Estonian Math Competitions 2021/2022

University of Tartu Youth Academy
Tartu 2022

WE THANK:

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Estonian Mathematical Olympiad

https://www.math.olympiaadid.ut.ee/

Mathematics Contests in Estonia

The Estonian Mathematical Olympiad is held annually in three rounds: at the school, town/regional and national levels. The best students of each round (except the final) are invited to participate in the next round.

In each round of the Olympiad, separate problem sets are given to the students of each grade from the 7th to the 12th. This year, the final round was organized also to grades 7 and 8; previously, these two grades participated at school and regional levels only. About 25 students of each grade reach the final round. Some towns, regions and schools organize math olympiads for even younger students. The school round usually takes place in December, the regional round in January or February and the final round in spring in Tartu. The problems for every grade are usually in compliance with the school curriculum of that grade but, in the final round, also problems requiring additional knowledge may be given.

The first problem solving contest in Estonia took place in 1950. The next one, which was held in 1954, is considered as the first Estonian Mathematical Olympiad.

In addition to the Olympiad, open contests take place in September and in December. In addition to students of Estonian middle and secondary schools, Estonian citizens who are studying abroad may also participate in these contests. The participants must have never enrolled in a university or other higher educational institution. The contestants compete in two categories: Juniors and Seniors. In the former category, only students up to the 10th grade may participate. Being successful in the open contests generally assumes knowledge outside the school curriculum.

Based on the results of all competitions during the year, about 20 IMO team candidates are selected. The IMO team selection contest for them is held in April or May in two rounds. Each round is an IMO-style two-day competition with 4.5 hours to solve 3 problems on both days. Some problems in our selection contest are at the level of difficulty of the IMO but easier problems are usually also included.

The problems of previous competitions can be downloaded at the Estonian Mathematical Olympiads website.

Problems Listed by Topic

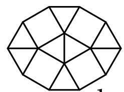
Number theory: O1, O3, O7, O9, O14, O19, F3, F4, F7, F8, F12, F17, F22, F26, S2, S3, S5
Algebra: O4, O10, O15, O20, F5, F13, F18, F23, S1
Geometry: O5, O8, O11, O16, O18, O21, F1, F6, F9, F11, F14, F16, F19, F24
Discrete mathematics: O2, O6, O12, O17, O22, F2, F10, F15, F20, F21, F25, S4

Problems

Selected Problems from Open Contests

O1 *(Juniors.)* Does there exist a positive integer whose
(a) digit sum is 100 more than the product of digits?
(b) product of digits is 100 more than the digit sum?
(c) product of digits is 100 times greater than the digit sum?

O2 *(Juniors.)* We have a board consisting of triangular and square tiles (see figure). Is it possible to start from a tile, travel through all tiles exactly once and return to the starting tile, if



(a) in each step we can travel from a tile to another tile sharing an edge with the previous tile?
(b) in each step we can travel from a tile to another tile sharing a corner, but not an edge with the previous tile?
(c) in each step we can travel from a tile to another tile sharing a corner or an edge with the previous tile?

O3 *(Juniors.)* Does there exist a positive integer n such that across all representations $n = ab$, the digits $0, 1, \ldots, 9$ are all present as the final digit of a^b at least once?

O4 *(Juniors.)* Find the value of

$$\frac{1 \cdot 3}{3 \cdot 5} + \frac{2 \cdot 4}{5 \cdot 7} + \frac{3 \cdot 5}{7 \cdot 9} + \frac{4 \cdot 6}{9 \cdot 11} + \ldots + \frac{1009 \cdot 1011}{2019 \cdot 2021}.$$

O5 *(Juniors.)* Let ω be a circle with center O and diameter AB. A circle with center B intersects ω at C and AB at D. The line CD intersects ω at the point E ($E \neq C$). The intersection of lines OE and BC is F.
(a) Prove that the triangle OBF is isosceles.
(b) Find the ratio $\frac{FB}{BD}$, given that D is the midpoint of OB.

O6 *(Juniors.)* A *field* is a 2020×2021 grid with a positive integer written into each cell, such that no number repeats in any row or column. An *onion* consists of 4 consecutive cells in a row or column, whose numbers add up to exactly $4 \cdot 2021$. Find the largest possible number of onions on the field.
Remark: Different onions may share some cells.

O7 *(Juniors.)* Find all two-digit positive integers whose sum of digits and product of digits add up to the number itself.

O8 *(Juniors.)* Let ABC be a triangle where $AB = AC$. Let E be the foot of its altitude from the vertex B. Given that either BEA or BEC is an isosceles triangle, find all possibilities of the size of the angle at the vertex A of the triangle ABC.

O9 *(Juniors.)* Do there exist integers x and y such that:
(a) $x^2 + (x+1)^2 + (x+2)^2 = y^2$?
(b) $x^2 + (x+1)^2 + (x+2)^2 + (x+3)^2 = y^2$?

O10 *(Juniors.)* Mari chooses five distinct positive integers not greater than 2021. From these five numbers, it must be possible to choose two numbers with sum 1919 in two different ways. Likewise, from these five numbers, it must be possible to choose two numbers with sum 2929 in two different ways. Find all possibilities of which five numbers Mari may choose.

Remark: Choices of two numbers which only differ by the order of the numbers are not considered different.

O11 *(Juniors.)* The largest angle of triangle ABC is located at its vertex C. Let ϱ be the circle with centre A and radius AC, and let σ be the circle with centre B and radius BC. The circle σ intersects the circumcircle of the triangle ABC and the circle ϱ at points D and F, respectively ($D \neq C$, $F \neq C$). Prove that A, D and F are collinear.

O12 *(Juniors.)* There are 25 green, 20 brown and 15 orange chameleons in a zoo. Whenever exactly two chameleons of distinct colours meet, both change their colour to the third one. Otherwise, the chameleons do not change their colours. Is it possible that:

(a) at some time instant, we have the same number of chameleons of each colour?

(b) at some time instant, all chameleons are of the same colour?

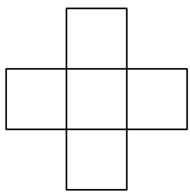
O13 *(Seniors.)* On the first line of a notebook Juku writes the number 43. On every following line he writes the number $x^2 - 66x + 1122$, where x is the number on the previous line. Find the number that Juku will write on the 2021st line.

O14 *(Seniors.)* We call a prime number p *cute* if there exists a prime q so that both $pq - 2$ and $pq + 2$ are also primes. We call p *wonderful* if both p and $p + 2$ are cute primes. Find all wonderful numbers.

O15 *(Seniors.)* A function $f : \mathbb{R} \to \mathbb{R}$ satisfies for all real x and y the equation $2f(x)\left(f(y)\right)^2 + y^2 f\left(-x|y|\right) = f\left(xy^2\right)$. Find all possible values of $f(1)$.

O16 *(Seniors.)* Let H be the orthocenter of an acute triangle ABC. Let M be the midpoint of BC. Let A' and H' be the reflections of the points A and H across the point M. Prove that the points B and C and the reflections of A' over the lines BH' and CH' are concyclic.

O17 *(Seniors.)* Let $m, n > 2$ be positive integers. In each cell of an $m \times n$ grid there is a lamp, which can either be turned on or off. In one switch, we can change the state of five lamps, which are placed in cells forming a cross (see the diagram). Initially all the lamps are turned off. How many possible arrangements of lamps can be formed with these switches? Arrangements obtained from each other by rotations and reflections are considered different.



O18 *(Seniors.)* Mother wants to divide a cake of triangular shape between three kids. She makes a straight cut from one vertex to the midpoint of the opposite side and then another straight cut from another vertex to the midpoint of the opposite side. She gives the piece of quadrilateral shape to Anna, the triangular piece opposite to it to Berta and the remaining two triangular pieces to Clara. Who gets the largest part of cake?

O19 *(Seniors.)* Find all pairs of integers (x, y) that satisfy the equation $y^4 = x(2x^2 + y)^3$.

O20 *(Seniors.)* Find the least possible sum of 2021 terms of the sequence $a_1, a_2, a_3, \ldots$, where $a_1 = 0$, $a_2 = a_3 = 1$ and $a_{i+j} > a_i + a_j$ for every $i, j \geqslant 2$.

O21 *(Seniors.)* Triangle ABC satisfies $AB = AC$. Medians AD and BE intersect at G. Let P be the midpoint of the line segment GE.

(a) Prove that if $GP = GD$ then the quadrilateral $CEPD$ is cyclic.

(b) Does it hold that if the quadrilateral $CEPD$ is cyclic then $GP = GD$?

O22 *(Seniors.)* Little Juku writes all integers from 1 to n on a blackboard, but as he does not know the digit 4 yet, he skips all numbers that contain 4. Juku's sister Mari erases two numbers on the blackboard and writes the absolute value of the difference of these numbers on the blackboard. Then Mari again erases two numbers on the blackboard and writes the absolute value of their difference on the blackboard, etc. Can it happen after a finite number of such steps that there are all integers from 1 to n that contain the digit 4 and only these on the blackboard, each one exactly once, if

(a) $n = 2021$;

(b) $n = 10000$?

Selected Problems from the Final Round of National Olympiad

F1 *(Grade 7.)* Points A, B and C are chosen in a rectangle of shape 7×10 in such a way that the distances from A to some three sides of the rectangle are 2, 3 and 4, the distances from B to some three sides of the rectangle are 3, 4 and 5, and the distances from C to some three sides of the rectangle are 4, 5 and 6. Find the largest possible area the triangle ABC can have.

F2 *(Grade 7.)* Juku has four cans of juice: A 1-litre can containing $\frac{1}{2}$ litres of juice, a $\frac{1}{2}$-litre can containing $\frac{1}{3}$ litres of juice, a $\frac{1}{3}$-litre can containing $\frac{1}{4}$ litres of juice and a $\frac{1}{4}$-litre can containing $\frac{1}{5}$ litres of juice. There are no volume markings on the cans. Juku wants to measure exactly $\frac{1}{30}$ litres of juice by a sequence of pourings of the juice from one can to another. Letting the juice spill is not allowed. Find out all cans into which Juku can get the desired exact amount of juice.

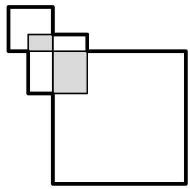
F3 *(Grade 7.)* Priit's collection consists of 1111 stamps which are all distributed into envelopes in such a way that every envelope contains more than one stamp, all envelopes contain the same number of stamps, and each envelope contains only stamps from one country. It is known that more than 40% of stamps in this collection are from Estonia, more than 30% of stamps in the collection are from Latvia and more than 20% of stamps in the collection are from Lithuania. Find the largest possible number of envelopes containing Estonian stamps and the largest possible number of envelopes containing Lithuanian stamps.

F4 *(Grade 8.)* We call positive integers n, m an *interesting pair* if $n < m$ and the greatest prime factor of n is equal to the greatest prime factor of m.

(a) For an interesting pair n, m, will there always exist a prime p such that $n < p < m$?

(b) Among the first 25 positive integers, how many don't form an interesting pair with any smaller positive integer?

F5 *(Grade 8.)* A dodecagon with perimeter 72 cm is constructed from three squares as shown in the figure. The two outer squares have a common vertex and both share a rectangular part with the central square, such that the perimeter of the shared part is 5 times less than the sum of the perimeters of the central square and the corresponding outer square. Find the side length of the central square.



F6 *(Grade 8.)* On the side AB of a triangle ABC, let D be a point such that $\angle BDC = \angle ACB$. Let K be the midpoint of CD and let E be the intersection of lines BK and AC. Given that $\angle BKD = 2\angle BCD$, find $\angle AEB$.

F7 *(Grade 8.)* There are three consecutive positive integers on a blackboard. A move consists of erasing the smallest number on the blackboard and replacing it by the sum of itself and the greatest number on the blackboard. Is it possible that the sum of the numbers on the blackboard will be a power of 10:

(a) after the 6th move?

(b) after the 8th move?

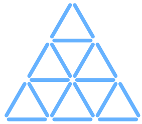
F8 *(Grade 9.)* Two positive integers together contain each digit $0, 1, \ldots, 9$ exactly once. Find the largest possible common divisor that these two numbers can have.

F9 *(Grade 9.)* The bisector of the internal angle at the vertex B of a triangle ABC intersects the circumcircle of the triangle ABC at a point P ($P \neq B$). The line through P perpendicular to the line AC intersects the circumcircle of the triangle ABC at a point P' ($P' \neq P$). Prove that the quadrilateral $APCP'$ is a square if and only if $\angle ABC = 90°$.

F10 *(Grade 9.)* There are sticks of length 1 with a number 1, 2 or 3 written on each of them. There is an unlimited supply of sticks with every number. Two triangles consisting of three sticks are considered different if neither of the triangles can be composed from sticks of the other triangle.

(a) How many different triangles consisting of three sticks are possible?

(b) From 18 sticks, one makes an equilateral triangle of side length 3, divided into 9 pairwise different equilateral triangles of side length 1. Find the largest possible sum of the numbers written on the 9 sticks on the boundary of the big triangle.



F11 *(Grade 9.)* A point M is chosen on the side AC of a triangle ABC and a point K is chosen on the line segment BM so that $AM = \frac{1}{3}AC$ and $BK = \frac{1}{4}BM$. Let N be the intersection of the line AK and the side BC. What percentage of the area of the triangle ABC is the area of the quadrilateral $MKNC$?

F12 *(Grade 10.)* When adding together positive integers a and b, Juku forgot to enter the final digit 7 of the number a and got a result of 2022. Had Juku instead forgotten to enter the final digit of b, the result would have been 5000. Find the sum of a and b.

F13 *(Grade 10.)* Find the sum

$$\sqrt{1+\frac{1}{1^2}+\frac{1}{2^2}}+\sqrt{1+\frac{1}{2^2}+\frac{1}{3^2}}+\ldots+\sqrt{1+\frac{1}{2021^2}+\frac{1}{2022^2}}.$$

F14 *(Grade 10.)* Points A, D, E and C lie on a line in this order. Point B is chosen such that triangles ADB and BEC are similar (in this specific order of vertices), moreover $EC = 2AD$ and $\angle ABC = 120°$. Find $\frac{AC}{AD}$.

F15 *(Grade 10.)* A fly farm contains 100001 fruit flies. A research group wishes to buy some flies from the farm to perform (once) either experiment A or experiment B (but not both).

For experiment A, the research group needs a set of flies in which no fly is a descendant of any other. For such a set, the research group would pay the farm 505 euros plus 5 euros per fly bought.

For experiment B, the research group needs a set of flies in which among every two flies, one is a descendant of the other. For such a set, the research group would pay the farm 1000 euros plus 10 euros per fly bought.

Prove that there exists a set of flies in the farm for which the research group would pay at least 3010 euros.

Remark: Any fly can have up to 2 parents.

F16 *(Grade 10.)* The radius of the circumcircle of an acute triangle ABC is R and its orthocenter is H. Show that $AH^2 + BC^2 = 4R^2$.

F17 *(Grade 11.)* Find all polynomials $P(x)$ with integral coefficients and the following property: for any pair (u, v) of positive integers, $\gcd(u, v) = 1$ implies $\gcd(|P(u)|, |P(v)|) = 1$.

F18 *(Grade 11.)* Find all triples (x, y, z) of real numbers that satisfy

$$\begin{cases} \frac{x}{y} + \frac{y}{z} + xy = 3, \\ \frac{y}{z} + \frac{z}{x} + yz = 3, \\ \frac{z}{x} + \frac{x}{y} + zx = 3. \end{cases}$$

F19 *(Grade 11.)* The orthocenter and the circumcenter of a non-equilateral triangle ABC are H and O, respectively. Let D be the foot of the altitude dropped from the vertex A of the triangle ABC. Prove that $\angle AHO = 90^\circ$ if and only if $\frac{AH}{HD} = 2$.

F20 *(Grade 11.)* A rectangle of integral side lengths is divided into 2022 unit squares. At least one unit square is coloured black. There are equally many black squares in every row and also equally many black squares in every column. Find all possibilities of how many black unit squares there can be in total.

F21 *(Grade 11.)* The teacher writes the digits 20212022 in a row on a blackboard. Juku must write each arithmetic operator $(+, -, \cdot, :)$ exactly once somewhere between these digits in such a way that the result is a correct mathematical expression with a real value, and find this value.

(a) Can Juku obtain the number 0 as the value of the expression?

(b) If the teacher allowed Juku to use parentheses, could Juku obtain expressions with larger values than it would be possible without parentheses?

(c) Prove that there exists a positive integer less than 1000 that cannot be obtained (without using parentheses) as the value of the expression.

F22 *(Grade 12.)* There are some distinct positive integers written on a blackboard. If we erase the smallest number written on the blackboard, then the ratio of the sum and the product of the remaining numbers will be 4 times greater than the ratio of the sum and the product of the numbers initially on the blackboard. Find all possibilities for the set of numbers that could have been on the blackboard initially.

F23 *(Grade 12.)* A function f maps every positive real number to a positive real number. There is a constant $c \neq 1$ such that $f(cx)^2 = f(x)f\left(c^2x\right)$ for all positive real numbers x. Must the same equality hold for any positive real numbers c and x?

F24 *(Grade 12.)* A triangle with perimeter P is divided into k triangular pieces for some $k \geqslant 2$.

(a) Show that there exists a piece with perimeter greater than $\frac{P}{k}$.

(b) Show that if the initial triangle is equilateral, then there exists a piece with perimeter at least $\frac{P}{\sqrt{k}}$.

F25 *(Grade 12.)* Let $n \geqslant 2$ be a positive integer and let $S = \{1, 2, \ldots, n\}$.

For $k = 1, 2, \dots, n-1$, we call two k-element subsets of S *neighbours*, if they have $k-1$ elements in common (i.e. differ by exactly one element). Let $f(n,k)$ be the size of the largest possible collection of k-element subsets of S, in which no two subsets are neighbours. Prove that $f(n,k) \leqslant \binom{n-1}{k-1}$.

F26 *(Grade 12.)* Show that there exist infinitely many positive integers n such that the integers $1, 2, 3, \dots, 2n$ can be split into pairs such that the sum of the products of the pairs is divisible by $2n$.

Selected Problems from the IMO Team Selection Contests

S1 Determine all functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy the inequality $f(x) + f(x+y) \leqslant f(xy) + f(y)$ for all real numbers x, y.

S2 Let p be a fixed prime number. Juku and Miku play the following game. One of the players chooses a natural number a such that $a > 1$ and a is not divisible by p, his opponent chooses any natural number n such that $n > 1$. Miku wins if the natural number written as n ones in the positional numeral system with radix a is divisible by p, otherwise Juku wins. Which player has a winning strategy if:
(a) Juku chooses the number a, tells it to Miku and then Miku chooses the number n;
(b) Juku chooses the number n, tells it to Miku and then Miku chooses the number a?

S3 (a) Is it true that, for arbitrary integer n greater than 1 and distinct positive integers i and j not greater than n, the set of any n consecutive integers contains distinct numbers i' and j' whose product $i'j'$ is divisible by the product ij?
(b) Is it true that, for arbitrary integer n greater than 2 and distinct positive integers i, j, k not greater than n, the set of any n consecutive integers contains distinct numbers i', j', k' whose product $i'j'k'$ is divisible by the product ijk?

S4 For any non-negative integer i, denote by d_i the first digit of the number 2^i. Let n be a positive integer. Prove that there exists a non-zero digit that occurs in the tuple $(d_0, d_1, \dots, d_{n-1})$ less than $\frac{n}{17}$ times.

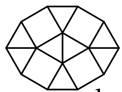
S5 Determine all triples (a, b, c) of integers which satisfy the equation $(a-b)^3(a+b)^2 = c^2 + 2(a-b) + 1$.

Problems with Solutions

Selected Problems from Open Contests

O1 *(Juniors.)* Does there exist a positive integer whose
(a) digit sum is 100 more than the product of digits?
(b) product of digits is 100 more than the digit sum?
(c) product of digits is 100 times greater than the digit sum?
Answer: (a) Yes; (b) Yes; (c) Yes.
Solution:
(a) The number $111\ldots1$, consisting of 101 ones, works. Its product of digits is 1 and digit sum 101.
(b) The number 1111112345 works. Its product of digits is 120 and digit sum 20.
(c) The number 1225555 has product of digits 2500 and digit sum 25.

O2 *(Juniors.)* We have a board consisting of triangular and square tiles (see figure). Is it possible to start from a tile, travel through all tiles exactly once and return to the starting tile, if



(a) in each step we can travel from a tile to another tile sharing an edge with the previous tile?
(b) in each step we can travel from a tile to another tile sharing a corner, but not an edge with the previous tile?
(c) in each step we can travel from a tile to another tile sharing a corner or an edge with the previous tile?
Answer: (a) No; (b) No; (c) Yes.
Solution:
(a) From the triangular tiles on the edge of the board it's only possible to move to adjacent tiles also on the edge of the board. Drawing these obligatory segments, we have formed a closed loop, but not visited the central two tiles. Therefore the desired journey is impossible.
(b) From the triangular tiles on the top and bottom of the board it's only possible to move to the central two triangular tiles. Drawing these obligatory segments, we have formed a closed loop, but not visited most of the tiles. Therefore the desired journey is impossible.
(c) One possible path is drawn in Fig. 1 with a blue line.

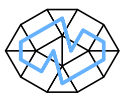


Fig. 1

O3 *(Juniors.)* Does there exist a positive integer n such that across all representations $n = ab$, the digits $0, 1, \ldots, 9$ are all present as the final digit of a^b at least once?

Answer: No.
Solution: If n is odd, both a and b will always be odd, but then a^b will never end with a 0. If n is even, then a or b will be even, so a^b will be even or a perfect square. In either case, it will never end in a 7.

O4 *(Juniors.)* Find the value of

$$\frac{1\cdot 3}{3\cdot 5}+\frac{2\cdot 4}{5\cdot 7}+\frac{3\cdot 5}{7\cdot 9}+\frac{4\cdot 6}{9\cdot 11}+\ldots+\frac{1009\cdot 1011}{2019\cdot 2021}.$$

Answer: $\frac{505\cdot 1009}{2021}=\frac{509545}{2021}=252+\frac{253}{2021}$.
Solution: The sum consists of 1009 terms, where the i-th term is of the form $\frac{i(i+2)}{(2i+1)(2i+3)}$. Let s be the desired sum. Notice that

$$\frac{i(i+2)}{(2i+1)(2i+3)}=\frac{1}{4}-\frac{3}{4}\cdot\frac{1}{(2i+1)(2i+3)}=\frac{1}{4}-\frac{3}{8}\cdot\left(\frac{1}{2i+1}-\frac{1}{2i+3}\right).$$

Therefore

$$\begin{aligned} s &= 1009\cdot\frac{1}{4}-\frac{3}{8}\cdot\left(\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{7}\right)+\ldots+\left(\frac{1}{2019}-\frac{1}{2021}\right)\right)\\ &= \frac{1009}{4}-\frac{3}{8}\cdot\left(\frac{1}{3}-\frac{1}{2021}\right)=\frac{1009}{4}-\frac{3}{8}\cdot\frac{2021-3}{3\cdot 2021}=\frac{1009}{4}-\frac{2018}{8\cdot 2021}\\ &= \frac{1009}{4}-\frac{1009}{4\cdot 2021}=\frac{1009}{4}\cdot\left(1-\frac{1}{2021}\right)=\frac{1009}{4}\cdot\frac{2020}{2021}=\frac{505\cdot 1009}{2021}=\frac{509545}{2021}. \end{aligned}$$

O5 *(Juniors.)* Let ω be a circle with center O and diameter AB. A circle with center B intersects ω at C and AB at D. The line CD intersects ω at the point E ($E\neq C$). The intersection of lines OE and BC is F.
(a) Prove that the triangle OBF is isosceles.
(b) Find the ratio $\frac{FB}{BD}$, given that D is the midpoint of OB.
Answer: (b) 4.
Solution:
(a) Let $\angle BCD=\alpha$, then the equal radii $BC=BD$ give us $\angle BDC=\alpha$ (Fig. 2) and $\angle CBD=180^\circ-2\alpha$. Then $\angle OBF=\angle CBD=180^\circ-2\alpha$. But on the other hand $\angle BOE=2\angle BCE=2\angle BCD=2\alpha$ and therefore $\angle BOF=180^\circ-2\alpha$. Since $\angle OBF=\angle BOF$, the triangle OBF is isosceles.

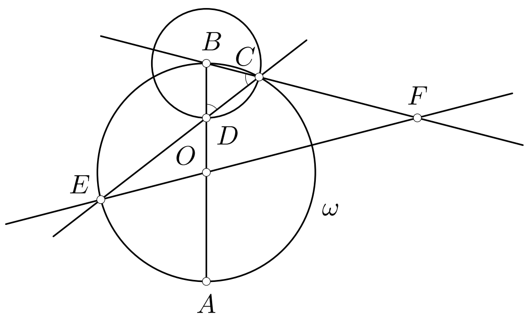


Fig. 2

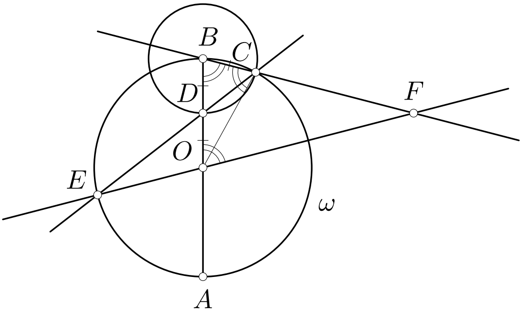


Fig. 3

(b) Due to equal radii the triangle CBO is isosceles (Fig. 3). Furthermore, it is similar to OBF as they share the angle at B. We have $OB = 2BD = 2BC$, so by the similarity also $FB = 2OB$. Therefore $FB = 4BD$ and $\frac{FB}{BD} = 4$.

O6 *(Juniors.)* A *field* is a 2020×2021 grid with a positive integer written into each cell, such that no number repeats in any row or column. An *onion* consists of 4 consecutive cells in a row or column, whose numbers add up to exactly $4 \cdot 2021$. Find the largest possible number of onions on the field.
Remark: Different onions may share some cells.
Answer: $1009 \cdot 4041$.
Solution: In a column of length 2020 there are 2017 potential onions. However there cannot be two consecutive onions, as they share 3 cells, meaning their fourth numbers would have to be equal. So a column can have at most 1009 onions. Analogously a row of length 2021 can also have at most 1009 onions. As there are $2020 + 2021 = 4041$ rows and columns in total, there can be at most $1009 \cdot 4041$ onions.
We will show there exists a field with $1009 \cdot 4041$ onions. We will first construct a 2020×2020 grid, choosing the first two rows to be

$$\begin{array}{c} 1,\ -1,\quad 2,\ -2,\quad 3,\ -3,\ \ldots,\ \ldots,\quad 1009,\ -1009,\quad 1010,\ -1010, \\ -1,\quad 1,\ -2,\quad 2,\ -3,\quad 3,\ \ldots,\ \ldots,\ -1009,\quad 1009,\ -1010,\quad 1010. \end{array}$$

The next two rows will be shifted the cells to the left:

$$\begin{array}{c} 2,\ -2,\quad 3,\ -3,\ \ldots,\ \ldots,\quad 1009,\ -1009,\quad 1010,\ -1010,\quad 1,\ -1, \\ -2,\quad 2,\ -3,\quad 3,\ \ldots,\ \ldots,\ -1009,\quad 1009,\ -1010,\quad 1010,\ -1,\quad 1. \end{array}$$

Similarly we construct the other rows. Finally we add the rightmost column consisting of the numbers

$$1011, -1011, 1012, -1012, \ldots, 2019, -2019, 2020, -2020.$$

We obtain the desired field by adding $2 \cdot 2021$ to all negative numbers in the grid.

O7 *(Juniors.)* Find all two-digit positive integers whose sum of digits and product of digits add up to the number itself.

Answer: $19, 29, 39, 49, 59, 69, 79, 89, 99$.

Solution: Let the number be $10A + B$ where A and B are its digits. The conditions of the problem imply $A + B + AB = 10A + B$, where collecting similar terms gives $AB = 9A$. As $A \neq 0$, we can reduce by A to obtain $B = 9$. Hence all numbers $19, 29, 39, 49, 59, 69, 79, 89, 99$ are possible.

O8 *(Juniors.)* Let ABC be a triangle where $AB = AC$. Let E be the foot of its altitude from the vertex B. Given that either BEA or BEC is an isosceles triangle, find all possibilities of the size of the angle at the vertex A of the triangle ABC.

Answer: $45°, 90°, 135°$.

Solution 1: Let $\alpha = \angle BAC$. Both triangles BEA and BEC have right angle at vertex E. Hence these triangles can be isosceles only if their other angles have size $45°$.

Suppose that BEC is isosceles (Fig. 4). Then $\angle BCE = 45°$. Since $\angle BCE = \angle BCA = \frac{180° - \alpha}{2}$, we have $\alpha = 180° - 2 \cdot 45° = 90°$.

Suppose now that BEA is isosceles. Then $\angle BAE = 45°$. If E lies on the line segment AC (Fig. 5) then $\angle BAE = \angle BAC = \alpha$, implying $\alpha = 45°$. If E lies outside the line segment AC (Fig. 6) then $\angle BAE = 180° - \angle BAC = 180° - \alpha$, implying $\alpha = 180° - 45° = 135°$.

Solution 2: Let $\alpha = \angle BAC$. Then $\angle ABC = \angle ACB = \frac{180° - \alpha}{2}$ and $\angle EBC = 90° - \frac{180° - \alpha}{2} = \frac{\alpha}{2}$. Both triangles BEA and BEC have right angle at vertex E. Hence these triangles can be isosceles only if their other two angles have equal size.

Suppose that BEC is isosceles. Then $\frac{180° - \alpha}{2} = \frac{\alpha}{2}$, implying $\alpha = 90°$.

Suppose now that BEA is isosceles. If the triangle ABC is acute, we must have $\angle BAE = \alpha$ and $\angle ABE = \frac{180° - \alpha}{2} - \frac{\alpha}{2} = 90° - \alpha$. Thus $\alpha = 90° - \alpha$, implying $\alpha = 45°$. If the triangle ABC is obtuse then $\angle BAE = 180° - \alpha$ and $\angle ABE = \frac{\alpha}{2} - \frac{180° - \alpha}{2} = \alpha - 90°$. Hence $180° - \alpha = \alpha - 90°$, implying $\alpha = 135°$. The triangle ABC is not right as, otherwise, the triangle BEA would have two right angles.

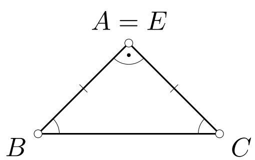


Fig. 4

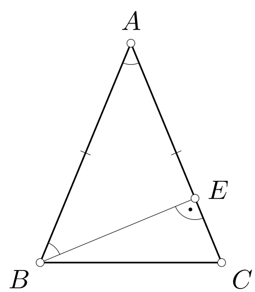


Fig. 5

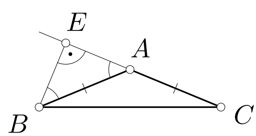


Fig. 6

O9 *(Juniors.)* Do there exist integers x and y such that:
(a) $x^2 + (x+1)^2 + (x+2)^2 = y^2$?
(b) $x^2 + (x+1)^2 + (x+2)^2 + (x+3)^2 = y^2$?
Answer: (a) No; (b) No.
Solution:
(a) The integers x, $x+1$, $x+2$ are congruent to 0, 1, 2 modulo 3 in some order. The squares of these integers are congruent to 0, 1, 1 modulo 3, respectively. Hence the l.h.s. of the equation is congruent to 2 while the r.h.s. is congruent to 0 or 1 modulo 3. Thus the equality cannot hold.
(b) Among the integers x, $x+1$, $x+2$, $x+3$, there are two even numbers and two odd numbers. The squares of even numbers are divisible by 4 while the squares of odd numbers are congruent to 1 modulo 4. Thus the l.h.s. of the equation is congruent to 2 while the r.h.s. is congruent to 0 or 1 modulo 4. Hence the equality cannot hold.

O10 *(Juniors.)* Mari chooses five distinct positive integers not greater than 2021. From these five numbers, it must be possible to choose two numbers with sum 1919 in two different ways. Likewise, from these five numbers, it must be possible to choose two numbers with sum 2929 in two different ways. Find all possibilities of which five numbers Mari may choose.
Remark: Choices of two numbers which only differ by the order of the numbers are not considered different.
Answer: $1, 908, 1011, 1918, 2021$ is the only possibility.
Solution 1: Let $(a, 1919-a)$ and $(b, 1919-b)$ be the two pairs of numbers with sum 1919. If these sums had a common addend then both addends would be the same whence the choices of two numbers would not be different. Thus a, b, $1919-a$ and $1919-b$ are pairwise distinct. Let x be the fifth chosen number; then the total sum of chosen numbers is $3838+x$.
Similarly, we see that the five numbers must be representable as c, $2929-c$, d, $2929-d$, y; their total sum is $5858+y$. From $3838+x = 5858+y$, one gets $x-y = 2020$. The only possibility for satisfying the latter equality is $x = 2021$, $y = 1$. Hence one of a, b, $1919-a$, $1919-b$ equals 1 and one of c, d, $2929-c$, $2929-d$ is 2021; w.l.o.g., $a = 1$ and $c = 2021$. Then the set of chosen numbers contains also $1919-1 = 1918$ and $2929-2021 = 908$. W.l.o.g., $b = 908$. The last chosen number must be $1919-908 = 1011$.
Solution 2: Let the chosen numbers be a, b, c, d, e. W.l.o.g., $a+b = 1919$. If the other way to obtain 1919 as the sum of two chosen numbers used either a or b as addend then the other addend would have to be b or a, respectively, which is not allowed. Hence, w.l.o.g., $c+d = 1919$.
Similarly, one must use four distinct numbers to get 2929 as the sum of two chosen numbers in two different ways. If these four numbers were a, b, c, d, then $a+b = c+d = 1919$ implies $a+b+c+d = 3838$ while also $a+b+c+d = 2929+2929 = 5858$. The contradiction shows that one must use the number e in at least one pair of numbers with sum 2929. W.l.o.g.,

$d + e = 2929$. The other pair of numbers with sum 2929 must use exactly two numbers among a, b, c, but $a + b = 1919$. Hence, w.l.o.g., $b + c = 2929$. These equalities together imply $b = 1919 - a$, $c = 2929 - b = 1010 + a$, $d = 1919 - c = 909 - a$ and $e = 2929 - d = 2020 + a$. The latter implies $a = 1$ and $e = 2021$. Substituting $a = 1$ into the other equalities gives $b = 1918$, $c = 1011$ and $d = 908$.

O11 *(Juniors.)* The largest angle of triangle ABC is located at its vertex C. Let ϱ be the circle with centre A and radius AC, and let σ be the circle with centre B and radius BC. The circle σ intersects the circumcircle of the triangle ABC and the circle ϱ at points D and F, respectively ($D \neq C$, $F \neq C$). Prove that A, D and F are collinear.

Solution 1: Denote $\angle BAC = \alpha$ (Figures 7 and 8 present two possible cases). Note that $AC = AF$ and $BC = BF$ as they are radii of circles ϱ and σ, respectively. Thus the triangles ABF are ABC equal by three equal sides. Hence $\angle BAF = \alpha$. From equal radii of σ, we also obtain $BD = BC$. Now by inscribed angles subtending equal chords of the circumcircle of the triangle ABC, we get $\angle BAD = \alpha$. Consequently, points A, D and F lie on one line.

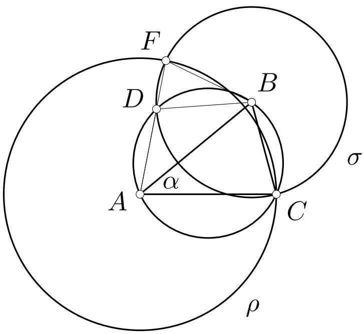


Fig. 7

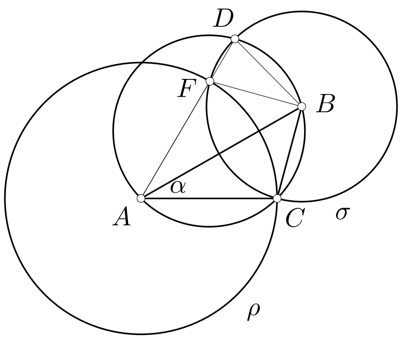


Fig. 8

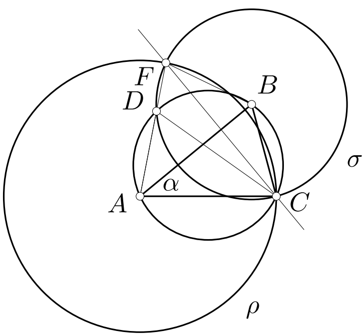


Fig. 9

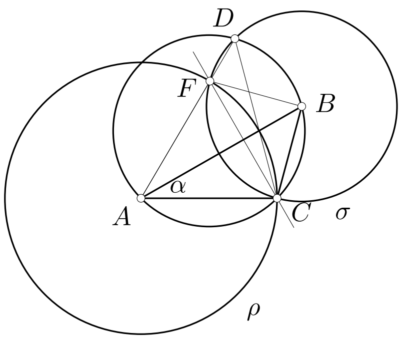


Fig. 10

Solution 2: From the inscribed angles of the circumcircle of the triangle ABC, we get $\angle ADC = \angle ABC$. As in Solution 1, we can show that the triangles ABF and ABC are equal; thus $\angle ABC = \angle ABF = \frac{\angle CBF}{2}$. If D and A lie on the same side of the line CF (Fig. 9) then $\frac{\angle CBF}{2} = 180^\circ - \angle CDF$ by the inscribed angle theorem. If D and A lie on different sides of the line CF (Fig. 10) then, analogously, $\frac{\angle CBF}{2} = \angle CDF$. Altogether, we have either $\angle ADC = 180^\circ - \angle CDF$ while D and A being on different sides of CF or $\angle ADC = \angle CDF$ while D and A being on different sides of CF. In both cases, the obtained equality implies that A, D and F lie on the same line.

O12 *(Juniors.)* There are 25 green, 20 brown and 15 orange chameleons in a zoo. Whenever exactly two chameleons of distinct colours meet, both change their colour to the third one. Otherwise, the chameleons do not change their colours. Is it possible that:

(a) at some time instant, we have the same number of chameleons of each colour?

(b) at some time instant, all chameleons are of the same colour?

Answer: (a) No; (b) No.

Solution 1: After each meet of two chameleons, the numbers of chameleons of two colours decrease by 1 and the number of chameleons of the remaining colour increases by 2. Thus the difference of the numbers of chameleons of each two colours is constant modulo 3. As the remainders modulo 3 of the numbers of chameleons of different colours are pairwise distinct in the beginning, the numbers of chameleons of no two colours can become equal. This also implies that all chameleons cannot be of the same colour since then the numbers of chameleons of two other colours would be equal to 0.

Solution 2:

(a) If there were equally many chameleons of each colour, the common number of chameleons of each colour would be 20. To keep the number of brown chameleons all in all unchanged, there must be 2 decreases by 1 per each increase by 2. Hence the total number of meets must be $3x$ for some integer x. For changing the number of green chameleons from 25 to 20, there must be 5 decreases by 1 and, in addition, 2 decreases by 1 per each increase by 2. Hence the total number of meets must be $5 + 3y$ for some integer y. Thus $3x = 5 + 3y$, but this equation has no integral solutions.

(b) Analogously, if the number of brown chameleons were 60 then the number of meets must be $20 + 3x$ for an integer x; then the number of orange chameleons would be 0, requiring $15 + 3y$ meets for an integer y, but the equation $20 + 3x = 15 + 3y$ has no integral solutions. If the number of brown chameleons were 0 then the number of meets must again be $20 + 3x$. If the number of orange chameleons were also 0 then the number of meets would be $15 + 3y$ for an integer y leading to the same equation; if the number of green chameleons were 0 then we would get the equation $20 + 3x = 25 + 3y$ which also does not have integral solutions.

O13 *(Seniors.)* On the first line of a notebook Juku writes the number 43. On every following line he writes the number $x^2 - 66x + 1122$, where x is the number on the previous line. Find the number that Juku will write on the 2021st line.

Answer: $10^{2^{2020}} + 33$.

Solution: Let x_i be the number written on the ith line, then for all $i = 1, 2, \ldots$ we have $x_{i+1} = x_i^2 - 66x_i + 1122$. Notice that this is equivalent to $x_{i+1} - 33 = x_i^2 - 66x_i + 1089 = (x_i - 33)^2$. Denoting $a_n = x_n - 33$, we acquire $a_{i+1} = a_i^2$ for all $i = 1, 2, \ldots$, which means that $a_1 = 43 - 33 = 10$, $a_2 = a_1^2 = 10^2$, $a_3 = a_2^2 = 10^4$ and in general $a_i = 10^{2^{i-1}}$. So $x_{2021} = a_{2021} + 33 = 10^{2^{2020}} + 33$.

O14 *(Seniors.)* We call a prime number p *cute* if there exists a prime q so that both $pq - 2$ and $pq + 2$ are also primes. We call p *wonderful* if both p and $p + 2$ are cute primes. Find all wonderful numbers.

Answer: 3 and 5.

Solution: Let p be a cute prime. The numbers $pq - 2$, pq and $pq + 2$ give all the possible remainders modulo 3, so one of them must be divisible by 3. If $3 \mid pq - 2$, then $pq - 2 = 3$ and $pq = 5$, which is impossible, as p, q are primes. Similarly if $3 \mid pq + 2$, then $pq + 2 = 3$ and $pq = 1$, which is also impossible. So $3 \mid pq$, which means that either $p = 3$ or $q = 3$. Given that in the case $p = 3$ we can have $q = 3$, we have shown that p is cute iff both $3p - 2$ and $3p + 2$ primes.

Therefore p is wonderful if p, $p + 2$, $3p - 2$, $3p + 2$, $3p + 4$ and $3p + 8$ are all primes. We know that the numbers $3p - 2$, $3p$, $3p + 2$, $3p + 4$ and $3p + 6$ give all of the possible remainders modulo 5, so one of those numbers is divisible by 5. If it's $3p - 2$, $3p + 2$ or $3p + 4$, then due to primality it must be equal to 5, but then p is not prime. If it's $3p$ or $3p + 6$, then either $5 \mid p$ or $5 \mid p + 2$ respectively. Thus $p = 3$ or $p = 5$. We can verify that both work.

O15 *(Seniors.)* A function $f : \mathbb{R} \to \mathbb{R}$ satisfies for all real x and y the equation $2f(x)\left(f(y)\right)^2 + y^2 f\left(-x|y|\right) = f\left(xy^2\right)$. Find all possible values of $f(1)$.

Answer: $-1, 0, 1$.

Solution: Denote $f(1) = a$ and $f(-1) = b$. Choosing $x = y = 1$ and $x = y = -1$ yields the system of equations

$$\begin{cases} 2a^3 + b = a \\ 2b^3 + a = b \end{cases}$$

Adding the equations and cancelling equal terms, we get $2b^3 = -2a^3$. Therefore $b = -a$. Substituting this into the first equation yields $2a^3 - 2a = 0$. So $f(1)$ is one of the numbers $-1, 0, 1$. These values can be obtained by the functions $f(x) = -x|x|$, $f(x) = 0$ and $f(x) = x|x|$ respectively, which satisfy the equations.

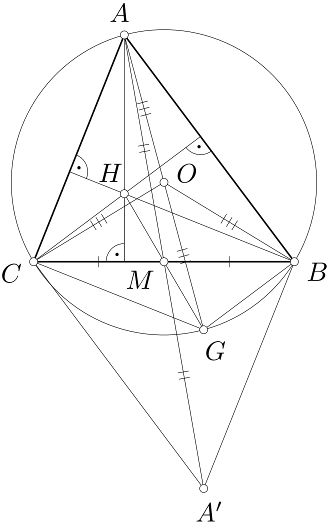


Fig. 11

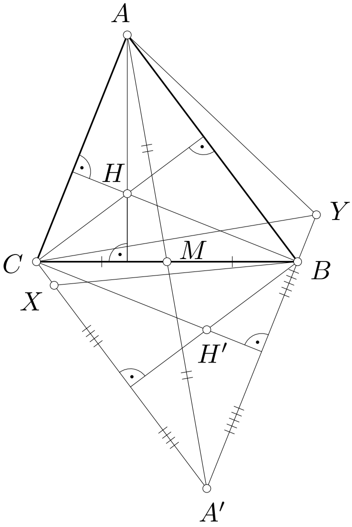


Fig. 12

O16 *(Seniors.)* Let H be the orthocenter of an acute triangle ABC. Let M be the midpoint of BC. Let A' and H' be the reflections of the points A and H across the point M. Prove that the points B and C and the reflections of A' over the lines BH' and CH' are concyclic.

Solution 1: Let O be the circumcenter of ABC and G the antipode of A (Fig. 11). Then $BG \perp AB$ and $CG \perp AC$. As $CH \perp AB$ and $BH \perp AC$, this means that $CG \parallel BH$ and $BG \parallel CH$. So $BHCG$ is a parallelogram. Therefore M, the midpoint of BC, is also the midpoint of HG. Consequently $H' = G$. Hence $BH' \perp AB$ and $CH' \perp AC$. Also, by the choice of A', $ABA'C$ is a parallelogram, meaning $A'B \parallel AC$ and $A'C \parallel AB$. Thus $A'B \perp CH'$ and $A'C \perp BH'$, which means that the reflections of A' over the lines BH' and CH' lie on lines $A'C$ and $A'B$ respectively; denote them by X and Y (Fig. 12). Then $CX \parallel AB$ and $BX = BA' = AC$, analogously also $BY \parallel AC$ and $CY = CA' = AB$. Therefore $ABCX$ and $ABCY$ are isosceles trapezoids. Isosceles trapezoids are cyclic quadrilaterals, thus both X and Y must lie on the circumcircle of ABC. The desired claim follows.

Solution 2: Like in the previous solution, notice that $ABA'C$ is a parallelogram. By symmetry with respect to M notice that triangles ABC and $A'CB$ are congruent and that H' is the orthocenter of $A'CB$.

Let X and Y be the reflections of A' over BH' and CH' respectively (Fig. 13). Then $A'X \perp BH'$ and $A'Y \perp CH'$ and since H' is the orthocenter of $A'CB$, we also have $A'C \perp BH'$ and $A'B \perp CH'$. So X and Y lie on the lines $A'C$ and $A'B$ respectively.

The choice of X and Y implies that the triangles $A'BX$ and $A'CY$ are isosceles. Therefore $\angle A'XB = \angle A'YC = \angle BA'C$. The desired claim follows.

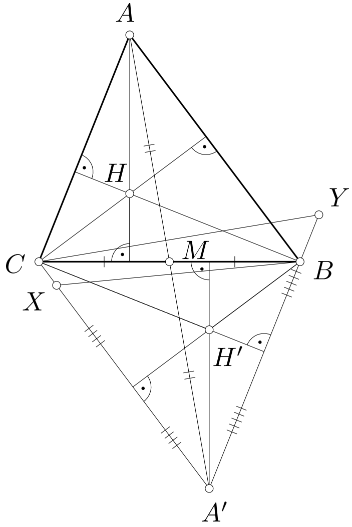
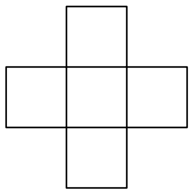


Fig. 13

O17 *(Seniors.)* Let $m, n > 2$ be positive integers. In each cell of an $m \times n$ grid there is a lamp, which can either be turned on or off. In one switch, we can change the state of five lamps, which are placed in cells forming a cross (see the diagram). Initially all the lamps are turned off. How many possible arrangements of lamps can be formed with these switches? Arrangements obtained from each other by rotations and reflections are considered different.



Answer: $2^{(n-2)(m-2)}$.

Solution: For convenience, assume that the switch for any cross is located at the central cell of that cross. Then switches are located in all cells not on the edge of the grid. There are $(m-2)(n-2)$ such cells.

The final state of each lamp is determined by whether there have been an even or odd number of switches affecting it, meaning we can cancel out all double presses of the same switch. So any possible arrangement can be achieved by using each switch at most one. There are $2^{(m-2)(n-2)}$ possibilities for the choice of switches to use.

Consider any two combinations differing by at least one switch used. Let L be the leftmost square, whose switch is used in one combination, but not in the other (if there are several options, choose at random). Then the left neighbour K of L will have a different end state for these arrangements. So different combinations yield different lamp arrangements. Therefore there are exactly $2^{(m-2)(n-2)}$ possible arrangements.

O18 *(Seniors.)* Mother wants to divide a cake of triangular shape between three kids. She makes a straight cut from one vertex to the midpoint of the opposite side and then another straight cut from another vertex to the midpoint of the opposite side. She gives the piece of quadrilateral shape to Anna, the triangular piece opposite to it to Berta and the remaining two triangular pieces to Clara. Who gets the largest part of cake?

Answer: all kids get the same amount.

Solution 1: Let S be the area of the initial triangle, S_A be the area of the quadrilateral piece, S_B be the area of Berta's triangle, and S_1 and S_2 be the areas of the remaining two triangles (Fig. 14). Then $S_1 + S_B = S_2 + S_B = \frac{1}{2}S$ (a common altitude while the ratio of the corresponding bases being $\frac{1}{2}$) and $S_B = 2S_1$ (for similar reasons). Thus $S_1 = S_2 = \frac{1}{6}S$ and $S_B = \frac{1}{3}S$, whence also $S_1 + S_2 = \frac{1}{3}S$ and $S_A = \frac{1}{3}S$.

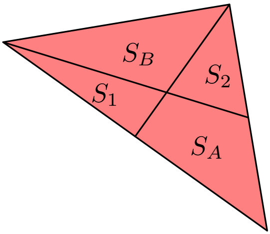


Fig. 14

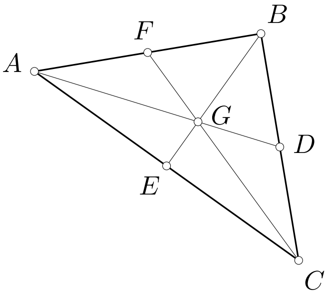


Fig. 15

Solution 2: Suppose that mother makes one more cut from the third vertex to the midpoint of the opposite side. As all medians of a triangle meet in one point, the new cut divides Anna's and Berta's pieces into two parts while not touching Clara's pieces. So every child gets exactly two pieces.

We show that medians of a triangle divide the triangle into six parts of equal area; this implies that all kids get the same amount of cake. Let the triangle be ABC, its medians be AD, BE and CF, and the centroid be G (Fig. 15). The length of the side BD of the triangle BGD is $\frac{1}{2}$ of the length of the side BC of the triangle ABC, the length of the corresponding altitude in the triangle BGD is $\frac{1}{3}$ of the length of the corresponding altitude in the triangle ABC (since $|AD| = 3|GD|$, the perpendicular drawn from the point A to the line BC is 3 times longer than the perpendicular drawn from the point G to the same line). Thus the area of the triangle BGD equals $\frac{1}{6}$ of the area of the triangle ABC. The same holds for other pieces. Consequently, all pieces have the same area.

O19 *(Seniors.)* Find all pairs of integers (x, y) that satisfy the equation $y^4 = x(2x^2 + y)^3$.

Answer: $\left(\frac{z^3(z-1)}{2}, \frac{z^6(z-1)}{2}\right)$ where z is arbitrary integer.

Solution. If $x = 0$ then $y = 0$. We now assume that $x \neq 0$.
Let $d = \gcd(x, y) > 0$ and $x = da$, $y = db$. Dividing the sides of the equation by d^4, we get $b^4 = a(2a^2d + b)^3$. Thus $a \mid b^4$. Since a and b are coprime, the only possibility is $|a| = 1$ and the equation reduces to

$$b^4 = \pm(2d + b)^3.$$

As the same integer is simultaneously a cube and a fourth power, it is a twelfth power of some integer, yielding $b = z^3$ for some integer z. Substituting $b = z^3$ into the equation and taking cube root gives $z^4 = \pm(2d + z^3)$ or, equivalently, $\pm z^4 = 2d + z^3$. Since $-z^4 \leqslant -|z^3| \leqslant z^3 < 2d + z^3$, the minus sign is not possible, therefore $d = \frac{z^4 - z^3}{2} = \frac{z^3(z-1)}{2}$. We conclude that $x = \frac{z^3(z-1)}{2}$ and $y = \frac{z^6(z-1)}{2}$, where z is an arbitrary integer different from 0 and 1. In this case $d > 0$ and the solution satisfies the original equation. If $z = 0$ or $z = 1$, we get the initial solution $(0, 0)$.

O20 *(Seniors.)* Find the least possible sum of 2021 terms of the sequence $a_1, a_2, a_3, \ldots$, where $a_1 = 0$, $a_2 = a_3 = 1$ and $a_{i+j} > a_i + a_j$ for every $i, j \geqslant 2$.
Answer: $2 \cdot 1010^2$.
Solution: We show that the least sum arises in the case of the sequence $0, 1, 1, 3, 3, 5, 5, \ldots$ ($a_{2i} = a_{2i+1} = 2i - 1$ for every $i \geqslant 1$). Firstly, we show that this sequence meets the conditions of the problem. Indeed, if j and k are of the same parity then $a_{j+k} = j + k - 1 > (j - 1) + (k - 1) \geqslant a_j + a_k$. If j and k have different parities then letting, w.l.o.g., j be even, we obtain $a_{j+k} = j + k - 2 > (j - 1) + (k - 2) = a_j + a_k$.
We prove now that if $a_1, a_2, a_3, \ldots$ is an arbitrary sequence of integers satisfying the conditions of the problems then $a_{2n} \geqslant 2n - 1$ ja $a_{2n+1} \geqslant 2n - 1$ for any positive integer n. We proceed by induction on n. The base cases $a_2 \geqslant 1$ and $a_3 \geqslant 1$ hold. If $a_{2n} \geqslant 2n - 1$ then, from the conditions of the problem,

$$a_{2n+2} \geqslant a_{2n} + a_2 + 1 \geqslant 2n - 1 + 1 + 1 = 2(n + 1) - 1,$$

which proves the induction step for even indices. The same holds for odd indices: If $a_{2n+1} \geqslant 2n - 1$ then

$$a_{2n+3} \geqslant a_{2n+1} + a_2 + 1 \geqslant 2n - 1 + 1 + 1 = 2(n + 1) - 1.$$

Thus the least sum of 2021 terms is $2(1 + 3 + \cdots + 2019)$, i.e., $2 \cdot 1010^2$.

O21 *(Seniors.)* Triangle ABC satisfies $AB = AC$. Medians AD and BE intersect at G. Let P be the midpoint of the line segment GE.
(a) Prove that if $GP = GD$ then the quadrilateral $CEPD$ is cyclic.
(b) Does it hold that if the quadrilateral $CEPD$ is cyclic then $GP = GD$?
Answer: (b) Yes.
Solution: Let $BD = DC = x$ and $GP = PE = y$. Then $BG = 2 \cdot 2y = 4y$, $BP = 4y + y = 5y$ and $BE = 4y + 2y = 6y$. Thus $GP = GD$ if and only if $y^2 = BG^2 - BD^2 = 16y^2 - x^2$ or, equivalently, $x^2 = 15y^2$. The quadrilateral $CEPD$ is cyclic if and only if $BP \cdot BE = BD \cdot BC$, i.e., $5y \cdot 6y = x \cdot 2x$. The

latter simplifies to $x^2 = 15y^2$, too. Hence $GP = GD$ if and only if the quadrilateral $CEPD$ is cyclic, which solves both parts of the problem.

Remark: The conditions of the problem are satisfied for the base angle γ such that $\sin^2 \gamma = \frac{3}{8}$ (then $\gamma \approx 37{,}76°$).

O22 *(Seniors.)* Little Juku writes all integers from 1 to n on a blackboard, but as he does not know the digit 4 yet, he skips all numbers that contain 4. Juku's sister Mari erases two numbers on the blackboard and writes the absolute value of the difference of these numbers on the blackboard. Then Mari again erases two numbers on the blackboard and writes the absolute value of their difference on the blackboard, etc. Can it happen after a finite number of such steps that there are all integers from 1 to n that contain the digit 4 and only these on the blackboard, each one exactly once, if

(a) $n = 2021$;

(b) $n = 10000$?

Answer: (a) No; (b) No.

Solution:

(a) The difference and the sum of two integers have equal parity, likewise are an integer and its absolute value of equal parity. Thus Mari's every step keeps the parity of the sum of all numbers on the blackboard unchanged. Among $1, 2, \ldots, 2021$, there are 1011 odd numbers. As 1011 is odd itself, the sum of all numbers $1, 2, \ldots, 2021$ is odd. This implies that the sum of all integers from 1 to 2021 that contain 4 and the sum of integers from 1 to 2021 that do not contain 4 have different parities. Hence the set of all natural numbers from 1 to 2021 that contain 4 can never appear on the blackboard.

(b) Let there be a positive integers from 1 to 10000 that do not contain 4. Then there are $10000 - a$ positive integers from 1 to 10000 that contain 4. As each Mari's step decreases the number of numbers on the blackboard by 1, she should make $a - (10000 - a)$ or, equivalently, $2a - 10000$ steps to reach the desired state. As only one new number can appear at each step, introducing $10000 - a$ new numbers assumes $2a - 10000 \geqslant 10000 - a$ which is equivalent to $a \geqslant \frac{20000}{3}$. But there are only 9^4 numbers among the first 10000 positive integers that do not contain 4; since $9^4 = 6561 < \frac{20000}{3}$, achieving the desired state is impossible.

Selected Problems from the Final Round of National Olympiad

F1 *(Grade 7.)* Points A, B and C are chosen in a rectangle of shape 7×10 in such a way that the distances from A to some three sides of the rectangle are 2, 3 and 4, the distances from B to some three sides of the rectangle are 3, 4 and 5, and the distances from C to some three sides of the rectangle are 4, 5 and 6. Find the largest possible area the triangle ABC can have.

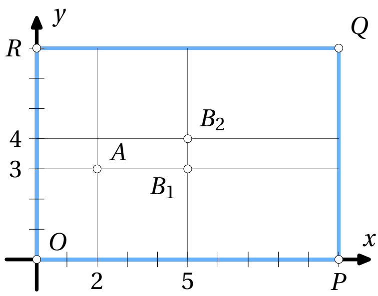


Fig. 16

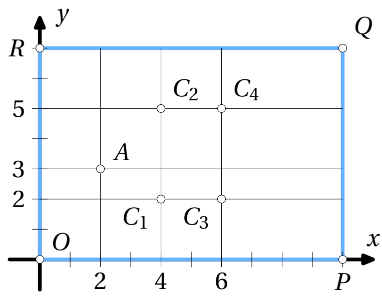


Fig. 17

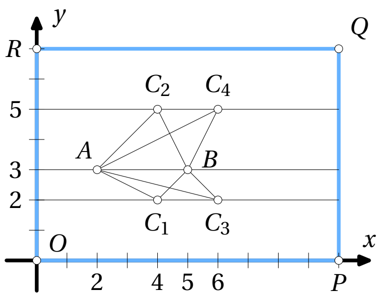


Fig. 18

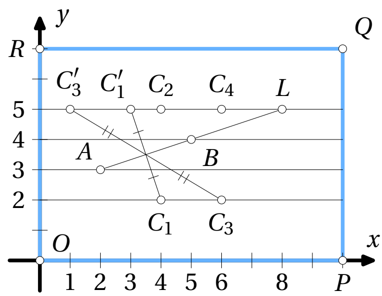


Fig. 19

Answer: $\frac{7}{2}$.

Solution: Introduce a Cartesian coordinate system with origin O at one vertex of the rectangle. Let the other vertices of the rectangle be $P = (10, 0)$, $Q = (10, 7)$ and $R = (0, 7)$. Given the distances from a point to three sides of the rectangle, two of these sides must be opposite sides and the corresponding distances sum up to the length of the perpendicular side of the rectangle. Thus A must lie at distances 3 and 4 from the horizontal sides and at distances 2 and 8 from the vertical sides. W.l.o.g., let $A = (2, 3)$. Similarly, B must lie at distances 3 and 4 from the horizontal sides and at distances 5 and 5 from the vertical sides. Thus $B = B_1 = (5, 3)$ or $B = B_2 = (5, 4)$ (Fig. 16). Finally, C must lie at distances 4 and 6 from the vertical sides and at distances 2 and 5 from the horizontal sides. Thus $C = C_1 = (4, 2)$ or $C = C_2 = (4, 5)$ or $C = C_3 = (6, 2)$ or $C = C_4 = (6, 5)$ (Fig. 17).

Altogether, the triangle ABC can be located in 8 ways. We study which one yields the largest area. If $B = B_1 = (5, 3)$, take the side AB with length 3 as base. The corresponding altitude has length 1 if $C = C_1 = (4, 2)$ or $C = C_3 = (6, 2)$ and 2 if $C = C_2 = (4, 5)$ or $C = C_4 = (6, 5)$ (Fig. 18). Thus the largest area in the observed cases is 3. Consider now the case $B = B_2 = (5, 4)$. Reflecting the points C_1 and C_3 from the midpoint of the side AB, we obtain points $C_1' = (3, 5)$ and $C_3' = (1, 5)$ which yield the same areas (Fig. 19). Points C_2, C_4, C_1' and C_3' lie on the line $y = 5$, whereby the one with the least x-coordinate, the point C_3', is the farthest from the line AB. Hence the largest area arises in the case $C = C_3 = (6,2)$. The size of the rectangle surrounded by lines $y = 2$, $y = 4$, $x = 2$ and $x = 6$ is 2×4. Removing three right triangles to extract the triangle ABC (Fig. 20) enables to express the area of the triangle ABC as $2 \cdot 4 - \frac{1}{2} \cdot 1 \cdot 3 - \frac{1}{2} \cdot 1 \cdot 2 - \frac{1}{2} \cdot 1 \cdot 4 = \frac{7}{2}$. As $\frac{7}{2} > 3$, the largest possible area of the triangle ABC is $\frac{7}{2}$.

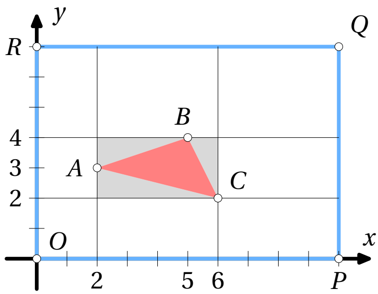


Fig. 20

F2 *(Grade 7.)* Juku has four cans of juice: A 1-litre can containing $\frac{1}{2}$ litres of juice, a $\frac{1}{2}$-litre can containing $\frac{1}{3}$ litres of juice, a $\frac{1}{3}$-litre can containing $\frac{1}{4}$ litres of juice and a $\frac{1}{4}$-litre can containing $\frac{1}{5}$ litres of juice. There are no volume markings on the cans. Juku wants to measure exactly $\frac{1}{30}$ litres of juice by a sequence of pourings of the juice from one can to another. Letting the juice spill is not allowed. Find out all cans into which Juku can get the desired exact amount of juice.

Answer: The second, the third, the fourth.

Solution: The second can contains $\frac{1}{6}$ litres of free space. Pouring juice from the fourth can over to the second can until the second can becomes full leaves $\frac{1}{30}$ litres of juice in the fourth can. If, after that, one pours all juice from either the second or the third can over to the first can (this is possible since the first can contains $\frac{1}{2}$ litres of free space and none of the other cans can contain more juice), then the juice in the fourth can can be poured over to either the second or the third can. Thus Juku can get exactly $\frac{1}{30}$ litres of juice into the fourth can, as well as into the third or the second one.

On the other hand, the total amount of juice in all cans is $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$ litres after any sequence of pourings. Thus even if the second, the third and the fourth can are full, the first can contains $\frac{1}{5}$ litres of juice; if the other cans contain free space then the amount of juice in the first can must be even larger. Consequently, it is impossible to get exactly $\frac{1}{30}$ litres of juice into the first can.

F3 *(Grade 7.)* Priit's collection consists of 1111 stamps which are all distributed into envelopes in such a way that every envelope contains more

than one stamp, all envelopes contain the same number of stamps, and each envelope contains only stamps from one country. It is known that more than 40% of stamps in this collection are from Estonia, more than 30% of stamps in the collection are from Latvia and more than 20% of stamps in the collection are from Lithuania. Find the largest possible number of envelopes containing Estonian stamps and the largest possible number of envelopes containing Lithuanian stamps.
Answer: 41 and 29.
Solution: As $1111 = 11 \cdot 101$ where the factors are prime, we have four cases:

- 1 envelope containing 1111 stamps;
- 1111 envelopes, each containing 1 stamp;
- 11 envelopes, each containing 101 stamps;
- 101 envelopes, each containing 11 stamps.

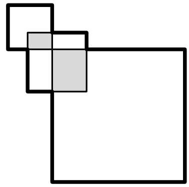
The first case is impossible since Priit has stamps of at least 3 countries and one envelope can contain only stamps of one country. The second case is excluded by the conditions of the problem explicitly. If there were 11 envelopes then at least 5 envelopes would have to contain Estonian stamps, at least 4 envelopes would have to contain Latvian stamps and at least 3 envelopes would have to contain Lithuanian stamps. This would require at least 12 envelopes in total which contradicts the assumption. Hence there must be 101 envelopes. Then at least 41 envelopes have to contain Estonian stamps, at least 31 envelopes have to contain Latvian stamps and at least 21 envelopes have to contain Lithuanian stamps. So the number of envelopes containing Lithuanian stamps cannot exceed 29. It is indeed possible to have 41 envelopes containing Estonian stamps and 29 envelopes containing Lithuanian stamps if 31 envelopes contain Latvian stamps and there are no stamps from other countries.

F4 *(Grade 8.)* We call positive integers n, m an *interesting pair* if $n < m$ and the greatest prime factor of n is equal to the greatest prime factor of m.
(a) For an interesting pair n, m, will there always exist a prime p such that $n < p < m$?
(b) Among the first 25 positive integers, how many don't form an interesting pair with any smaller positive integer?
Answer: (a) No; (b) 10.
Solution:
(a) The pair $24, 27$ is interesting, as the greatest prime factor of them both is 3. However, there are no primes between them.
(b) The number 1 and prime numbers cannot form a pair with any smaller integers. However each composite number forms a pair with its greatest prime factor, which is clearly smaller. Among the first 25 positive integers, the primes are 2, 3, 5, 7, 11, 13, 17, 19 and 23. Taking into account also the

number 1, we obtain 10 numbers not forming an interesting pair with any smaller positive integer.

Remark: In part (a) there are other counterexamples, the next ones (by size) are $120, 125$ (with greatest prime factor 5) and $140, 147$ (with greatest prime factor 7).

F5 *(Grade 8.)* A dodecagon with perimeter 72 cm is constructed from three squares as shown in the figure. The two outer squares have a common vertex and both share a rectangular part with the central square, such that the perimeter of the shared part is 5 times less than the sum of the perimeters of the central square and the corresponding outer square. Find the side length of the central square.



Answer: 6 cm.

Solution: Denote the vertices of the dodecagon as on Fig. 21. Since $QI = AS$ and $IS = QA$, we have

$$DQ + QI + IS + SG = DQ + AS + QA + SG = DA + AG.$$

Similarly we obtain $ER + RK + KP + PB = EA + AB$. Therefore

$$\begin{aligned} CD + DQ + QI + IS + SG + GF + FE + ER + RK + KP + PB + BC \\ = CD + DA + AG + GF + FE + EA + AB + BC \\ = AB + BC + CD + DA + AE + EF + FG + GA. \end{aligned}$$

So the sum of the perimeters of the squares $ABCD$ and $AEFG$ is equal to the perimeter of the dodecagon (72 cm).

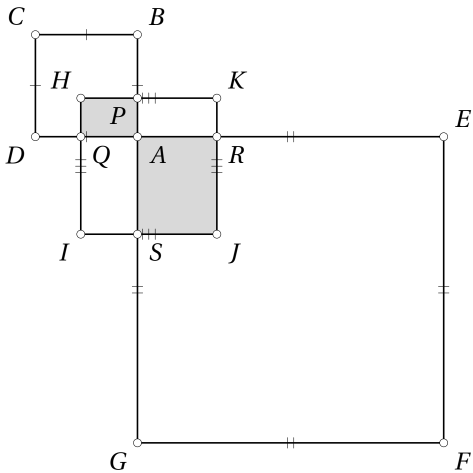


Fig. 21

Since $QA = IS$, $AP = RK$, $SA = IQ$ and $AR = PK$, we have

$$\begin{aligned} AP + PH + HQ + QA + AR + RJ + JS + SA \\ &= RK + PH + HQ + IS + PK + RJ + JS + IQ \\ &= HI + IJ + JK + KH. \end{aligned}$$

So the perimeter of the square $HIJK$ is equal to the sum of the perimeters of the rectangles $APHQ$ and $ARJS$. However we are given that the sum of the perimeters of $APHQ$ ja $ARJS$ is equal to one fifth of the sum of the perimeter of $ABCD$, the perimeter of $AEFG$ and twice the perimeter of $HIJK$.

Denoting the side length of $HIJK$ by x cm, its perimeter will be $4x$ cm. Then we can combine the relations of the previous two paragraphs into the equation

$$4x = \frac{1}{5}(72 + 2 \cdot 4x).$$

Solving it, we obtain $4x = 24$, which gives $x = 6$. So the side length of the square $HIJK$ is 6 cm.

F6 *(Grade 8.)* On the side AB of a triangle ABC, let D be a point such that $\angle BDC = \angle ACB$. Let K be the midpoint of CD and let E be the intersection of lines BK and AC. Given that $\angle BKD = 2\angle BCD$, find $\angle AEB$.

Answer: 90°.

Solution: Denote $\angle CAB = \alpha$ and $\angle BCA = \gamma$. Then $\angle BDC = \gamma$ and triangles CBD and ABC will be similar due to having two equal angles (Fig. 22). Therefore $\angle BCD = \alpha$ and $\angle BKD = 2\alpha$. But then $\angle KBC = 2\alpha - \alpha = \alpha = \angle KCB$, which yields $KB = KC$. However $KC = KD$ by the choice of K, which yields $\angle KBD = \angle KDB = \gamma$. Now triangle KBD gives us $\gamma + \gamma + 2\alpha = 180^\circ$ or $\alpha + \gamma = 90^\circ$. Finally, since two angles in triangle ABE are α and γ, the third angle AEB must be $180^\circ - (\alpha + \gamma) = 90^\circ$.

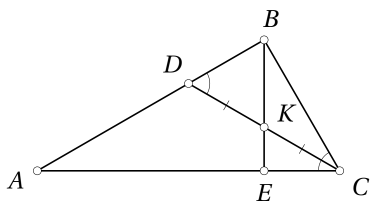


Fig. 22

F7 *(Grade 8.)* There are three consecutive positive integers on a blackboard. A move consists of erasing the smallest number on the blackboard and replacing it by the sum of itself and the greatest number on the blackboard. Is it possible that the sum of the numbers on the blackboard will be a power of 10:

(a) after the 6th move?

(b) after the 8th move?

Answer: (a) Yes; (b) No.
Solution: Let x be the smallest of the three numbers initially on the blackboard. We will find the numbers on the blackboard after having made 0, 1, 2, 3, 4, 5, 6, 7 and 8 moves, and their sums:

Number of moves	Numbers on the blackboard	Sum of numbers
0	x , $x+1$, $x+2$	$3x+3$
1	$x+1$, $x+2$, $2x+2$	$4x+5$
2	$x+2$, $2x+2$, $3x+3$	$6x+7$
3	$2x+2$, $3x+3$, $4x+5$	$9x+10$
4	$3x+3$, $4x+5$, $6x+7$	$13x+15$
5	$4x+5$, $6x+7$, $9x+10$	$19x+22$
6	$6x+7$, $9x+10$, $13x+15$	$28x+32$
7	$9x+10$, $13x+15$, $19x+22$	$41x+47$
8	$13x+15$, $19x+22$, $28x+32$	$60x+69$

(a) The total sum of the numbers on the blackboard after 6 moves will be $28x+32$. Either by direct computation or by considerations modulo 4 and 7, we observe that the equation $28x+32=10^4$ has an integer solution. Therefore it is possible that after the 6th move, the sum of the numbers on the blackboard will be a power of 10.
(b) The total sum of the numbers on the blackboard after 8 moves will be $60x+69$. As $60x$ and 69 end with 0 and 9, respectively, their sum will also end with the digit 9. But such a number cannot be equal to a power of 10.

F8 *(Grade 9.)* Two positive integers together contain each digit $0, 1, \ldots, 9$ exactly once. Find the largest possible common divisor that these two numbers can have.
Answer: 48651.
Solution: Suppose that both numbers contain 5 digits. A common divisor of two different numbers cannot exceed half of the larger one; thus the greatest common divisor of two such numbers must be less than 50000. It means that if the greatest common divisor has 5 digits then the first digit is at most 4. Let the greatest common divisor be d and suppose that it has 5 digits, the first of which is 4. Then the two numbers under consideration are d and $2d$. If the second digit of d were 9 then the first two digits of $2d$ would be either 98 or 99, but in both cases, 9 would occur in these numbers repeatedly. Thus the second digit of d is at most 8; suppose that it is 8. Then the first digit of $2d$ is 9. If the third digit of d were 7 then the second digit of $2d$ would be 7, too. Thus the third digit of d is at most 6 and the fourth digit of d is at most 5; suppose that the third and the fourth digit are 6 and 5, respectively. Then the second and the third digit of $2d$ are 7 and 3, respectively, and the fourth digit is 0 as the remaining digits 1 and 2 cannot cause a carry. Finally, 1 and 2 must be the last digits of d and $2d$, respectively, yielding $d=48651$ and $2d=97302$. These numbers satisfy the conditions

of the problem.

If the given integers are not 5-digit numbers then the greatest common divisor can have at most 4 digits. Consequently, there cannot be solutions greater than that found in the previous paragraph.

F9 *(Grade 9.)* The bisector of the internal angle at the vertex B of a triangle ABC intersects the circumcircle of the triangle ABC at a point P ($P \neq B$). The line through P perpendicular to the line AC intersects the circumcircle of the triangle ABC at a point P' ($P' \neq P$). Prove that the quadrilateral $APCP'$ is a square if and only if $\angle ABC = 90°$.

Solution 1: As $\angle ABC = \angle AP'C$ (Fig. 23), it suffices to show that the quadrilateral $APCP'$ is a square if and only if $\angle AP'C = 90°$.

Since $\angle ABP = \angle CBP$, we have $AP = CP$. Thus PP' is the perpendicular bisector of the side AC. Hence the line PP' passes through the circumcentre of the triangle ABC, i.e., the chord PP' is a diameter. By Thales' theorem, $\angle PAP' = \angle PCP' = 90°$.

If $\angle AP'C = 90°$ then also $\angle APC = 180° - 90° = 90°$. Thus the quadrilateral $APCP'$ is a rectangle and, by equality of adjacient sides AP and CP, a square. On the other hand, if $APCP'$ is a square then obviously $\angle AP'C = 90°$. This completes the proof.

Solution 2: Since $\angle ABC = \angle AP'C$, it suffices to show that the quadrilateral $APCP'$ is a square if and only if $\angle AP'C = 90°$.

As $\angle ABP = \angle CBP$, we have $AP = CP$. Thus PP' bisects the line segment AC.

If $\angle AP'C = 90°$ then, by Thales' theorem, AC is the diameter of the circumcircle of the triangle ABC. Thus the chord PP' bisecting AC passes through the circumcentre of the triangle ABC, implying that AC bisects the chord PP'. As $PP' \perp AC$, the diagonals of the quadrilateral $APCP'$ are perpendicular and bisect each other. Hence the quadrilateral $APCP'$ is a rhombus and, because of $\angle AP'C = 90°$, a square. On the other hand, if $APCP'$ is a square then obviously $\angle AP'C = 90°$. This completes the proof.

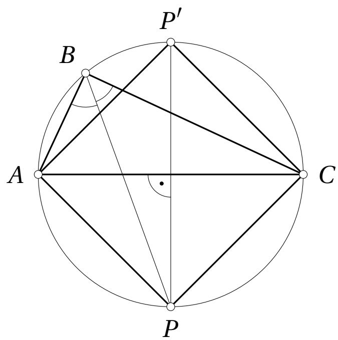
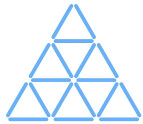


Fig. 23

F10 *(Grade 9.)* There are sticks of length 1 with a number 1, 2 or 3 written on each of them. There is an unlimited supply of sticks with every number. Two triangles consisting of three sticks are considered different if neither of the triangles can be composed from sticks of the other triangle.

(a) How many different triangles consisting of three sticks are possible?

(b) From 18 sticks, one makes an equilateral triangle of side length 3, divided into 9 pairwise different equilateral triangles of side length 1. Find the largest possible sum of the numbers written on the 9 sticks on the boundary of the big triangle.



Answer: (a) 10; (b) 26.

Solution 1:

(a) There are 3 triangles having the same number on each side (111, 222, 333). There are 6 triangles having one number on two sides and another number on the third side (112, 113, 221, 223, 331, 332). Only 1 triangle has a different number on every side (123). Thus there are 10 different triangles consisting of three sticks in total.

(b) The largest sum of numbers written on 9 sticks is 27. Suppose that the sum of the numbers on the sticks on the boundary of the big triangle is 27. This means that all sticks on the boundary have 3 on it. There are 6 different triangles consisting of three sticks having 3 on some side. As the number of small triangles having a side on the boundary of the big triangle is also 6, all 6 different triangles consisting of three sticks and having 3 on some side must occur at the corners or in the middle of a side of the big triangle. As one of these 6 triangles has 3 on every side, but not all sides of a small triangle can lie on the boundary of the big triangle, at least one stick with number 3 lies in the interior of the big triangle. The other small triangle with this stick as a side lies entirely in the interior of the big triangle. This contradicts the previously proved claim that all triangles consisting of three sticks and having 3 on one side lie on the boundary of the big triangle. The contradiction shows that the sum 27 is impossible.

Figure 24 shows that the sum can be 26.

Solution 2:

(a) The question of the problem is equivalent to the question how many three-digit numbers whose each digit is 1, 2 or 3 and digits are in non-decreasing order do there exist. There are 10 such numbers (111, 112, 113, 122, 123, 133, 222, 223, 233, 333).

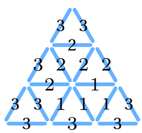


Fig. 24

(b) The largest sum of numbers written on 9 sticks is 27. Suppose that the sum of the numbers on the sticks on the boundary of the big triangle is 27. This means that all sticks on the boundary have 3 on it. In all 10 triangles consisting of three sticks, the number 3 occurs 10 times in total. To have 9 occurrences on the boundary, the triangle with 3 on every side must definitely be used. As not all sides of a small triangle can lie on the boundary of the big triangle, at least one stick with number 3 lies in the interior of the big triangle. Thus all 10 occurrences must be used. But the stick in the interior of the big triangle and having 3 on it is a side of another triangle entirely in the interior of the big triangle, whence we have 11 occurrences of 3 when counted by triangles. The contradiction shows that the sum 27 is impossible.
Any of Figures 25, 26, 27, 28 and 29 shows that the sum can be 26.
Remark: Figures 24–29 contain all possibilities, modulo rotations and reflections, for obtaining the sum 26.

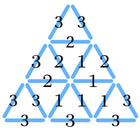


Fig. 25

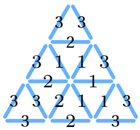


Fig. 26

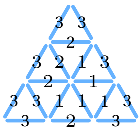


Fig. 27

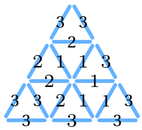


Fig. 28

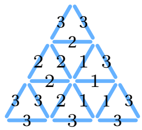


Fig. 29

F11 *(Grade 9.)* A point M is chosen on the side AC of a triangle ABC and a point K is chosen on the line segment BM so that $AM = \frac{1}{3}AC$ and $BK = \frac{1}{4}BM$. Let N be the intersection of the line AK and the side BC. What percentage of the area of the triangle ABC is the area of the quadrilateral $MKNC$?
Answer: 65%.
Solution 1: Let the area of the triangle ABC be S and the areas of triangles AKM, BKN, MKN ja CMN be S_1, S_2, S_3 and S_4, respectively (Fig. 30). Then:

- $S_2 + S_3 + S_4 = \frac{2}{3}S$ since the l.h.s. is the area of the triangle MBC while triangles MBC and ABC having equal altitudes and the ratio of the lengths of the corresponding bases being $\frac{2}{3}$;
- $S_1 + S_3 + S_4 = \frac{3}{2}S_4$ since the l.h.s. is the area of the triangle ANC while triangles ANC and MNC having equal altitudes and the ratio of the lengths of the corresponding bases being $\frac{3}{2}$;
- $S_2 = \frac{1}{3}S_3$ because of triangles BNK and MNK having equal altitudes and the ratio of the lengths of the corresponding bases being $\frac{1}{3}$;
- $S_1 = \frac{1}{4}S$ because the ratio of altitudes of the triangles AKM and ABC is $\frac{3}{4}$ while the ratio of the lengths of the corresponding bases being $\frac{1}{3}$.

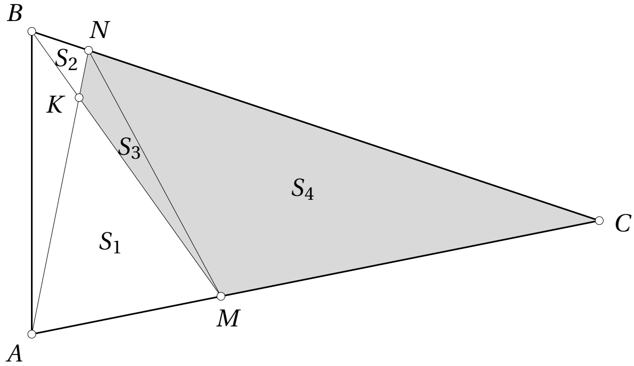


Fig. 30

Substituting S_1 and S_2 from the latter two equations to the former two equations, we obtain the system of equations

$$\begin{cases} \frac{1}{3}S_3 + S_3 + S_4 = \frac{2}{3}S, \\ \frac{1}{4}S + S_3 + S_4 = \frac{3}{2}S_4. \end{cases}$$

After adding $\frac{1}{3}S_4$ to both sides of the first equation and bringing the term with S to the l.h.s., we obtain the equivalent system

$$\begin{cases} \frac{4}{3}\left(S_3 + S_4\right) - \frac{2}{3}S = \frac{1}{3}S_4, \\ \left(S_3 + S_4\right) + \frac{1}{4}S = \frac{3}{2}S_4. \end{cases}$$

Solving w.r.t. $S_3 + S_4$ and S_4 yields $S_3 + S_4 = 0.65S$. Hence the area of the quadrilateral $MKNC$ equals 65% of the area of the triangle ABC.

Solution 2: Denote the area of a figure Π by S_Π. From the conditions of the problem, we obtain the following:

- $\frac{S_{ABM}}{S_{ABC}} = \frac{|AM|}{|AC|} = \frac{1}{3}$, whence $S_{ABM} = \frac{1}{3}S_{ABC}$ and $S_{CBM} = \frac{2}{3}S_{ABC}$;
- $\frac{S_{ABK}}{S_{ABM}} = \frac{|BK|}{|BM|} = \frac{1}{4}$, whence $S_{ABK} = \frac{1}{4} \cdot \frac{1}{3}S_{ABC} = \frac{1}{12}S_{ABC}$ and, consequently, $S_{AMK} = \left(\frac{1}{3} - \frac{1}{12}\right)S_{ABC} = \frac{1}{4}S_{ABC}$;
- $\frac{S_{BCK}}{S_{BCM}} = \frac{|BK|}{|BM|} = \frac{1}{4}$ (Fig. 31), whence $S_{BCK} = \frac{1}{4} \cdot \frac{2}{3}S_{ABC} = \frac{1}{6}S_{ABC}$ and $S_{MCK} = \left(\frac{2}{3} - \frac{1}{6}\right)S_{ABC} = \frac{1}{2}S_{ABC}$.

Since $\frac{S_{ABN}}{S_{ACN}} = \frac{|BN|}{|CN|} = \frac{S_{KBN}}{S_{KCN}}$, we obtain

$$\frac{S_{ABN}}{S_{ACN}} = \frac{S_{ABN} - S_{KBN}}{S_{ACN} - S_{KCN}} = \frac{S_{ABK}}{S_{AMK} + S_{MCK}} = \frac{\frac{1}{12}}{\frac{1}{4} + \frac{1}{2}} = \frac{\frac{1}{12}}{\frac{3}{4}} = \frac{1}{9}.$$

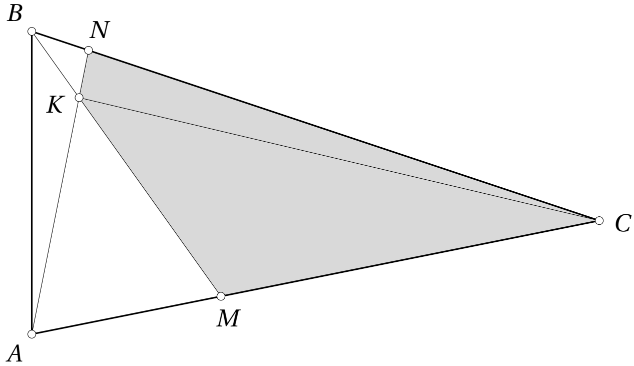


Fig. 31

Hence $\frac{S_{ABN}}{S_{ABC}} = \frac{S_{ABN}}{S_{ABN}+S_{ACN}} = \frac{S_{ABN}}{S_{ABN}+9S_{ABN}} = \frac{1}{10}$, implying $S_{ABN} = \frac{1}{10}S_{ABC}$. Altogether, we obtain

$$\frac{S_{MKNC}}{S_{ABC}} = \frac{S_{ABC} - S_{ABN} - S_{AMK}}{S_{ABC}} = 1 - \frac{1}{10} - \frac{1}{4} = 0.65,$$

meaning that the area of the quadrilateral $MKNC$ equals 65% of the area of the triangle ABC.

F12 *(Grade 10.)* When adding together positive integers a and b, Juku forgot to enter the final digit 7 of the number a and got a result of 2022. Had Juku instead forgotten to enter the final digit of b, the result would have been 5000. Find the sum of a and b.

Answer: 6385.

Solution: Let $a = \overline{x7}$ and $b = \overline{yd}$, where d is the final digit of b and x, y are the numbers a and b without their final digits. Then $a = 10x + 7$ ja $b = 10y + d$. From the given information, we compile the system of equations

$$\begin{cases} x + 10y + d = 2022, \\ y + 10x + 7 = 5000. \end{cases} \tag{1}$$

Subtracting the first equation from the second one, we get the equation

$$9(x - y) - d = 2971. \tag{2}$$

As x, y are integers, $9(x - y)$ is divisible by 9. As the number 2971 gives a remainder of 1 upon division by 9, the number d must give a remainder of 8; the only option for this is $d = 8$. Substituting this into (2), we get $x - y = 331$, which we can substitute into either equation of (1) to get $x = 484$ and $y = 153$. Therefore $a + b = 4847 + 1538 = 6385$.

F13 *(Grade 10.)* Find the sum

$$\sqrt{1 + \frac{1}{1^2} + \frac{1}{2^2}} + \sqrt{1 + \frac{1}{2^2} + \frac{1}{3^2}} + \ldots + \sqrt{1 + \frac{1}{2021^2} + \frac{1}{2022^2}}.$$

Answer: $2021 + \frac{2021}{2022}$.

Solution: We will denote

$$s = \sqrt{1+\frac{1}{1^2}+\frac{1}{2^2}} + \sqrt{1+\frac{1}{2^2}+\frac{1}{3^2}} + \ldots + \sqrt{1+\frac{1}{2021^2}+\frac{1}{2022^2}}.$$

Notice that

$$1+\frac{1}{k^2}+\frac{1}{(k+1)^2} = \frac{k^2(k+1)^2+(k+1)^2+k^2}{k^2(k+1)^2} = \frac{(k(k+1)+1)^2}{(k(k+1))^2},$$

which yields

$$\sqrt{1+\frac{1}{k^2}+\frac{1}{(k+1)^2}} = \frac{k(k+1)+1}{k(k+1)} = 1+\frac{1}{k(k+1)} = 1+\frac{1}{k}-\frac{1}{k+1}.$$

Therefore

$$s = 2021 + \left(\frac{1}{1}-\frac{1}{2}\right) + \left(\frac{1}{2}-\frac{1}{3}\right) + \ldots + \left(\frac{1}{2021}-\frac{1}{2022}\right) = 2021 + \frac{2021}{2022}.$$

F14 *(Grade 10.)* Points A, D, E and C lie on a line in this order. Point B is chosen such that triangles ADB and BEC are similar (in this specific order of vertices), moreover $EC = 2AD$ and $\angle ABC = 120^\circ$. Find $\frac{AC}{AD}$.

Answer: $3+\sqrt{2}$.

Solution 1: Denote $\angle EDB = \alpha$ (Fig. 32). The similarity of ADB and BEC yields $\angle DAB = \angle EBC$ and $\angle BDA = \angle CEB$; the second of which gives $\angle DEB = 180^\circ - \angle CEB = 180^\circ - \angle BDA = \angle EDB = \alpha$. Therefore $DB = BE$. Hence the similarity of ADB and BEC yields $\frac{BE}{AD} = \frac{EC}{DB} = \frac{EC}{BE}$; thus $\left(\frac{BE}{AD}\right)^2 = \frac{BE}{AD} \cdot \frac{EC}{BE} = \frac{EC}{AD} = 2$, which gives $\frac{BE}{AD} = \sqrt{2}$ or $BE = \sqrt{2}AD$.

The isosceles triangle BDE gives $\angle DBE = 180^\circ - 2\alpha$. On the other hand,

$$\begin{aligned} \angle DBE &= 120^\circ - \angle ABD - \angle EBC = 120^\circ - \angle ABD - \angle DAB \\ &= 120^\circ - (\angle ABD + \angle DAB) = 120^\circ - \alpha. \end{aligned}$$

The equation $180^\circ - 2\alpha = 120^\circ - \alpha$ yields $\alpha = 60^\circ$. So the triangle BDE is equilateral, which yields $DE = BE = \sqrt{2}AD$.

Therefore $\frac{AC}{AD} = \frac{AD+DE+EC}{AD} = \frac{AD+\sqrt{2}AD+2AD}{AD} = 1+\sqrt{2}+2 = 3+\sqrt{2}$.

Solution 2: The similarity of ADB and BEC yields $\angle DAB = \angle EBC$ and $\angle ABD = \angle BCE$. So the triangles ADB and BEC are also similar to ABC. Therefore $\angle ADB = \angle BEC = \angle ABC = 120^\circ$. But this yields $\angle BDE = \angle DEB = 180^\circ - 120^\circ = 60^\circ$, meaning that the triangle BDE is equilateral.

The similarity of ADB and ABC yields $\frac{AD}{AB} = \frac{AB}{AC}$. The similarity of BEC and ABC yields $\frac{EC}{BC} = \frac{BC}{AC}$. Combining these with $2AD = EC$, we see that

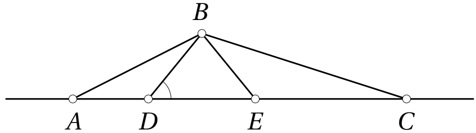


Fig. 32

$BC^2 = EC \cdot AC = 2AD \cdot AC = 2AB^2$. So $BC = \sqrt{2}AB$, meaning that the scale factor between triangles ADB and BEC is $\sqrt{2}$. Thus $DE = BD = \sqrt{2}AD$.

Therefore $\frac{AC}{AD} = \frac{AD+DE+EC}{AD} = \frac{AD+\sqrt{2}AD+2AD}{AD} = 1 + \sqrt{2} + 2 = 3 + \sqrt{2}$.

F15 *(Grade 10.)* A fly farm contains 100001 fruit flies. A research group wishes to buy some flies from the farm to perform (once) either experiment A or experiment B (but not both).

For experiment A, the research group needs a set of flies in which no fly is a descendant of any other. For such a set, the research group would pay the farm 505 euros plus 5 euros per fly bought.

For experiment B, the research group needs a set of flies in which among every two flies, one is a descendant of the other. For such a set, the research group would pay the farm 1000 euros plus 10 euros per fly bought.

Prove that there exists a set of flies in the farm for which the research group would pay at least 3010 euros.

Remark: Any fly can have up to 2 parents.

Solution: For $k = 1, 2, \ldots$, we call a fly a *k-th order parent*, if the greatest suitable set for experiment B containing this fly as the oldest fly consists of exactly k flies.

If there exists a suitable set for experiment B with at least 201 flies, then the research group would pay at least $1000 + 10 \cdot 201 = 3010$ euros for it, which we wanted to show. Now, assuming there is no such set, each suitable set for experiment B contains 200 or fewer flies. Then each fly will be a parent of order at most 200. Since $100001 : 200 > 500$, the pigeonhole principle implies that there exists a k such that at least 501 flies are parents of order exactly k. Out of two flies who are parents of the same order, one can clearly never be a descendant of the other, meaning the set of all parents of order exactly k will be suitable for experiment A. For such a set, the research group would pay at least $505 + 5 \cdot 501 = 3010$ euros, which we wanted to show.

F16 *(Grade 10.)* The radius of the circumcircle of an acute triangle ABC is R and its orthocenter is H. Show that $AH^2 + BC^2 = 4R^2$.

Solution 1: Let O and G be the circumcenter and centroid of ABC respectively and let K be the midpoint of BC (Fig. 33). We know that $AG = 2GK$. Also we know that H, G and O are collinear with $HG = 2GO$ (Euler line). So triangles AHG and KOG are similar with scale factor 2 (by 2 proportional sides and an equal angle between them). Therefore $AH = 2KO$. Now the Pythagorean theorem in triangle KOB yields $KO^2 + KB^2 = OB^2 = R^2$ and $AH^2 + BC^2 = (2KO)^2 + (2KB)^2 = 4\left(KO^2 + KB^2\right) = 4R^2$.

Solution 2: Let B' be the other end of the diameter to the circumcircle of ABC drawn from B (Fig. 34). Since AB' and AB are perpendicular and CH and AB are perpendicular, the lines AB' and CH are parallel. Similarly

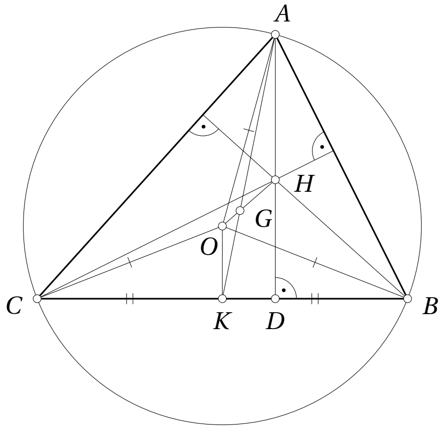


Fig. 33

we see that CB' and AH parallel, meaning that $AHCB'$ is a parallellogram. Thus $CB' = AH$. Now the Pythagorean theorem in triangle BCB' yields $CB'^2 + CB^2 = BB'^2$. As $CB' = AH$ and $BB' = 2R$, the desired result follows.

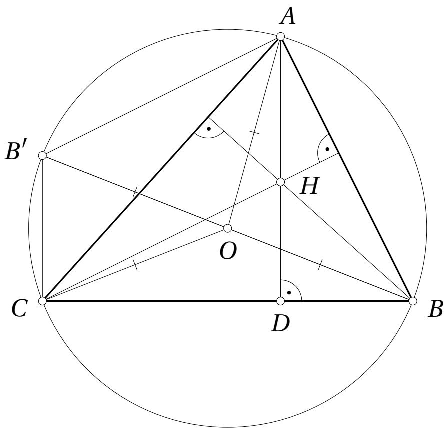


Fig. 34

F17 *(Grade 11.)* Find all polynomials $P(x)$ with integral coefficients and the following property: for any pair (u, v) of positive integers, $\gcd(u, v) = 1$ implies $\gcd(|P(u)|, |P(v)|) = 1$.

Answer: All polynomials of the form $P(x) = \pm x^l$ where l is a non-negative integer.

Solution 1: Firstly, we show that all prime divisors of $P(n)$ are divisors of n. Suppose that there is a prime number q dividing $P(n)$ but not dividing n. Obviously $P(n+q) \equiv P(n) \equiv 0 \pmod q$; but then $\gcd(n, n+q) = 1$ while

$\gcd(|P(n)|, |P(n+q)|) \geqslant q > 1$, contradicting the assumption.
Next we show that only polynomials of the form $P(x) = \pm x^l$ where $l \geqslant 0$ satisfy the conditions of the problem. We proceed by induction on the degree of $P(x)$. If $P(x)$ is a constant polynomial then the constant must be coprime with itself in order to satisfy the conditions of the problem. This is possible only if $P(x) = \pm 1 = \pm x^0$. Assume now that $P(x)$ is a non-constant polynomial of degree l. Then $|P(q)| > 1$ for infinitely many prime numbers q. In each such case, namely q must divide $P(q)$. Hence the constant term of $P(x)$ must be divisible by q. As the constant term does not depend on q, it must be divisible by infinitely many prime numbers. Hence the constant term equals 0, i.e., $P(x) = xP_1(x)$ where $P_1(x)$ is a polynomial with integral coefficients. The polynomial $P_1(x)$ satisfies the conditions of the problem, because $\gcd(|P_1(u)|, |P_1(v)|) > 1$ for coprime u and v would imply $\gcd(|P(u)|, |P(v)|) = \gcd(|uP_1(u)|, |vP_1(v)|) > 1$, contradicting the choice of $P(x)$. By the induction hypothesis, $P_1(x) = \pm x^{l-1}$. Thus $P(x) = \pm x^l$.
On the other hand, all such polynomials satisfy the condition of the problem as $\gcd(u, v) = 1$ implies $\gcd(u^l, v^l) = 1$.
Solution 2: As in Solution 1, we show that all prime divisors of $P(n)$ are divisors of n. Letting q be any prime number, we conclude that $P(q) = \pm q^k$ for some natural number k. Let the degree of $P(x)$ be l. Then $x^{l+1} > |P(x)|$ for every $x > N$ where N is a large enough integer. Thus for infinitely many prime numbers q, there exists a natural number k not exceeding l such that $P(q) = \pm q^k$. As there are finitely many such natural numbers, one can fix k such that either $P(q) = q^k$ for infinitely many primes q or $P(q) = -q^k$ for infinitely many primes q. This implies that either $P(x) = x^k$ or $P(x) = -x^k$ (as a matter of fact, $k = l$). All such polynomials satisfy the conditions of the problem as $\gcd(u, v) = 1$ implies $\gcd(u^l, v^l) = 1$.

F18 *(Grade 11.)* Find all triples (x, y, z) of real numbers that satisfy

$$\begin{cases} \frac{x}{y} + \frac{y}{z} + xy = 3, \\ \frac{y}{z} + \frac{z}{x} + yz = 3, \\ \frac{z}{x} + \frac{x}{y} + zx = 3. \end{cases}$$

Answer: $(1, 1, 1)$, $(-1, -1, -1)$.
Solution: Numbers x, y and z must have the same sign, because if exactly one or two of them are negative then there exists an equation in the system whose all terms in the l.h.s. are negative and cannot sum up to 3. It is also easy to see that (x, y, z) being a solution implies $(-x, -y, -z)$ being a solution, too.
Hence, w.l.o.g., assume that x, y and z are positive. Subtracting the second equation from the first one, the third equation from the second one, and the

first equation from the third one, we obtain the following new system:

$$\begin{cases} x\left(y+\frac{1}{y}\right)=z\left(y+\frac{1}{x}\right), \\ y\left(z+\frac{1}{z}\right)=x\left(z+\frac{1}{y}\right), \\ z\left(x+\frac{1}{x}\right)=y\left(x+\frac{1}{z}\right). \end{cases}$$

The first equation of the new system shows that $x > z$ holds if and only if $y+\frac{1}{y} < y+\frac{1}{x}$, where the latter inequality obviously holds if and only if $y > x$. Similarly, the second equation implies that $y > x$ if and only if $z > y$, and the third equation implies that $z > y$ if and only if $x > z$. Thus any of the inequalities $x > z$, $y > x$ and $z > y$ yields the impossible cycle $x > z > y > x$. Consequently, we must have $x \leqslant z \leqslant y \leqslant x$ which implies $x = y = z$. Every equation of the original system now reduces to $1+1+x^2 = 3$. Hence $x = y = z = 1$.

Besides the positive solution, the system has the corresponding negative solution $(-1,-1,-1)$.

F19 *(Grade 11.)* The orthocenter and the circumcenter of a non-equilateral triangle ABC are H and O, respectively. Let D be the foot of the altitude dropped from the vertex A of the triangle ABC. Prove that $\angle AHO = 90^\circ$ if and only if $\frac{AH}{HD} = 2$.

Solution 1: Let O' be the other endpoint of the diameter of the circumcircle of the triangle ABC drawn from the vertex A and let H' be the reflection of the point H through the line BC (Fig. 35). Then $HD = H'D$, whence the condition $\frac{AH}{HD} = 2$ is equivalent to the condition $\frac{AH}{AH'} = \frac{1}{2} = \frac{AO}{AO'}$. Thus $\frac{AH}{HD} = 2$ if and only if $OH \parallel O'H'$.

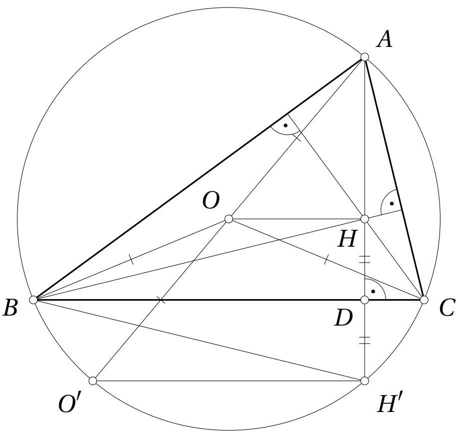


Fig. 35

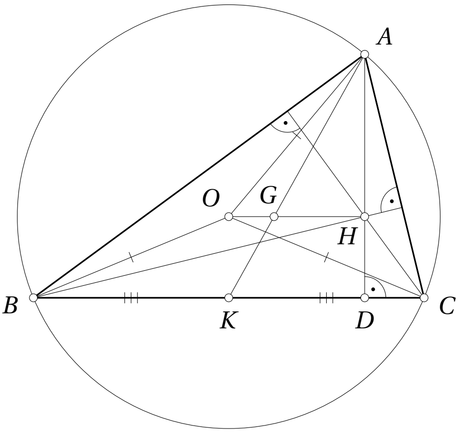


Fig. 36

It is known that H' lies on the circumcircle of the triangle ABC. Thus by Thales' theorem, $\angle AH'O' = 90°$. Hence the condition $OH \parallel O'H'$ is equivalent to the condition $\angle AHO = 90°$. Consequently, $\frac{AH}{HD} = 2$ if and only if $\angle AHO = 90°$.

Solution 2: W.l.o.g., assume $AC \leqslant AB$. Let G be the centroid of the triangle ABC and K be the midpoint of the side BC (Fig. 36). Then $\frac{AG}{GK} = 2$. Thus $\frac{AH}{HD} = 2$ if and only if $\frac{AH}{HD} = \frac{AG}{GK}$, which in turn is valid if and only if $HG \parallel DK$. The latter condition is equivalent to $\angle AHG = \angle ADK = 90°$. But H, G and O lie on the same line (Euler line). Thus $\angle AHG = 90°$ if and only if $\angle AHO = 90°$. Hence $\frac{AH}{HD} = 2$ if and only if $\angle AHO = 90°$.

F20 *(Grade 11.)* A rectangle of integral side lengths is divided into 2022 unit squares. At least one unit square is coloured black. There are equally many black squares in every row and also equally many black squares in every column. Find all possibilities of how many black unit squares there can be in total.

Answer: 2022.

Solution: Let the rectangle be of size $a \times b$. Let there be k black unit squares in every row and l black unit squares in every column; then $ak = bl$. Since $ab = 2022 = 2 \cdot 3 \cdot 337$ where the factors are primes, numbers a and b must be coprime. Thus $b \mid k$, implying $ak = abk' = 2022k'$ for some integer k'. Since at least one black square exists and the total number of unit squares is 2022, the only possibility is $ak = 2022$, i.e., all unit squares are black.

F21 *(Grade 11.)* The teacher writes the digits 20212022 in a row on a blackboard. Juku must write each arithmetic operator $(+, -, \cdot, :)$ exactly once somewhere between these digits in such a way that the result is a correct mathematical expression with a real value, and find this value.

(a) Can Juku obtain the number 0 as the value of the expression?

(b) If the teacher allowed Juku to use parentheses, could Juku obtain expressions with larger values than it would be possible without parentheses?
(c) Prove that there exists a positive integer less than 1000 that cannot be obtained (without using parentheses) as the value of the expression.

Answer: (a) Yes; (b) Yes.

Solution:

(a) One possibility is $2 + 0 \cdot 2120 : 2 - 2 = 0$.

(b) If the teacher allowed using parentheses, Juku could write the expression $(2 - 0 + 2) : 1 \cdot 2022$ whose value is 8088. We show that it is impossible to achieve so big value without using parentheses. For that, we show that the value of multiplication must be less than 5000. Note that placing four operators between eight digits leaves there exactly three pairs of digits lying next to each other without an operator between them. Thus if both factors are numerals then the larger factor contains at most 4 digits and if the larger factor contains 3 digits then the smaller factor contains at most 2 digits. As 2 is the largest digit in use, the product cannot exceed $2222 \cdot 2$ in the first case and $222 \cdot 22$ in the second case. Both bounds are less than 5000. If the first factor is the ratio and the second is a numeral then the value cannot be larger, because the ratio cannot be larger than the dividend. Analogously, we see that an addend that does not involve multiplication can contain at most 4 digits and must be less than 3000, because division or subtraction inside the addend cannot increase it. Consequently, the value of any expression that can be written without parentheses is less than 8000. Thus Juku can obtain larger values when parentheses are allowed.

(c) Placing operators between digits can be done in $7 \cdot 6 \cdot 5 \cdot 4$ ways in total. The number of different values Juku can obtain cannot exceed this number. As $7 \cdot 6 \cdot 5 \cdot 4 = 840 < 1000$, there exists a positive integer less than 1000 that cannot be obtained.

F22 *(Grade 12.)* There are some distinct positive integers written on a blackboard. If we erase the smallest number written on the blackboard, then the ratio of the sum and the product of the remaining numbers will be 4 times greater than the ratio of the sum and the product of the numbers initially on the blackboard. Find all possibilities for the set of numbers that could have been on the blackboard initially.

Answer: $\{5, 20\}$, $\{5, 6, 14\}$, $\{5, 7, 13\}$, $\{5, 8, 12\}$, $\{5, 9, 11\}$, $\{6, 12\}$.

Solution: Clearly there must be at least 2 numbers. Let n be the smallest number, s the sum and k the product of the remaining numbers. We get the equation $4 \cdot \frac{n+s}{nk} = \frac{s}{k}$. Multiplying by nk, we get $4(n + s) = ns$, which rearranges to $(n - 4)(s - 4) = 16$. We know that $16 = 1 \cdot 16 = 2 \cdot 8 = 4 \cdot 4 = 8 \cdot 2 = 16 \cdot 1$ and $n < s$, as n was the smallest number on the blackboard. This leaves the options $n - 4 = 1, s - 4 = 16$ and $n - 4 = 2, s - 4 = 8$ (note that negative factors would also yield negative n and/or s). So $n = 5$ and $s = 20$ or $n = 6$ and $s = 12$.

- Let $n = 5$ and $s = 20$. If there are initially 2 numbers, they must be 5 and 20. If there are 3 numbers, they could be 5, 6 and 14 or 5, 7 and 13 or 5, 8 and 12 or 5, 9 and 11. There can't be 4 or more numbers, because $6 + 7 + 8 > 20$.
- Let $n = 6$ and $s = 12$. If there are initially 2 numbers, they must be 6 and 12. There can't be 3 or more numbers, because $7 + 8 > 12$.

F23 *(Grade 12.)* A function f maps every positive real number to a positive real number. There is a constant $c \neq 1$ such that $f(cx)^2 = f(x)f\left(c^2x\right)$ for all positive real numbers x. Must the same equality hold for any positive real numbers c and x?
Answer: No.
Solution 1: Let $f(x) = \mathrm{e}^{\sin\ln x}$. Then

$$\begin{aligned}(f(cx))^2 &= \left(\mathrm{e}^{\sin\ln(cx)}\right)^2 = \mathrm{e}^{2\sin(\ln x+\ln c)},\\ f(x)f\left(c^2x\right) &= \mathrm{e}^{\sin\ln x}\cdot\mathrm{e}^{\sin\ln\left(c^2x\right)} = \mathrm{e}^{\sin\ln x+\sin(\ln x+2\ln c)}.\end{aligned}$$

Taking $c = e^{2\pi}$, we get

$$\begin{aligned}2\sin(\ln x+\ln c) &= 2\sin(\ln x+2\pi) = 2\sin\ln x = \sin\ln x+\sin\ln x\\ &= \sin\ln x+\sin(\ln x+4\pi)\\ &= \sin\ln x+\sin(\ln x+2\ln c),\end{aligned}$$

so the desired condition is satisfied. However, taking $c = \mathrm{e}^{\frac{\pi}{2}}$ and $x = 1$, we obtain $2\sin(\ln 1+\ln\mathrm{e}^{\frac{\pi}{2}}) = 2\sin\frac{\pi}{2} = 2$, whereas $\sin\ln 1+\sin(\ln 1+2\ln\mathrm{e}^{\frac{\pi}{2}}) = \sin 0+\sin\pi = 0$. Hence the equality does not hold for $c = \mathrm{e}^{\frac{\pi}{2}}$ and $x = 1$.
Solution 2: Let

$$f(x) = \begin{cases} 2, & \text{if } x \text{ is rational,}\\ 1, & \text{if } x \text{ is irrational.}\end{cases}$$

For all positive rational c we have $f(cx) = f(x)$, as x and cx are either both rational or both irrational. So $(f(cx))^2 = (f(x))^2$ and $f(x)f\left(c^2x\right) = f(x)f(cx) = (f(x))^2$. Hence the desired equality holds for all positive rational c. However, taking $c = \sqrt{2}$ and $x = 1$, we obtain $f(cx) = f\left(\sqrt{2}\right) = 1$, whereas $f(x)f\left(c^2x\right) = f(1)f(2) = 2\cdot 2 = 4$. So the equality does not hold.

F24 *(Grade 12.)* A triangle with perimeter P is divided into k triangular pieces for some $k \geqslant 2$.
(a) Show that there exists a piece with perimeter greater than $\frac{P}{k}$.
(b) Show that if the initial triangle is equilateral, then there exists a piece with perimeter at least $\frac{P}{\sqrt{k}}$.
Solution:
(a) The sides of the initial triangle are distributed between the pieces. As $k \geqslant 2$, the sides of the pieces must also pass through the interior of the

initial triangle. So the sum of the perimeters of the k pieces is greater than P. Hence there must exist a piece with perimeter greater than $\frac{P}{k}$.
(b) Let the area of the equilateral triangle be S. Then there must exist a piece Δ with area at least $\frac{S}{k}$. If Δ were equilateral, it would be similar to the initial triangle with a scale factor of $\frac{1}{\sqrt{k}}$, so its perimeter would be $\frac{P}{\sqrt{k}}$. However, among triangles with a fixed area, an equilateral triangle has the smallest perimeter. Therefore the perimeter of Δ is at least $\frac{P}{\sqrt{k}}$.

F25 *(Grade 12.)* Let $n \geqslant 2$ be a positive integer and let $S = \{1, 2, \ldots, n\}$. For $k = 1, 2, \ldots, n-1$, we call two k-element subsets of S *neighbours*, if they have $k-1$ elements in common (i.e. differ by exactly one element). Let $f(n,k)$ be the size of the largest possible collection of k-element subsets of S, in which no two subsets are neighbours. Prove that $f(n,k) \leqslant \binom{n-1}{k-1}$.
Solution 1: For any two subsets belonging to such a collection, the sets of the $k-1$ smallest elements must be different (or else they would be neighbours). There are $\binom{n-1}{k-1}$ ways to choose the $k-1$ smallest elements, since the number n cannot be one of them. Therefore $f(n,k) \leqslant \binom{n-1}{k-1}$, as desired.
Solution 2: For all $n \geqslant 2$ we have $f(n,1) = 1$, since any two 1-element sets are neighbours. So for all $n \geqslant 2$ and $k = 1$ the statement holds. We will prove the general statement by induction by n, taking the base case to be $n = 2$. We will assume that the statement holds for $n-1$ and consider the n-element set $S = \{1, 2, \ldots, n\}$. Let $1 < k \leqslant n-1$. The number of k-element subsets containing the number n can be at most $f(n-1, k-1)$ in a neighbour-free collection, since removing the number n from all of those subsets yields a neighbour-free collection of $k-1$-element subsets of $\{1, 2, \ldots, n-1\}$. The number of k-element subsets not containing n can be at most $f(n-1,k)$. Therefore $f(n,k) \leqslant f(n-1,k-1) + f(n-1,k)$. By the induction assumption $f(n-1,k-1) \leqslant \binom{n-2}{k-2}$ and $f(n-1,k) \leqslant \binom{n-2}{k-1}$. Therefore by Pascal's rule $f(n,k) \leqslant f(n-1,k-1) + f(n-1,k) \leqslant \binom{n-2}{k-2} + \binom{n-2}{k-1} = \binom{n-1}{k-1}$, as desired.

F26 *(Grade 12.)* Show that there exist infinitely many positive integers n such that the integers $1, 2, 3, \ldots, 2n$ can be split into pairs such that the sum of the products of the pairs is divisible by $2n$.
Solution 1: For each prime p we can split the numbers $1, 2, 3, \ldots, 2p$ into the pairs $(1, p+1), (2, p+2), \ldots, (p, 2p)$. The products of the pairs are congruent to $1^2, 2^2, \ldots, (p-1)^2, p^2$ modulo p, so the sum of the products is congruent to $1^2 + 2^2 + \ldots + p^2 = \frac{p(p+1)(2p+1)}{6}$. So for $p > 3$, p divides $1^2 + 2^2 + \ldots + p^2$, as it doesn't divide the denominator. Also, in each pair one of the numbers is even, so the product is even. Hence the sum of the products is divisible by 2 and therefore also by $2p$, as desired.
Solution 2: We use the identity $1 \cdot 2 + 3 \cdot 4 + \ldots + (2n-1) \cdot 2n = \frac{n(n+1)(4n-1)}{3}$.

If n is not divisible by 2 or 3, then $2n \mid \frac{n(n+1)(4n-1)}{3}$, since $2 \mid n+1$ and either $3 \mid n+1$ or $3 \mid 4n-1$, depending on whether $n \equiv -1 \pmod 3$ or $n \equiv 1 \pmod 3$. Therefore splitting the numbers $1, 2, 3, \ldots, 2n$ into the pairs $(1,2), (3,4), \ldots, (2n-1, 2n)$ works for infinitely many n.
Solution 3: For each prime p we can split the numbers $1, 2, 3, \ldots, 2p$ into the pairs $(1, p+1), (2, p+2), \ldots, (p, 2p)$. The products of the pairs are congruent to $1^2, 2^2, \ldots, (p-1)^2, p^2$ modulo p.
Let $p \equiv 1 \pmod 4$ (by Dirichlet's theorem, there are infinitely many such primes), then there exists an integer a such that $a^2 \equiv -1 \pmod p$. If $p > 2$, then clearly $a \not\equiv 1, -1 \pmod p$. Then for all $i = 1, 2, \ldots, p-1$ the numbers $i, ai, a^2 i, a^3 i$ give different remainders modulo p, whereas $a^4 i \equiv i \pmod p$. So the remainders $1, 2, \ldots, p-1$ are split into 4-cycles. The sum of squares of each 4-cycle is divisible by p, as $i^2 + (ai)^2 = (1+a^2)i^2 \equiv (1-1)i^2 = 0 \pmod p$. So the sum $1^2 + 2^2 + \ldots + (p-1)^2$ is divisible by p. Adding also the final term p^2, the divisibility will still hold. Also, in each pair one of the numbers is even, so the product is even. Therefore the sum of the products is divisible by 2 and therefore also by $2p$, as desired.

Selected Problems from the IMO Team Selection Contests

S1 Determine all functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy the inequality $f(x) + f(x+y) \leqslant f(xy) + f(y)$ for all real numbers x, y.
Answer: All constant functions $f(x) = c$ where c is arbitrary real number.
Solution 1: Denote the given inequality by $V(x,y)$. Then $V(x,0)$ together with simplification gives

$$f(x) \leqslant f(0) \tag{3}$$

for every real number x. On the other hand, adding $V(x,y)$ and $V(y,x)$ gives $f(x+y) \leqslant f(xy)$, where taking $y = -x$ leads to $f(0) \leqslant f\left(-x^2\right)$. Along with (3) this implies that

$$f(z) = f(0) \tag{4}$$

for any non-positive real number z. Now $V(z,-1)$ with non-positive z, simplified by (4), gives $f(0) \leqslant f(-z)$. The latter along with (3) implies $f(x) = f(0)$ for all positive real numbers x.
Thus $f(x) = f(0)$ for every real number x, i.e., f is a constant function. All constant functions clearly satisfy the conditions of the problem.
Solution 2: Denote the given inequality by $V(x,y)$.
Firstly, note that $V(x,1)$ along with simplification leads to $f(x+1) \leqslant f(1)$. As $x+1$ takes all real values, the function f obtains its maximum value at 1.
Secondly, note that $V(1,y)$ leads to $f(1) + f(y+1) \leqslant 2f(y)$. Along with the inequality $f(y) \leqslant f(1)$ obtained above, this implies $f(y+1) \leqslant f(y)$

for all real numbers y. By applying the latter inequality to both $y = 0$ and $y = 1$ and taking into account that $f(1)$ is the maximum value of f, one gets $f(1) = f(0) = f(-1)$.
Thirdly, note that $V(-1, y)$ gives $f(-1) + f(y-1) \leqslant f(-y) + f(y)$. As $f(y) \leqslant f(y-1)$ by the above, the inequality $f(-1) \leqslant f(-y)$ must hold for every real number y. Since $-y$ obtains all real values, the function f obtains its minimum value at -1. As $f(1) = f(-1)$, the maximum and minimum value coincide which means that f is a constant function.
All constant functions clearly satisfy the conditions of the problem.

S2 Let p be a fixed prime number. Juku and Miku play the following game. One of the players chooses a natural number a such that $a > 1$ and a is not divisible by p, his opponent chooses any natural number n such that $n > 1$. Miku wins if the natural number written as n ones in the positional numeral system with radix a is divisible by p, otherwise Juku wins. Which player has a winning strategy if:
(a) Juku chooses the number a, tells it to Miku and then Miku chooses the number n;
(b) Juku chooses the number n, tells it to Miku and then Miku chooses the number a?
Answer: (a) Miku; (b) Juku.
Solution: The positional representation with radix a consisting of n ones denotes the sum $a^{n-1} + a^{n-2} + \ldots + a + 1$ which equals $\frac{a^n-1}{a-1}$.
(a) Let $a \equiv 1 \pmod{p}$. If $p > 2$ or $a \equiv 1 \pmod{4}$ then, by the lifting-the-exponent lemma, the exponent of the prime p in the canonical representation of $\frac{a^n-1}{a-1}$ equals that in the canonical representation of n. Thus to win, Miku may choose any number n that is divisible by p. If $p = 2$ and $a \equiv -1 \pmod{4}$ then the exponent of the prime 2 in the canonical representation of $a - 1$ is 1, while that in the canonical representation of $a^2 - 1$ is larger as $a^2 \equiv 1 \pmod{4}$. Thus to win, Miku may choose $n = 2$.
Let now $a \not\equiv 1 \pmod{p}$. By Fermat's little theorem, $a^{p-1} \equiv 1 \pmod{p}$. Hence $a^n \equiv 1 \pmod{p}$ whenever n is a multiple of $p - 1$. Then the numerator of the fraction $\frac{a^n-1}{a-1}$ is divisible by p while the denominator is not, whence the value of the fraction is divisible by p. Consequently, Miku can win by choosing any multiple of $p - 1$ greater than 1 as n.
(b) We show that Juku wins by choosing $n = 2p - 1$. Let a be the number chosen by Miku.
Let $a \equiv 1 \pmod{p}$. If $p > 2$ or $a \equiv 1 \pmod{4}$ then, by the lifting-the-exponent lemma, the exponent of p in the canonical representation of $\frac{a^n-1}{a-1}$ equals that in the canonical representation of n. As p does not divide $2p-1$, it does not divide $\frac{a^n-1}{a-1}$ either. If $p = 2$ and $a \equiv -1 \pmod{4}$ then $a^n \equiv -1 \pmod{4}$, whence 2 does not divide $a^n - 1$. Therefore 2 does not divide $\frac{a^n-1}{a-1}$.
Let now $a \not\equiv 1 \pmod{p}$. By Fermat's little theorem, $a^{p-1} \equiv 1 \pmod{p}$ and

$a^p \equiv a \pmod{p}$, giving $a^n = a^{2p-1} \equiv a \pmod{p}$. Thus $a^n \not\equiv 1 \pmod{p}$. As p does not divide the numerator of the fraction $\frac{a^n-1}{a-1}$, it cannot divide the value of the fraction either.

S3 (a) Is it true that, for arbitrary integer n greater than 1 and distinct positive integers i and j not greater than n, the set of any n consecutive integers contains distinct numbers i' and j' whose product $i'j'$ is divisible by the product ij?

(b) Is it true that, for arbitrary integer n greater than 2 and distinct positive integers i, j, k not greater than n, the set of any n consecutive integers contains distinct numbers i', j', k' whose product $i'j'k'$ is divisible by the product ijk?

Answer: (a) Yes; (b) No.

Solution:

(a) Consider n consecutive integers $k+1, k+2, \ldots, k+n$. As $i \leqslant n$, there must be a multiple of i among them; denote it i'.

As $j \leqslant n$, there must also be a multiple x of j among them. If $x \neq i'$ then one may choose $j' = x$; then the product $i'j'$ is divisible by the product ij. Suppose in the rest that $x = i'$; then i' as a common multiple of i and j is divisible by $\operatorname{lcm}(i, j)$.

Note that if $\gcd(i, j) > \frac{n}{2}$ then $\gcd(i, j)$ could not have two distinct multiples i and j among $1, 2, \ldots, n$. Thus $\gcd(i, j) \leqslant \frac{n}{2}$, which in turn implies that $\gcd(i, j)$ must have two distinct multiples among $k+1, k+2, \ldots, k+n$. At least one of them differs from i'; let that be j'. Then the product $i'j'$ is divisible by the product $\operatorname{lcm}(i, j)\gcd(i, j)$ which equals ij.

(b) Let $n = 143$ and $i = 77 = 7 \cdot 11$, $j = 91 = 7 \cdot 13$, and $k = 143 = 11 \cdot 13$; then $ijk = 7^2 \cdot 11^2 \cdot 13^2$. We show that among 143 consecutive integers $p-71, p-70, \ldots, p+70, p+71$, where $p = 6006 = 2 \cdot 3 \cdot 7 \cdot 11 \cdot 13$, no three distinct numbers can have a product divisible by $7^2 \cdot 11^2 \cdot 13^2$. For that, note that the largest number less than p that is divisible by at least two numbers among 7, 11 and 13 is $p - 77$, and similarly, the least number greater than p that is divisible by at least two numbers among 7, 11 and 13 is $p + 77$. Both lie outside the region under consideration. Consequently, at most one prime among 7, 11 and 13 can belong to the canonical representation of any of the numbers $p-71, p-70, \ldots, p-1, p+1, \ldots, p+70, p+71$. Now choose any three numbers among $p-71, p-70, \ldots, p+70, p+71$.

- Let p be among these three numbers. The primes 7, 11 and 13 have exponent 1 in its canonical representation. As shown above, only one of these three primes can occur in the canonical representation of either of the other two chosen numbers. Thus the product of the chosen three numbers cannot be divisible by $7^2 \cdot 11^2 \cdot 13^2$.
- Let all these three numbers differ from p. As shown above, each of these numbers can be divisible by at most one prime among 7, 11 and 13. Hence, for their product to be divisible by $7^2 \cdot 11^2 \cdot 13^2$, one of the

numbers should be divisible by 13^2. But $\frac{p}{13} = 6 \cdot 77 \equiv -6 \pmod{13}$ which implies $p \equiv -6 \cdot 13 \pmod{13^2}$. Thus the nearest to p numbers divisible by 13^2 are $p - 7 \cdot 13$ and $p + 6 \cdot 13$ which lie outside the region under consideration. Consequently, the product of the chosen three number cannot be divisible by $7^2 \cdot 11^2 \cdot 13^2$.

S4 For any non-negative integer i, denote by d_i the first digit of the number 2^i. Let n be a positive integer. Prove that there exists a non-zero digit that occurs in the tuple $(d_0, d_1, \ldots, d_{n-1})$ less than $\frac{n}{17}$ times.

Solution. The claim obviously holds for $n = 1$, whence assume in the following that $n \geqslant 2$. Let k be the minimal number of occurrences of a non-zero digit in the tuple $(d_0, d_1, \ldots, d_{n-1})$. Then the total number of occurrences of digits 5, 6, 7, 8, 9 is at least $5k$. As each digit 5, 6, 7, 8, 9 that is not the last digit of the tuple is followed by a digit 1, and also $d_0 = 1$, the number of occurrences of digit 1 is at least $5k$. Each digit 1 that is not the last digit of the tuple is followed by either 2 or 3. Thus if the last digit is not 1 then the total number of occurrences of 2 and 3 is at least $5k$. But if the last digit of the tuple is 1 then the first 1 of the tuple was previously not counted, whence the total number of occurrences of digits 2 and 3 is at least $5k$ in this case, too. Therefore, the number of occurrences of the only digit not counted yet, the digit 4, is at most $n - 15k$. As each occurrence of 8 or 9 follows a digit 4, the total number of occurrences of 8 and 9 is also at most $n - 15k$. Hence $n - 15k \geqslant 2k$, implying $k \leqslant \frac{n}{17}$.

To prove that $k < \frac{n}{17}$, suppose that $k = \frac{n}{17}$, i.e., $n = 17k$. This implies that, in the argument above, every inequality must hold as an equality, i.e., each of the digits 5, 6, 7, 8, 9 occurs exactly k times and the digit 1 occurs exactly $5k$ times. As $d_3 = d_{13} = d_{23} = 8$, the digit 8 occurs more than once in the tuple $(d_0, d_1, \ldots, d_{17-1})$ and more than twice in the tuple $(d_0, d_1, \ldots, d_{2 \cdot 17-1})$. Hence $k \geqslant 3$.

We show that each segment consisting of exactly 17 consecutive terms of the tuple contains at least 5 occurrences of the digit 1. Indeed, the last term of the tuple $(2^i, 2^{i+1}, \ldots, 2^{i+16})$ is exactly 65536 times the first term, whence the last term contains at least 4 more digits than the first term. As the first power of 2 containing a certain number of digits definitely starts with 1, the tuple $(2^{i+1}, \ldots, 2^{i+16})$ contains at least 4 terms starting with 1. If also 2^i starts with 1 then there are at least 5 such terms altogether; but if 2^i starts with a larger digit then the last term contains at least 5 more digits than 2^i, implying that there are still 5 terms that start with 1.

It remains to notice that the tuple $(d_0, d_1, \ldots, d_{3 \cdot 17-1})$ contains the digit 1 at least 16 times because $d_0 = 1$ and $2^{3 \cdot 17-1} = 2^{50} = \left(2^{1024}\right)^5 > \left(10^3\right)^5 = 10^{15}$, implying that the number $2^{3 \cdot 17-1}$ has at least 16 digits. As every segment $(d_{17i}, d_{17i+1}, \ldots, d_{17(i+1)-1})$ contains the digit 1 at least 5 times, the digit 1 occurs in the tuple $(d_0, d_1, \ldots, d_{17k-1})$ more than $5k$ times. The contradiction shows that $k < \frac{n}{17}$.

S5 Determine all triples (a, b, c) of integers which satisfy the equation $(a-b)^3(a+b)^2 = c^2 + 2(a-b) + 1$.
Answer: $(0, 1, 0)$, $(-1, 0, 0)$.
Solution. Substituting $x = a - b$, $y = a + b$ we obtain the equation

$$x^3y^2 = c^2 + 2x + 1,$$

which suffices to be solved in integers such that x and y have equal parity. We shall consider the equivalent equation

$$x(x^2y^2 - 2) = c^2 + 1.$$

If both x and y are even then $x^2y^2 - 2$ is even, whence the l.h.s. is divisible by 4. Thus $c^2 \equiv -1 \pmod 4$, but this is impossible since -1 is not a quadratic residue modulo 4.
Let now both x and y be odd; then x^2y^2 is a positive odd number. Consider two cases:

- If $x^2y^2 = 1$ then $x^2 = 1$ and $y^2 = 1$. Since $x^2y^2 - 2 = -1 < 0$ whereas the product $x(x^2y^2 - 2)$ equals the positive number $c^2 + 1$, we must have $x = -1$. The possibilities $x = -1, y = 1$ and $x = -1, y = -1$ give $a = 0, b = 1$ and $a = -1, b = 0$, respectively. In both cases $c = 0$.
- If $x^2y^2 > 1$ then $x^2y^2 - 2 > 0$. Squares of odd numbers give remainder 1 upon division by 4, whence $x^2y^2 - 2 \equiv 1 \cdot 1 - 2 = -1 \pmod 4$. Consequently there exists a prime divisor p of $x^2y^2 - 2$ such that $p \equiv -1 \pmod 4$. But then $c^2 \equiv -1 \pmod p$, which is impossible since -1 is not a quadratic residue modulo p.