Y | | | | Y | | |
| | | Y | | | | Y | | |
| | | Y | Y | Y | Y | Y | | |
| X | X | | | | | | X | X |
| X | X | | | | | | X | X |

*Remark:* A solved sudoku is a board of size $9 \times 9$ whose every row, every column and every part of size $3 \times 3$ bounded by bold rules contains every digit from 1 to 9 exactly once.

*Solution:* As digits 1 through 9 occur exactly once in every row, every col-

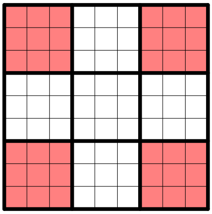


Fig. 33

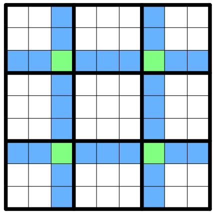


Fig. 34

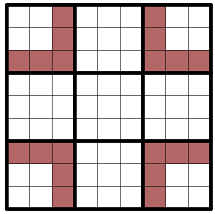


Fig. 35

umn and every part of size $3 \times 3$ bounded by bold rules, each of these digits occurs exactly 4 times in the squares colored in Figures 33 and 34, if digits in green squares are counted twice.

Removing from both (multi)sets of digits those digits that occur in squares colored in Figure 35, we obtain precisely the sets of digits that occur in squares marked "X" and "Y", respectively. Thus these sets must also equal.

**F11** *(Grade 11.)* Every term of the sequence $a_1, a_2, a_3, \ldots$ is either 0 or 1. It is known that both 0 and 1 occur at least 1010 times among every 2021 consecutive terms of the sequence. May one be sure that the sequence is periodic from some place on, i.e., there exist positive integers $n$ and $p$ such that $a_{n+i} = a_{n+i+p}$ for every natural number $i$?

*Answer:* No.

*Solution:* Consider the tuples $\underbrace{11\ldots1}_{1011\text{ times}}\underbrace{00\ldots0}_{1010\text{ times}}$ and $\underbrace{11\ldots1}_{1010\text{ times}}\underbrace{00\ldots0}_{1011\text{ times}}$. Concatenating infinitely many instances of these tuples in any order produces a sequence that satisfies the conditions of the problem. Indeed, consider any segment of 2021 consecutive terms of such sequence. As any two consecutive full blocks of zeros or ones contain at least 2020 terms in total, the segment under consideration contains terms of at most three such blocks. If it contains terms of three blocks then it contains the middle block fully. If the segment contained only two partial blocks of zeros or ones, it could contain at most 2020 terms in total. Hence the segment under consideration must contain one full block even if it contains only terms of two consecutive blocks. In any case, the digit of the full block occurs either 1010 or 1011 times and the other digit must occur either 1011 or 1010 times, respectively.

Combining the tuples defined at the beginning of the solution according to some non-periodic pattern (e.g., $xyxxyxxxyxxxxy\ldots$), the resulting sequence is not periodic from any place on. Indeed, suppose the contrary; let the length of the period be $p$. We can find a pattern of length $\operatorname{lcm}(p, 2021)$ starting from the place where periodicity starts, containing the tuples defined at the beginning of the solution a full number of times. But the middle term of these tuples is not repeating periodically, implying that the entire pattern also cannot be periodic. Consequently, there exist non-periodic sequences that satisfy the conditions of the problem.

**F12** *(Grade 12.)* Find all pairs $(a, b)$ of positive integers such that $a \geqslant b$ and

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{2021}.$$

*Answer:* $(2021 \cdot 2022, 2022)$, $(2021 \cdot 48, 43 \cdot 48)$, $(2021 \cdot 44, 47 \cdot 44)$, $(2021 \cdot 2, 2021 \cdot 2)$, $(47 \cdot 90, 43 \cdot 90)$.

*Solution 1:* Let $d = \gcd(a, b)$, $a = da'$, $b = db'$. The given equation reduces to $\frac{1}{da'} + \frac{1}{db'} = \frac{1}{2021}$ which is equivalent to

$$2021(a' + b') = da'b'. \tag{4}$$

As $a'$ and $b'$ are relatively prime to each other, they both are relatively prime to $a' + b'$, implying that $a'b'$ and $a' + b'$ are relatively prime. By (4), $a'b' \mid 2021(a' + b')$, implying $a'b' \mid 2021$. As $2021 = 43 \cdot 47$ where the factors are prime, the number 2021 has exactly four positive factors 1, 43, 47 and 2021. Taking into account that $a \geqslant b$ holds if and only if $a' \geqslant b'$, consider all cases:

- If $a' = 2021$ and $b' = 1$ then (4) implies $d = 2022$. Consequently, $(a, b) = (2021 \cdot 2022, 2022)$.
- If $a' = 47$ and $b' = 43$ then (4) implies $d = 90$. Consequently, $(a, b) = (47 \cdot 90, 43 \cdot 90)$.
- If $a' = 47$ and $b' = 1$ then (4) implies $d = 43 \cdot 48$. Consequently, $(a, b) = (2021 \cdot 48, 43 \cdot 48)$.
- If $a' = 43$ and $b' = 1$ then (4) implies $d = 47 \cdot 44$. Consequently, $(a, b) = (2021 \cdot 44, 47 \cdot 44)$.
- If $a' = 1$ and $b' = 1$ then (4) implies $d = 2021 \cdot 2$. Thus $(a, b) = (2021 \cdot 2, 2021 \cdot 2)$.

*Solution 2:* The given equation is equivalent to $\frac{a+b}{ab} = \frac{1}{2021}$ which reduces to $ab - 2021a - 2021b = 0$. After adding $2021^2$ to both sides and factorizing in the l.h.s, we get

$$(a - 2021)(b - 2021) = 2021^2. \tag{5}$$

As both $a$ and $b$ are positive, the factors in the l.h.s. of (5) are greater than $-2021$. Thus if both factors were negative then the absolute value of their product would be less than $2021^2$ and (5) could not hold. Hence both factors are positive. Since $a \geqslant b$, we must have $a - 2021 \geqslant b - 2021$. As $2021 = 43 \cdot 47$ with factors being prime, we obtain variants $(a - 2021, b - 2021) = (2021^2, 1)$, $(a - 2021, b - 2021) = (2021 \cdot 47, 43)$, $(a - 2021, b - 2021) = (2021 \cdot 43, 47)$, $(a - 2021, b - 2021) = (47^2, 43^2)$ and $(a - 2021, b - 2021) = (2021, 2021)$. Consequently, $(a, b) = (2021 \cdot 2022, 2022)$, $(a, b) = (2021 \cdot 48, 43 \cdot 48)$, $(a, b) = (2021 \cdot 44, 47 \cdot 44)$, $(a, b) = (47 \cdot 90, 43 \cdot 90)$, or $(a, b) = (2021 \cdot 2, 2021 \cdot 2)$.

**F13** *(Grade 12.)* Anna, Anne and Anni seek for real solutions $(x, y)$ to the system of equations

$$\begin{cases} 4x^3y - x^4 - 3x^2y^2 = 2021, \\ 4y^3x - y^4 - 3y^2x^2 = 2021. \end{cases}$$

Anna claims that the system of equations has a solution. Anne claims that the system of equations has no solution but at least one of the two equations has solutions. Anni claims that the system of equations has no solution and, even worse, neither of the two equations alone has a solution. Who is right?

*Answer:* Anne.

*Solution:* The system does not have a solution since adding the equation gives $-(x-y)^4 = 4042$ whose l.h.s. is non-positive but r.h.s. is positive.

We show that the first equation $4x^3y - x^4 - 3x^2y^2 = 2021$ has solutions (the same could be done for the second equation by symmetry). Dividing the equation by $x^4$ and reordering the terms in the l.h.s. results in

$$-3\left(\frac{y}{x}\right)^2 + 4\left(\frac{y}{x}\right) - 1 = \frac{2021}{x^4}. \tag{6}$$

As the discriminant of the quadratic equation $-3t^2 + 4t - 1 = 0$ is positive, there exists a real number $t$ such that $-3t^2 + 4t - 1$ equals a positive number $\varepsilon$. Define $x = \sqrt[4]{\frac{2021}{\varepsilon}}$ and $y = xt$; then $x$ and $y$ satisfy (6) and also the first equation of the given system of equations. Hence Anne is right.

**F14** *(Grade 12.)* Point $D$ inside an acute triangle $ABC$ satisfies

$$\angle ADC = \angle BDA = 180^\circ - \angle CAB.$$

Prove that the point symmetric to point $A$ w.r.t. point $D$ lies on the circumcircle of the triangle $ABC$.

*Solution 1:* Let the line $AD$ intersect the circumcircle of the triangle $ABC$ second time at point $D'$; by the assumptions, $\angle CDD' = \angle D'DB = \angle CAB$ (Fig. 36). We show that $AD = DD'$.

As the quadrilateral $ABD'C$ is cyclic, $\angle ABC = \angle AD'C = \angle DD'C$ and $\angle BCA = \angle BD'A = \angle BD'D$. Thus the triangled $ABC$, $DD'C$ and $DBD'$ are similar. Hence $\frac{|DD'|}{|DB|} = \frac{|DC|}{|DD'|}$, implying $|DD'|^2 = |BD| \cdot |CD|$.

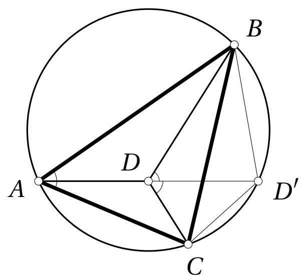



Fig. 36

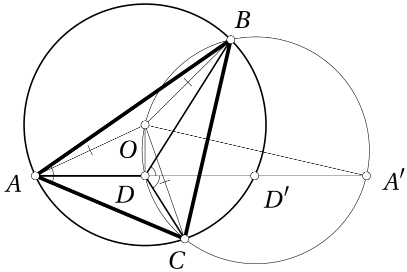



Fig. 37

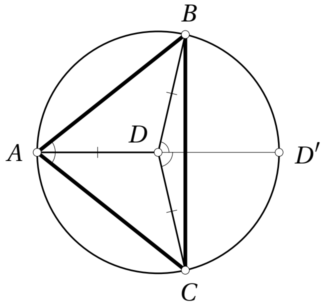



Fig. 38

By similarity of the triangles $ABC$ and $DD'C$, we obtain $\angle BCA = \angle D'CD$, which implies $\angle DCA = \angle D'CB = \angle D'AB = \angle DAB$. By similarity of the triangles $ABC$ and $DBD'$ we analogously get $\angle ABD = \angle CBD' = \angle CAD' = \angle CAD$. Hence the triangles $ABD$ and $CAD$ are similar. Consequently, $\frac{AD}{CD} = \frac{BD}{AD}$, which implies $AD^2 = BD \cdot CD$.

Altogether, we have proven $AD^2 = DD'^2$, which implies the desired result.

*Solution 2:* Let the line $AD$ intersect the circumcircle of the triangle $ABC$ second time at point $D'$; by the assumptions, $\angle CDD' = \angle D'DB = \angle CAB$. We show that $AD = DD'$.

Let $O$ be the circumcenter of the triangle $ABC$. Let the line $AD$ intersect the circumcircle of the triangle $BCD$ second time at point $A'$ (Fig. 37). By the equality $\angle CDA' = \angle A'DB$, the arcs $BA'$ and $A'C$ of the circumcircle of the triangle $BCD$ are equal.

By the conditions of the problem, $\angle CDB = 2\angle CAB$. Hence point $O$ lies on the circumcircle of the triangle $BCD$. Since $OC = OB$, the arcs $CO$ and $OB$ of the circumcircle of the triangle $BCD$ are equal.

Consequently, $OA'$ is a diameter of the circumcircle of the triangle $BCD$. If $O \neq D$ then, by Thales' theorem, $\angle A'DO = 90^\circ$. Thus $OD \perp AD'$, which implies $AD = DD'$ since a radius perpendicular to a chord bisects the chord. If $O = D$ (Fig. 38) then $AD'$ is a diameter of the circumcircle of the triangle $ABC$, obviously bisected by the center $O$.

**F15** *(Grade 12.)* There are 6 distinct lines chosen in the space. Find the largest possible number of points where at least 3 chosen lines intersect.

*Answer:* 4.

*Solution 1:* Call a point *rich* if at least 3 chosen lines intersect there. Let the number of rich points be $n$. We count pairs $(P, s)$ where $P$ is a rich point and $s$ is a chosen line that passes through $P$. The number of these pairs is at least $3n$ since each of the $n$ rich points occurs in at least 3 pairs. On the other hand, after fixing a chosen line, the other lines passing through distinct rich points on the fixed line are distinct (two lines cannot intersect in more than one point). Hence there can be at most $\left\lfloor \frac{6-1}{2} \right\rfloor = 2$ rich points on every

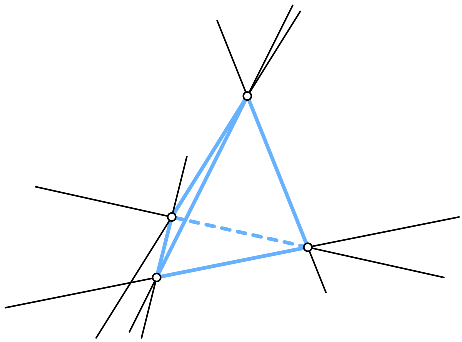


Fig. 39

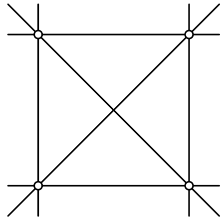


Fig. 40

chosen line. This implies that the number of pairs under consideration does not exceed $6 \cdot 2$, i.e., 12. Altogether, we have established $3n \leqslant 12$, which implies $n \leqslant 4$.

A configuration with 4 rich points can be obtained by choosing 6 lines determined by the 6 edges of a tetrahedron; those lines intersect at 4 vertices of the tetrahedron, 3 in each one (Figure 39).

*Solution 2:* Call a point *rich* if at least 3 chosen lines intersect there. Let the number of rich points be $n$. We count pairs $(P, p)$ where $P$ is a rich point and $p$ is a pair of distinct chosen lines both passing through $P$ (the order of lines in the pair is irrelevant). The number of such pairs is at most 15 since $C_6^2 = 15$ and every two lines intersect in at most one rich point. On the other hand, the number of these pairs must be at least $3n$, as at least 3 chosen lines intersect at every rich point and these give rise to $C_3^2 = 3$ pairs. Consequently, $3n \leqslant 15$ which implies $n \leqslant 5$. In the case $n = 5$, all inequalities in the argument above should hold as equalities, i.e., every two chosen lines intersect in a rich point and in every rich point exactly 3 chosen lines intersect. Thus every chosen line must intersect at rich points with exactly 5 chosen lines whereby at every rich point with exactly 2 of them, which is impossible as 5 is not divisible by 2. Consequently, $n \leqslant 4$.

To find a configuration with 4 rich points, choose arbitrarily 4 points, no 3 of which lie on one line. Pairs of these points determine 6 distinct lines. Every chosen point belongs to 3 distinct pairs, whence 3 such lines pass through every chosen point (in Fig. 40, rich points are marked with bubbles).

*Solution 3:* Call a point *rich* if at least 3 chosen lines intersect there. As in Solution 1 or Solution 2, we show that 4 rich points is possible. To prove that there can be no more rich points, let $n$ be the number of rich points in a configuration that satisfies the conditions of the problem with the largest possible number of rich points; by the above, we may assume that $n \geqslant 4$. Consider arbitrary two distinct rich points in this configuration and let $A$ and $B$ be the corresponding sets of chosen lines that meet in these points.

The sets $A$ and $B$ cannot have two or more common members as two lines cannot intersect in two distinct points. Suppose that the sets $A$ and $B$ are disjoint. As both $A$ and $B$ contain at least 3 chosen lines and the total number of chosen lines is 6, every chosen line must belong to either $A$ or $B$. Let $C$ be the set of chosen lines intersecting in a third rich point. As the set $C$ contains at least 3 lines and each of them belongs to either $A$ or $B$, at least 2 lines of the set $C$ belong to one of $A$ and $B$. But then $C$ would have 2 common members with this set which we saw is impossible. Consequently, every two rich points are connected with a chosen line. As 3 rich points cannot lie on one line since this would require $3 \cdot 2 + 1 = 7$ chosen lines, lines that connect different pairs of rich points are distinct. Thus there must be at least $C_n^2$ chosen lines, i.e., $C_n^2 \leqslant 6$. Consequently, $n \leqslant 4$.

**F16** *(Grade 12.)* Two regular polygons have a common circumcircle. The sum of the areas of the incircles of these polygons equals the area of their common circumcircle. Find all possibilities of how many vertices can the two polygons have.
*Answer:* 3 and 6 or 4 and 4.
*Solution:* The ratio of the inradius and the circumradius of a regular $n$-gon is $\cos \frac{180^\circ}{n}$. Hence the ratio of the areas of the incircle and the circumcircle of a regular $n$-gon is $\cos^2 \frac{180^\circ}{n}$.
Let a regular $n$-gon and a regular $m$-gon with a common circumcircle be given. W.l.o.g., let $n \leqslant m$ and the area of the common circumcircle be 1. By the above, the areas of the incircles of these polygons are $\cos^2 \frac{180^\circ}{n}$ and $\cos^2 \frac{180^\circ}{m}$, respectively. As their sum must equal the area of the common circumcircle, we get the equation $\cos^2 \frac{180^\circ}{n} + \cos^2 \frac{180^\circ}{m} = 1$ which is equivalent to $\cos^2 \frac{180^\circ}{n} = \sin^2 \frac{180^\circ}{m}$. As both $n$ and $m$ are larger than 2, both $\frac{180^\circ}{n}$ and $\frac{180^\circ}{m}$ are less than $90^\circ$, whence the equation reduces to $\cos \frac{180^\circ}{n} = \sin \frac{180^\circ}{m}$. Thus $\frac{180^\circ}{n} + \frac{180^\circ}{m} = 90^\circ$, implying that $\frac{1}{n} + \frac{1}{m} = \frac{1}{2}$. If $n = 3$ then $m = 6$; if $n = 4$ then $m = 4$; if $n > 4$ then $m < 4$, contradicting the assumption $n \leqslant m$.

# Selected Problems from the IMO Team Selection Contests

**S1** Juku has the first 100 volumes of the Harrie Totter book series at his home. For every $i$ and $j$, where $1 \leqslant i < j \leqslant 100$, call the pair $(i, j)$ *reversed* if volume No $j$ is before volume No $i$ on Juku's shelf. Juku wants to arrange all volumes of the series to one row on his shelf in such a way that there does not exist numbers $i$, $j$, $k$, where $1 \leqslant i < j < k \leqslant 100$, such that pairs $(i, j)$ and $(j, k)$ are both reversed. Find the largest number of reversed pairs that can occur under this condition.
*Answer:* 2500.

*Solution 1:* Let all 100 volumes be placed to a shelf $a$ in some order. For every $i = 1, 2, \ldots, 100$, let $a_i$ be the number of the volume that occurs as the $i$th in this order. In the order the volumes occur on shelf $a$, we start relocating the volumes to two new shelves $b$ and $c$. We place a volume to the end of shelf $b$ if, as the intermediate result, all volumes on shelf $b$ would be increasingly sorted by volume numbers. Otherwise, we place the volume to the end of shelf $c$ if, as the intermediate result, all volumes on shelf $c$ would be increasingly sorted by volume numbers. Continuing this way, we either can relocate all volumes onto two shelves in such a way that all volumes on either shelf are increasingly sorted by volume numbers or get stuck on some step $k$ of the process because $a_k$ is less than the number of the last volume on both new shelves. In the last case, let the number of the last volume on shelf $c$ be $a_j$; then, by assumptions, $j < k$ and $a_j > a_k$. Since volume No $a_j$ has been relocated to shelf $c$, some volume with number $a_i$ must occur on shelf $b$ such that $i < j$ and $a_i > a_j$. This means that pairs $(i, j)$ and $(j, k)$ were initially reversed. Hence we can conclude that, in the case of Juku's favourite orderings, all volumes can be relocated to two shelves, i.e, there exist two tuples $a_{i_1}, a_{i_2}, \ldots, a_{i_s}$ and $a_{j_1}, a_{j_2}, \ldots, a_{j_t}$ with $s + t = 100$, where $i_1 < \ldots < i_s$, $a_{i_1} < \ldots < a_{i_s}$ and $j_1 < \ldots < j_t$, $a_{j_1} < \ldots < a_{j_t}$. The number of pairs $(i, j)$ such that $i < j$ and the numbers $a_i$ and $a_j$ are in distinct tuples is exactly $st$. Each reversed pair $(i, j)$ must be one of these $st$ pairs. Thus the number of reversed pairs does not exceed $st$. As $st \leqslant \left(\frac{s+t}{2}\right)^2 = 50^2 = 2500$, the number of reversed pairs cannot exceed 2500. The number 2500 is achieved by the order $51, 52, \ldots, 100, 1, 2, \ldots, 50$.

*Solution 2:* Let all 100 volumes be placed to the shelf in some order. Consider the graph whose vertices are numbers $1, 2, \ldots, 100$ and an edge occurs between vertices $i$ and $j$ if and only if either $(i, j)$ or $(j, i)$ (depending on whether $i < j$ or $j < i$) is reversed. Suppose that the graph contains an odd cycle. Such cycle must contain three consecutive vertices $i$, $j$ and $k$ that are in either increasing or decreasing order; w.l.o.g., assume that $i < j < k$. Then pairs $(i, j)$ and $(j, k)$ are both reversed. Thus under the conditions of the problem, the graph cannot contain an odd cycle. Hence the graph is bipartite. Let the numbers of vertices in the two parts be $s$ and $t$, respectively; then the maximal number of edges is $st$. By AM-GM, $st \leqslant \left(\frac{s+t}{2}\right)^2 = 50^2 = 2500$. If the volumes are in the order $51, 52, \ldots, 100, 1, 2, \ldots, 50$, exactly 2500 reversed pairs arise indeed.

*Remark:* The problem can be solved even more immediately with Mantel's theorem that states that the maximum number of edges in a graph that does not contain cycles of length 3 is $\left\lfloor \frac{n^2}{4} \right\rfloor$, where $n$ is the number of vertices.

**S2** Find all polynomials $P(x)$ with integral coefficients whose values at points $x = 1, 2, \ldots, 2021$ are numbers $1, 2, \ldots, 2021$ in some order.

*Answer:* All polynomials of the form $P(x) = x + R(x)(x-1)(x-2)\ldots(x-$

2021) and the form $P(x) = 2022 - x + R(x)(x-1)(x-2)\dots(x-2021)$, where $R(x)$ is an arbitrary polynomial with integral coefficients.
*Solution:* Since $P(x)$ has integral coefficients, it satisfies

$$a - b \mid P(a) - P(b) \tag{7}$$

for arbitrary integers $a$ and $b$. By substituting $a = 2021$ and $b = 1$ into (7), we get $2020 \mid P(2021) - P(1)$. As $P(1), \dots, P(2021)$ must be 2021 distict numbers, $P(1)$ and $P(2021)$ cannot be equal, whence their difference must be at least 2020. The only possibility is $P(1)$ and $P(2021)$ being 1 and 2021 in some order. Consider these two cases separately.

- If $P(1) = 1$ and $P(2021) = 2021$ then substituting $a = 2020$ and $b = 1$ into (7) gives $2019 \mid P(2020) - 1$ that can be satisfied only if $P(2020) = 2020$. Similarly by substituting $a = k$ and $b = 1$ into (7) for $k = 2019, 2018, \dots, 2$, we each time get $P(k) = k$ as the only possibility, since all larger values of the polynomial that could satisfy $k - 1 \mid P(k) - 1$ are in use already. Hence $P(x) = x$ for every $x = 1, \dots, 2021$. As $1, \dots, 2021$ are roots of the polynomial $P(x) - x$, we must have $P(x) - x = (x-1)\dots(x-2021)R(x)$, where $R(x)$ can be any polynomial with integral coefficients. From here, we obtain solutions of the form $P(x) = x + R(x)(x-1)(x-2)\dots(x-2021)$.
- If $P(1) = 2021$ and $P(2021) = 1$ then by substituting $a = k$ and $b = 2021$ into (7) for $k = 2, \dots, 2020$, we each time get $2021 - k \mid P(k) - 1$ that can be satisfied only if $P(k) = 2022 - k$ analogously to the previous case. Hence $P(x) = 2022 - x$ for every $x = 1, \dots, 2021$. As $1, \dots, 2021$ are roots of the polynomial $P(x) - (2022 - x)$, we must have $P(x) - (2022 - x) = (x-1)\dots(x-2021)R(x)$, where $R(x)$ is any polynomial with integral coefficients. This leads to solutions of the form $P(x) = 2022 - x + R(x)(x-1)(x-2)\dots(x-2021)$.

Both families of solutions meet the required conditions: $P(1), \dots, P(2021)$ are $1, \dots, 2021$, respectively, in the first case and $2021, \dots, 1$, respectively, in the second case.

**S3** (a) There are $2n$ rays marked in a plane, with $n$ being a natural number. Given that no two marked rays have the same direction and no two marked rays have a common initial point, prove that there exists a line that passes through none of the initial points of the marked rays and intersects with exactly $n$ marked rays.
(b) Would the claim still hold if the assumption that no two marked rays have a common initial point was dropped?
*Answer:* (b) Yes.
*Solution:* Consider any circle such that all of the endpoints of the marked rays are inside the circle. Choose a tangent line of the circle that is not parallel to any of the marked rays. Let this tangent line be $l_0$ and let $l_\alpha$ be the tangent line we get by rotating $l_0$ counterclockwise by angle $\alpha$ w.r.t. the

center of the circle. For any $\alpha$, let $f(\alpha)$ be the number of marked rays that intersect with $l_\alpha$. Since $l_0$ and $l_\pi$ are two parallel lines and all of the endpoints of the marked rays are between them, we know from the definition of $l_0$ that $f(0)+f(\pi)=2n$. W.l.o.g., let $f(0) \leqslant n$ and $f(\pi) \geqslant n$. Because no two rays have the same direction, when $\alpha$ increases continuously $f(\alpha)$ can at any time instance only change by at most one. Thus $f(\alpha)$ ranges over all integral values between $f(0)$ and $f(\pi)$. Hence $f(\alpha)=n$ for some $\alpha$.

The argument above does not use the assumption that the initial points of the marked rays are distinct, whence it solves both parts of the problem.

*Remark:* Part a) of the problem can be solved otherwise. Choose an arbitrary line $l$ that is not parallel to any of the marked rays or any line connecting the initial points of two marked rays and from which the initial points of all marked rays lie on one side. Let $l'$ be a line parallel to $l$ such that the initial points of all marked rays lie between $l$ and $l'$. Then every marked ray intersects exactly one of the lines $l$ and $l'$, whence the numbers of the intersection points always sum up to exactly $2n$. W.l.o.g., let the number of intersection points on the line $l$ be less than or equal to the number of intersection points on the line $l'$. When shifting the line $l$ towards the line $l'$ while keeping its direction unchanged, the number of intersection points on it can change by at most one at any time instance. Thus there exists a position where the number of intersection points equals $n$.

**S4** Positive real numbers $a, b, c$ satisfy $abc=1$. Prove that

$$\frac{a}{1+b}+\frac{b}{1+c}+\frac{c}{1+a} \geqslant \frac{3}{2}.$$

*Solution 1:* The given inequality is equivalent to

$$2a(1+c)(1+a)+2b(1+a)(1+b)+2c(1+b)(1+c) \geqslant 3(1+a)(1+b)(1+c).$$

By removing parentheses and collecting similar terms, it is reduced to

$$2a^2+2b^2+2c^2+2a^2c+2b^2a+2c^2b \geqslant 3+a+b+c+ab+bc+ca+3abc. \quad (8)$$

Using AM-GM, we can make the following observations:

$$a^2+b^2+c^2 = \tfrac{a^2+b^2}{2}+\tfrac{b^2+c^2}{2}+\tfrac{c^2+a^2}{2} \geqslant \sqrt{a^2b^2}+\sqrt{b^2c^2}+\sqrt{c^2a^2} = ab+bc+ca;$$

$$\begin{aligned} a^2+b^2+c^2 &= \tfrac{a^2+a^2+a^2+a^2+b^2+c^2}{6}+\tfrac{b^2+b^2+b^2+b^2+c^2+a^2}{6}+\tfrac{c^2+c^2+c^2+c^2+a^2+b^2}{6} \\ &\geqslant \sqrt[6]{a^8b^2c^2}+\sqrt[6]{b^8c^2a^2}+\sqrt[6]{c^8a^2b^2} = a^{\frac{4}{3}}b^{\frac{1}{3}}c^{\frac{1}{3}}+b^{\frac{4}{3}}c^{\frac{1}{3}}a^{\frac{1}{3}}+c^{\frac{4}{3}}a^{\frac{1}{3}}b^{\frac{1}{3}} \\ &= (abc)^{\frac{1}{3}}(a+b+c); \end{aligned}$$

$$a^2c+b^2a+c^2b = 3\cdot\tfrac{a^2c+b^2a+c^2b}{3} \geqslant 3\sqrt[3]{a^3b^3c^3} = 3abc.$$

Taking into account that $abc=1$, we obtain

$$\begin{aligned} &2a^2+2b^2+2c^2+2a^2c+2b^2a+2c^2b \\ &\quad = \left(a^2+b^2+c^2\right)+\left(a^2+b^2+c^2\right)+\left(a^2c+b^2a+c^2b\right)+\left(a^2c+b^2a+c^2b\right) \\ &\quad \geqslant (ab+bc+ca)+(abc)^{\frac{1}{3}}(a+b+c)+3abc+3abc \\ &\quad = 3+a+b+c+ab+bc+ca+3abc. \end{aligned}$$

This proves (8), whence the desired result follows.

*Remark:* Instead of the second out of the three observations made in Solution 1, one can prove the inequality $a^2+b^2+c^2 \geqslant a+b+c$ using the assumption $abc=1$ as follows: Apply AM-GM to obtain $a^2+b^2+c^2 = 3\cdot\frac{a^2+b^2+c^2}{3} \geqslant 3\sqrt[3]{a^2b^2c^2}=3$ and AM-HM to get $\sqrt{\frac{a^2+b^2+c^2}{3}} \geqslant \frac{a+b+c}{3}$; then $a^2+b^2+c^2 = \sqrt{(a^2+b^2+c^2)(a^2+b^2+c^2)} \geqslant \sqrt{3(a^2+b^2+c^2)} \geqslant a+b+c$. The rest is analogous to Solution 1.

*Solution 2:* The given inequality is equivalent to $\frac{a^2}{a(1+b)}+\frac{b^2}{b(1+c)}+\frac{c^2}{c(1+a)} \geqslant \frac{3}{2}$. Applying the so-called Titu's lemma to $(a,b,c)$ and $\left(\frac{1}{a(1+b)},\frac{1}{b(1+c)},\frac{1}{c(1+a)}\right)$ gives $\frac{a^2}{a(1+b)}+\frac{b^2}{b(1+c)}+\frac{c^2}{c(1+a)} \geqslant \frac{(a+b+c)^2}{a(1+b)+b(1+c)+c(1+a)} = \frac{(a+b+c)^2}{(a+b+c)+(ab+bc+ca)}$. Hence it suffices to prove the inequality $\frac{(a+b+c)^2}{(a+b+c)+(ab+bc+ca)} \geqslant \frac{3}{2}$; we prove the equivalent inequality $\frac{1}{a+b+c}+\frac{ab+bc+ca}{(a+b+c)^2} \leqslant \frac{2}{3}$. By AM-GM, $a+b+c \geqslant 3\sqrt[3]{abc}=3$, implying $\frac{1}{a+b+c} \leqslant \frac{1}{3}$. Hence it suffices to prove the inequality $\frac{ab+bc+ca}{(a+b+c)^2} \leqslant \frac{1}{3}$. By removing parentheses and collecting similar terms, the latter reduces to $a^2+b^2+c^2 \geqslant ab+bc+ca$, which can be established by applying AM-GM repeatedly (similarly to Solution 1).

**S5** Find all polynomials $P(x,y)$ with real coefficients which for all real numbers $x$ and $y$ satisfy $P(x+y,x-y)=2P(x,y)$.

*Answer:* $P(x,y)=(a+b)x^2+axy+by^2$, where $a,b$ are any real numbers.

*Solution:* By the given equality,

$$P(2x,2y)=P((x+y)+(x-y),(x+y)-(x-y))=2P(x+y,x-y)=4P(x,y). \quad (9)$$

As (9) holds for arbitrary real numbers $x$ and $y$, we can consider it as an equality of polynomials. Hence the corresponding coefficients are equal. Let $i$ and $j$ be arbitrary non-negative integers. The coefficient of $x^iy^j$ in the l.h.s. of (9) is $a_{i,j}2^i2^j$ or, equivalently, $2^{i+j}a_{i,j}$, whereas the coefficient of the $x^iy^j$ in the r.h.s. of (9) is $4a_{i,j}$. Hence $2^{i+j}a_{i,j}=4a_{i,j}$ or, equivalently, $\left(2^{i+j}-4\right)a_{i,j}=0$, which provides us two possibilities: either $i+j=2$ or $a_{i,j}=0$. Consequently, the only terms in the polynomial $P(x,y)$ have exponents of variables that sum up to 2. In other words, $P(x,y)=cx^2+axy+by^2$ for some coefficients $a,b,c$. After substituting into the initial equality and collecting similar terms in both sides, we obtain

$$(c+a+b)x^2+2(c-b)xy+(c-a+b)y^2=2cx^2+2axy+2by^2.$$

Again, the coefficients in the l.h.s. and r.h.s. must be equal. From the coefficients of $x^2$, we get $c+a+b=2c$ which implies $c=a+b$. This makes also the corresponding coefficients of the other terms equal. Consequently, all polynomials of the form $P(x,y)=(a+b)x^2+axy+by^2$ where $a$ and $b$ are arbitrary real numbers satisfy the conditions of the problem.