Estonian Math Competitions
2022/2023

University of Tartu Youth Academy
Tartu 2023

WE THANK:

Estonian Ministry of Education and Research

University of Tartu

SA Solenum

Cybernetica AS

Robert Kitt

Kadi Liis Saar & Alexey Morgunov

Problem authors: Juhan Aru, Kaarel Hänni, Maksim Ivanov, Urve Kangro, Oleg Koshik, Richard Luhtaru, Härmel Nestra, Erik Paemurru, Ago-Erik Riet, Rasmus Saame, Kaur Aare Saar, Sandra Schumann, Marko Tsengov, Triinu Veeorg, Birgit Veldi, Hendrik Vija, Raili Vilt

Translators: Urve Kangro, Härmel Nestra, Hendrik Vija

Editor: Härmel Nestra

Estonian Mathematical Olympiad

`https://www.math.olympiaadid.ut.ee/`

Mathematics Contests in Estonia

The Estonian Mathematical Olympiad is held annually in three rounds: at the school, town/regional and national levels. The best students of each round (except the final) are invited to participate in the next round.

In each round of the Olympiad, separate problem sets are given to the students of each grade from the 7th to the 12th. In the last two years, the final round was organized also to grades 7 and 8; previously, these two grades participated at school and regional levels only. About 25 students of each grade reach the final round. Some towns, regions and schools organize math olympiads for even younger students. The school round usually takes place in December, the regional round in January or February and the final round in spring in Tartu. The problems for every grade are usually in compliance with the school curriculum of that grade but, in the final round, also problems requiring additional knowledge may be given.

The first problem solving contest in Estonia took place in 1950. The next one, which was held in 1954, is considered as the first Estonian Mathematical Olympiad.

In addition to the Olympiad, open contests take place in September and in December. In addition to students of Estonian middle and secondary schools, Estonian citizens who are studying abroad may also participate in these contests. The participants must have never enrolled in a university or other higher educational institution. The contestants compete in two categories: Juniors and Seniors. In the former category, only students up to the 10th grade may participate. Being successful in the open contests generally assumes knowledge outside the school curriculum.

Based on the results of all competitions during the year, about 20 IMO team candidates are selected. The IMO team selection contest for them is held in April or May in two rounds. Each round is an IMO-style two-day competition with 4.5 hours to solve 3 problems on both days. Some problems in our selection contest are at the level of difficulty of the IMO but easier problems are usually also included.

The problems of previous competitions can be downloaded at the Estonian Mathematical Olympiads website.

Problems Listed by Topic

Number theory: O2, O8, O14, O18, O19, O20, F1, F5, F10, F19, F24, S1, S4
Algebra: O4, O6, O9, O12, O15, O21, F2, F6, F11, F15, F20, F25
Geometry: O1, O3, O10, O13, O16, O22, F3, F7, F12, F14, F16, F21, F26, S3
Discrete mathematics: O5, O1, O5, O17, O23, F4, F8, F9, F13, F17, F18, F22, F23, F27, S2

Problems

Selected Problems from Open Contests

O1 *(Juniors.)* Point X is chosen on the line perpendicular to the bisector of angle AOB and passing through point O ($X \neq O$). One of the sizes of the angles AOB and AOX is half of the other one. Find all possibilities for $\angle AOB$. (Assume that $\angle AOB < 180^\circ$ and $\angle AOX < 180^\circ$.)

O2 *(Juniors.)* Determine all pairs (a, b) of positive integers that satisfy $a! = b^2 + 44$.

O3 *(Juniors.)* A circle passing through vertices B and C of a triangle ABC intersects the sides AB and AC at points D and E, respectively. Let E' be the point symmetric to E w.r.t. the line AB. Let X be a point on the line segment AD. Let Y be the second intersection point of the line CD and the circumcircle of the triangle ACX. Prove that $BE' \parallel XY$.

O4 *(Juniors.)* Find the least possible value of $\frac{(x^2+1)(4y^2+1)(9z^2+1)}{6xyz}$ if the variables x, y and z obtain positive values (not necessarily integers).

O5 *(Juniors.)* Let n be a positive integer. After the IMO, the contestants, team leaders and their new friends, which makes n people altogether, stay for some days in the host country as tourists. They have various expenses and often can a person's expenses be covered by some of the other $n-1$ people. After the trip, they decide to pay all debts. Find the least number of bank transfers sufficient to pay all debts regardless of the distribution of debts, so that nobody's account balance would become negative at any time moment. It is known that everybody's account balance is non-negative in the beginning and would have stayed non-negative if everyone had paid for their own expenses.

O6 *(Juniors.)* The least 100 positive integers are written in the increasing order in a row to form one large number 12345678910111213 . . . . . . 9899100. Then 100 digits of this number are removed. Which digits should be chosen for removal in order to leave the greatest possible number there?

O7 *(Juniors.)* Pippi, Tommy and Annika landed on a desolate islet where cannibals immediately served them one after another for lunch. Marine police arriving in the evening found out the following facts:

(1) Pippi was not served right after Tommy;
(2) Tommy was not served right before Annika;
(3) Annika was not served first.

Can policemen determine based on these facts in which order the cannibals served children for lunch?

O8 *(Juniors.)* Let a, b and x be positive integers. Prove that $x \cdot \gcd(a, b)$ is divisible by $\gcd(x, a) \cdot \gcd(x, b)$.

O9 *(Juniors.)* Solve the system of equations

$$\begin{cases} x + y = z^2, \\ y + z = x^2, \\ z + x = y^2. \end{cases}$$

Reckon with the possibility that the variables might not have integer values.

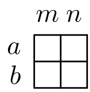
O10 *(Juniors.)* A point D on the bisector of the internal angle of a triangle ABC at its vertex C and a point E on the side AB of the triangle ABC satisfy $\angle BED < 90^\circ$. The circumcircle of the triangle ADE intersects the side AC of the triangle ABC at point F ($F \neq A$, $F \neq C$). The circumcircle of the triangle BDE intersects the side BC of the triangle ABC at point G ($G \neq B$, $G \neq C$). Prove that $DF = DG$.

O11 *(Juniors.)* Let n be a positive integer. How many n-digit palindromes are there? A positive integer is called a *palindrome* if reversing the order of its digits gives the same number.

O12 *(Seniors.)* Let x and y be positive real numbers that satisfy the equality $x + y = x^2 + y^2$. May one be sure that $x = y$?

O13 *(Seniors.)* Find all positive integers n for which there exists a quadrilateral on the coordinate plane whose all vertices have integral coordinates, area is n and two shortest side lengths sum up to n, too.

O14 *(Seniors.)* Let m, n, a, b be distinct positive integers such that $ab = mn$. Into every cell of the 2×2 table in the figure, one writes the greatest common divisor of the number in front of the corresponding row and the number on the top of the corresponding column.



(a) Can it happen that neither of the row products equals the number in front of the row and neither of the column products equals the number on the top of the column?

(b) May one be sure that, whenever $\gcd(m, n) = 1$, both row products equal the number in front of the row and both column products equal the number on the top of the column?

O15 *(Seniors.)* Find all pairs (x, y) of real numbers such that

$$x^9 + 4x^6y + 6x^3y^2 + 4y^3 = 0.$$

O16 *(Seniors.)* A convex quadrilateral $ABCD$ has $\angle ABC = \angle ADC$ and $\angle BAD > \angle DCB$. The bisector of the angle ADC and the perpendicular bisector of the line segment AC meet at point P. Point E on the line segment AC satisfies $\frac{AB}{BC} = \frac{CE}{EA}$. Prove that $\angle EPC = \angle ACB$.

O17 *(Seniors.)* Players M_1 and M_2 play the following game. On the first move, M_1 chooses a positive integer k. Then M_2 chooses integers r_1 and r_2. After that, M_1 and M_2 alternately move a piece on a strip of width 1 that is infinite in both directions and divided into unit squares. The player M_1 starts by moving the piece by 1 unit in one or another direction. On every subsequent move, one must move the piece in one or another direction by 1 more units than the opponent just did. The game ends when one player, say M_i, moves the piece on a square that lies at distance more than k units from the original location of the piece. Thereby, M_i wins if the sum of r_i and the number of the last move (only that player's moves are counted) is even, otherwise the opponent wins. Which player has a winning strategy?

O18 *(Seniors.)* Find all triples (a, b, c) of positive integers satisfying the equality $a! + b! = c!$.

O19 *(Seniors.)* Joonatan got 9 years old. His uncle is 36 years old and grandmother is 64 years old. Uncle claims that such a situation where the age in full years of all three are squares of integers would happen never more even if they all lived infinitely.

(a) May one be sure based on these data that uncle's claim is true?
(b) May one be sure based on these data that uncle's claim is false?

O20 *(Seniors.)*

(a) Has the equation $x^3 + y^3 + z^3 = 2022$ solutions in positive integers?
(b) Has the equation $x^3 + y^3 + z^3 - 3xyz = 2022$ solutions in integers?

O21 *(Seniors.)* Prove for all positive real numbers x and y the inequality

$$\frac{x^5 + y^5}{x + y} + 1 \geqslant x^2 + y^2.$$

O22 *(Seniors.)* In a triangle ABC, the midpoint of side BC is M and the orthocenter is H. Points P and Q are chosen on line BC such that points P, B, M, C, Q occur in this order and $PB = CQ$. The line passing through B perpendicular to PH and the line passing through C perpendicular to QH intersect the line AH at points D and E, respectively. Prove that the lines AM, PD and QE meet in one point.

O23 *(Seniors.)* Andres, Bearu and Crõõt jointly count natural numbers 1 through 2022 in increasing order. Each of them must say exactly 674 numbers, but none of them may say two consecutive numbers. Prove that they have at least 10^{400} possibilities to perform counting according to the rules.

Selected Problems from the Final Round of National Olympiad

F1 *(Grade 7.)* On the table there are 24 unopened decks of cards containing the same number of cards. Janek opens some of the decks and takes $\frac{1}{4}$ of the cards from each opened deck. Then Kaarel opens some decks and takes $\frac{1}{5}$ of the cards from the decks he opened. Together they have now $\frac{1}{6}$ of the cards originally on the table. Find all possibilities of the number of unopened decks still on the table.

F2 *(Grade 7.)* Anna and Paul together have 3 times more candies than Kati. Among the three kids there are two who have together exactly 80 candies. If Anna gave 10 candies to Kati then Anna would have as many candies as Paul and Kati together. How many candies does each kid have?

F3 *(Grade 7.)* On the side AB of a triangle ABC a point D is chosen so that $\angle BCD = 3\angle ACD$, and on the side AC a point E satisfying $CE = DE$ is chosen. Find the size of the angle ACD if CD ja BE are perpendicular and the sum of angles CAD, CDB and CBE is 147°.

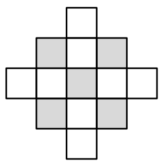
F4 *(Grade 7.)* A rectangular board with positive integer side lengths is divided into unit squares. Two different unit squares are called *neighboring squares* if they share a common side.
Initially, all unit squares are empty. Mati arbitrarily chooses one unit square and writes down the number of its neighboring squares. Then Mati selects another empty unit square and writes down the number of its empty neighboring squares. Mati repeats this step until there are no more empty unit squares left. The sum of all the numbers written by Mati is 22. Find all possible dimensions of the game board.

F5 *(Grade 8.)* Let a, b and c be positive integers satisfying $a + b = 2c$, $2a - 2b = 3c$ and $ab > 2023$. Find the smallest possible c.

F6 *(Grade 8.)* Scooter A travels the first third of the road between Kükametsa and Kapa-Kohila at a speed of 60 km/h. For the rest of the road, scooter A travels at a slower constant speed. Scooter B covers the first third of the same road at a speed of 100 km/h and then continues on the remaining part of the road at a speed higher than 50 km/h. Can scooters A and B reach Kapa-Kohila from Kükametsa in the same amount of time?

F7 *(Grade 8.)* Consider a rectangle $ABCD$. Let E be the midpoint of AB and let F be a point on BC. Denote the point of intersection of CE and DF by G. Find the size of the angle CEF if $CF = CG$ and $DE = DG$.

F8 *(Grade 8.)* Natural numbers from 1 to 13 are written in the squares of a diamond-shaped grid, such that each square contains exactly one number. The sum of numbers in the squares of the longest row, the sum of numbers in the squares of the longest column, and the sum of numbers in the five gray-colored squares are equal. Determine all possible values for the number in the middle square.



F9 *(Grade 8.)* In the figure, there is a grid of dimensions 5×5 filled with numbers 0, 2, and 3. How many different possibilities are there to choose 4 distinct unit squares such that the selected squares can be traversed in a sequence where the numbers in the squares combine to form the number 2023? When traversing the squares, one can move from one square to another only if they share a common side or vertex.

3	3	3	3	3
3	2	2	2	3
3	2	0	2	3
3	2	2	2	3
3	3	3	3	3

F10 *(Grade 9.)* Let a, b and c be integers which satisfy the conditions $ab + bc + ca = 1$ and $a + b = c$. Prove that the product abc is divisible by 3.

F11 *(Grade 9.)* Given numbers x, y, z such that $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{x+y+z}$, prove that $\frac{1}{x^{2023}} + \frac{1}{y^{2023}} + \frac{1}{z^{2023}} = \frac{1}{x^{2023} + y^{2023} + z^{2023}}$.

F12 *(Grade 9.)* Let c be a circle, O its center and AB its diameter. Let M be the midpoint of the segment AO and CD be a chord of the circle c passing through the point M. Let $H \neq D$ be a point on the chord CD, such that $DM = MH$. Find the angle ABC, given that $BH \perp CD$.

F13 *(Grade 9.)* A strip of width 1 is divided into unit squares. Juku and Miku take turns placing pieces one by one on the squares of the strip. A piece can be placed on any square except those already containing a piece and those adjacent to such squares. Juku places the first piece and the player who cannot place a piece loses. Prove that whenever Miku has a winning strategy for a strip of length k, Juku has a winning strategy for strips of lengths $k + 1$, $k + 2$ and $k + 3$.

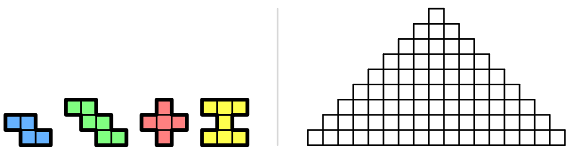
F14 *(Grade 9.)* Juku draws two isosceles triangles. In both triangles, the altitude drawn onto a leg divides the angle at the base into two parts, which have a ratio of $1 : x$, where x is the same positive number for both triangles (it is not known if x is an integer). In both triangles, a vertex can be chosen such that the angles at the chosen vertices are equal. Can we be certain that the two triangles Juku drew are similar?

F15 *(Grade 10.)* A positive integer n is written as a sum of at least two positive integers in such a way that the sum of the squares of the addends is a perfect square. Prove that $(n - x)^2 > 2x$ holds for every addend x.

F16 *(Grade 10.)* Let O be the circumcenter of an acute triangle ABC. Points D and E are chosen on the lines AB and AO respectively, such that the points A, B, D and A, O, E lie on the lines in this order and E lies inside both the triangle ABC and the circumcircle of BCD. Let F be the intersection of the circumcircles of ACE and BCD ($F \neq C$, $F \neq E$). Find $\angle DFE$.

F17 *(Grade 10.)*

(a) In the figure on the left are depicted four types of jigsaw pieces. Is it possible to construct a 9-level pyramid, depicted in the figure on the right, from such pieces? There is an unlimited supply of each type of piece, the pieces can be rotated and reflected, but they may not overlap.



(b) What about a 10-level pyramid?

F18 *(Grade 10.)* Juku and Miku play the following game on a coordinate plane. Both players have a piece which is initially located at the origin. In their turn, a player has to move their piece by one unit in either the positive or negative direction of either the x or y-axis. Juku starts the game by taking k turns, then Juku and Miku take alternating turns in this order with both players taking n turns. Miku wins, if the final distance between the two pieces is an integer, and Juku wins otherwise. Find all pairs (n,k) of positive integers for which Miku has a winning strategy.

F19 *(Grade 11.)* How many consecutive zeros are there at the end of $2022! + 2023!$?

F20 *(Grade 11.)* Do there exist positive real numbers a and b satisfying the system of inequalities

$$\begin{cases} \sqrt{119}a + \sqrt{17}b \leqslant 2ab, \\ a^2 + b^2 \leqslant 2\sqrt{2023}? \end{cases}$$

F21 *(Grade 11.)* The internal and external angle bisectors of the angle at the vertex A of a triangle ABC intersect the circumcircle of ABC at $D \neq A$ and $E \neq A$ respectively. Let F be the intersection of the lines AD and BC and D' the reflection of D over the point F. Prove that the points B, D', E are collinear if and only if $\angle BAC = 2\angle ACB$.

F22 *(Grade 11.)* Let n and m be positive integers. There are n cards in a deck and m distinct symbols on each card, such that every two cards have at most one symbol in common. Prove that there are at least $\frac{1+\sqrt{1+4nm(m-1)}}{2}$ distinct symbols among all of the cards.

F23 *(Grade 11.)* For any positive real number x, we can perform the following operations with the hands of a clock:

(1) Rotate the minute hand by x degrees in either direction, in which case the hour hand rotates by $\frac{x}{12}$ degrees in the same direction.

(2) Rotate both hands by x degrees in either direction.

Find the smallest real number α for which the clock hands can be turned from any position (not necessarily one showing a valid time) into any other position in such a way that the minute hand rotates by a total of α degrees (taking into account rotation in both directions).

F24 *(Grade 12.)* Let n and k be positive integers satisfying $0 < k < n$. Prove that C_n^k is divisible by at least one of the prime factors of n.

F25 *(Grade 12.)* Let $P(x)$ be a polynomial of degree 2023, each of whose coefficients is either 0 or 1. Furthermore let $P(0) = 1$. Prove that every real root of $P(x)$ is less than $\frac{1-\sqrt{5}}{2}$.

F26 *(Grade 12.)* Let ABC be a triangle with $\angle ACB = 90°$. Let F be the foot of the altitude drawn from C. Let the incenters of triangles ABC, ACF

and BCF be I, I_1 and I_2 respectively and let M, M_1 and M_2 be the tangency points of the incircles of ABC, ACF and BCF with the sides AB, CA and CB respectively. Prove that M is both the circumcenter of II_1I_2 and the intersection of M_1I_1 and M_2I_2.

F27 *(Grade 12.)* We call a permutation $(\sigma_1, \sigma_2, \ldots, \sigma_n)$ of the numbers $1, 2, \ldots, n$ *alternating*, if $(-1)^i \sigma_i < (-1)^i \sigma_{i+1}$ holds for all $i = 1, \ldots, n-1$. For all positive integers n let α_n be the proportion of alternating permutations among all permutations of $1, 2, \ldots, n$. (For example $\alpha_1 = \frac{1}{1}$, $\alpha_2 = \frac{1}{2}$, $\alpha_3 = \frac{2}{6}$ etc.) Prove that there exist real numbers c_1, c_2 in the interval $(0; 1)$ and a positive integer N, such that for all positive integers $n \geqslant N$ the inequality $(c_1)^n < \alpha_n < (c_2)^n$ holds.

Selected Problems from the IMO Team Selection Contests

S1 Given a prime number p and integers x and y, find the remainder of the sum $x^0 y^{p-1} + x^1 y^{p-2} + \ldots + x^{p-2} y^1 + x^{p-1} y^0$ upon division by p.

S2 For any natural number n and positive integer k, we say that n is *k-good* if there exist non-negative integers $a_1, \ldots, a_k$ such that

$$n = a_1^2 + a_2^4 + a_3^8 + \ldots + a_k^{2^k}.$$

Is there a positive integer k for which every natural number is k-good?

S3 A convex quadrilateral $ABCD$ has $\angle BAC = \angle ADC$. Let M be the midpoint of the diagonal AC. The side AD contains a point E such that $ABME$ is a parallelogram. Let N be the midpoint of the line segment AE. Prove that the line AC touches the circumcircle of the triangle DMN at point M.

S4 We say that distinct positive integers n, m are *friends* if $|n - m|$ is a divisor of both n and m. Prove that, for any positive integer k, there exist k distinct positive integers such that any two of these integers are friends.

Problems with Solutions

Selected Problems from Open Contests

O1 *(Juniors.)* Point X is chosen on the line perpendicular to the bisector of angle AOB and passing through point O ($X \neq O$). One of the sizes of the angles AOB and AOX is half of the other one. Find all possibilities for $\angle AOB$. (Assume that $\angle AOB < 180^\circ$ and $\angle AOX < 180^\circ$.)

Answer: $36^\circ, 60^\circ, 90^\circ$.

Solution: Denote $\alpha = \angle AOB$.

If $\angle AOB = \frac{1}{2}\angle AOX$ then A and X lie either on the same side or on different sides of the bisector of AOB. In the first case (Fig. 1), $90^\circ - \frac{1}{2}\alpha = 2\alpha$, implying $\alpha = 36^\circ$. In the second case (Fig. 2), $90^\circ + \frac{1}{2}\alpha = 2\alpha$, giving $\alpha = 60^\circ$.

If $\angle AOX = \frac{1}{2}\angle AOB$ then points A and X must lie on different sides of the bisector of the angle AOB (Fig. 3), otherwise $\angle AOX > \angle AOB$ contradicting the assumption. Hence $\alpha = 2\left(90^\circ - \frac{1}{2}\alpha\right)$, giving $\alpha = 90^\circ$.

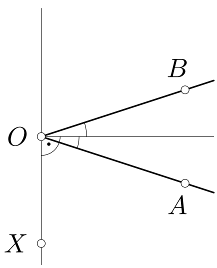


Fig. 1

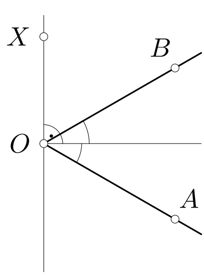


Fig. 2

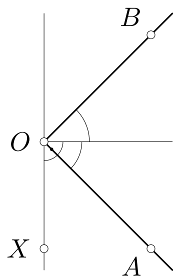


Fig. 3

O2 *(Juniors.)* Determine all pairs (a, b) of positive integers that satisfy $a! = b^2 + 44$.

Answer: $(6, 26)$.

Solution 1: As $b^2 + 44 > 44$ while $1! < 2! < 3! < 4! = 24 < 44$, we have $a > 4$. Hence the product $1 \cdot 2 \cdot \ldots \cdot a$ contains factors 2 and 4, implying that $8 \mid a!$. As 44 is even, b^2 must be even. Hence b is even, i.e., $b = 2c$ for a positive integer c. Thus $b^2 + 44 = (2c)^2 + 44 = 4c^2 + 44 = 4\left(c^2 + 11\right)$. For this number to be divisible by 8, the number $c^2 + 11$ must be even, implying that c^2 must be odd. Hence c is odd, i.e., $c = 2d - 1$ for a positive integer d. Thus $c^2 + 11 = (2d-1)^2 + 11 = 4d^2 - 4d + 12 = 4\left(d(d-1) + 3\right)$. Since $d(d-1)$ is even, $d(d-1) + 3$ is definitely odd, implying that $8 \nmid 4\left(d(d-1)+3\right)$. Consequently, the r.h.s. of the original equation is divisible by 16 but not by 32. Thus the product $1 \cdot 2 \cdot \ldots \cdot a$ must contain factors 2, 4 and 6 but not the factor 8. This leaves the cases $a = 6$ and $a = 7$. If $a = 6$ then $b^2 = 6! - 44 = 720 - 44 = 676$, giving $b = 26$. If $a = 7$ then no solution exists since $7! - 44 = 5040 - 44 = 4996$ and 4996 is not a perfect square.

Solution 2: If $a \geqslant 7$ then $a!$ contains factor 7. As $44 \equiv 2 \pmod 7$, the equation can hold only if $b^2 \equiv 5 \pmod 7$. But 5 is not a quadratic residue modulo 7. Hence $a < 7$. As $b^2 \geqslant 0$, we must have $a! \geqslant 44$, leaving only the cases $a = 5$ and $a = 6$. If $a = 5$ then $b^2 = 120 - 44 = 76$ but 76 is not a perfect square. If $a = 6$ then $b^2 = 720 - 44 = 676 = 26^2$, giving $b = 26$.

O3 *(Juniors.)* A circle passing through vertices B and C of a triangle ABC intersects the sides AB and AC at points D and E, respectively. Let E' be the point symmetric to E w.r.t. the line AB. Let X be a point on the line segment AD. Let Y be the second intersection point of the line CD and the circumcircle of the triangle ACX. Prove that $BE' \parallel XY$.

Solution: We calculate:

$\angle E'BX$

$=$ (because E' and E are symmetric w.r.t. the line AB)

$\angle EBX$

$=$ (because A is on the ray BX)

$\angle EBA$

$=$ (because B, C, D, E are concyclic, BD and CE meet in A)

$\angle DCA$

$=$ (because C, X, Y, A are concyclic, CY and XA meet in D)

$\angle DXY$

$=$ (because B is on the ray XD)

$\angle BXY$.

As E' and Y lie on different sides of the line BX (Figures 4 and 5 show the two possible cases), this implies that the lines BE' and XY are parallel.

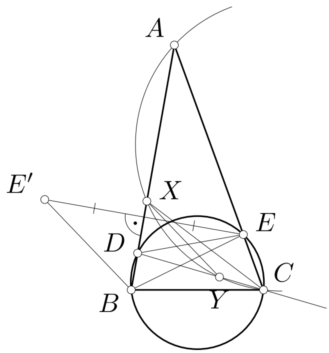


Fig. 4

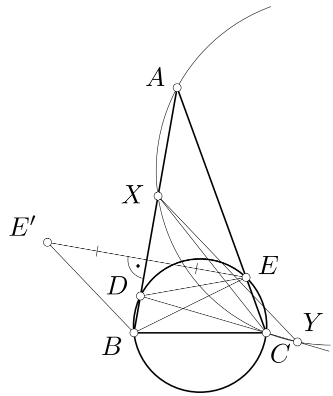


Fig. 5

O4 *(Juniors.)* Find the least possible value of $\frac{(x^2+1)(4y^2+1)(9z^2+1)}{6xyz}$ if the variables x, y and z obtain positive values (not necessarily integers).
Answer: 8.

Solution: If $x = 1, y = \frac{1}{2}, z = \frac{1}{3}$ then the value of the numerator is 8 while that of the denominator is 1. Thus the value of the expression can be 8.
We show that smaller values are impossible. The given expression can be written equivalentlty as $\frac{x^2+1}{x} \cdot \frac{(2y)^2+1}{2y} \cdot \frac{(3z)^2+1}{3z}$. The value of every factor in the last expression is at least 2 by AM-GM. Hence the value of the whole expression is at least 8.

O5 *(Juniors.)* Let n be a positive integer. After the IMO, the contestants, team leaders and their new friends, which makes n people altogether, stay for some days in the host country as tourists. They have various expenses and often can a person's expenses be covered by some of the other $n-1$ people. After the trip, they decide to pay all debts. Find the least number of bank transfers sufficient to pay all debts regardless of the distribution of debts, so that nobody's account balance would become negative at any time moment. It is known that everybody's account balance is non-negative in the beginning and would have stayed non-negative if everyone had paid for their own expenses.
Answer: $n-1$.
Solution 1: Call the difference of a traveller's bank account balance before and after paying all debts the traveller's *total debt*.
We show at first that $n-1$ transfers is sufficient for paying all debts. As n traveller's money is redistributed between themselves, the sum of their account balances does not change during paying the debts. Hence the sum of the traveller's total debts is zero. Suppose that there exists a traveller with a non-zero total debt. Then there must exist a traveller whose total debt is positive and a traveller whose total debt is negative. Let a traveller with a positive total debt make a transfer equal to the traveller's total debt to a traveller with a negative total debt. After that, the total debt of the payer is zero. Moreover, the total debt of all travellers whose total debt was zero before this transfer is still zero. Hence, with $n-1$ bank transfers, the travellers can reach a state where the number of travellers with non-zero total debt is at most one. But as the sum of all total debts is zero, there cannot be exactly one traveller with a non-zero total debt. Consequently, with $n-1$ bank transfers, the travellers can pay all debts.
We now show that $n-1$ bank transfers can be necessary. Suppose that all travellers made expenses and all expenses were covered by one person. Then there are $n-1$ travellers with a positive total debt. As a bank transfer can decrease only the payer's total debt, at least $n-1$ bank transfers are needed to make these $n-1$ traveller's total debt zero.
Solution 2: Define a traveller's total debt as in Solution 1.
We show at first that $n-1$ transfers is sufficient for paying all debts. As n traveller's money is redistributed between themselves, the sum of their account balances does not change during paying the debts. Hence the sum

of the traveller's total debts is zero. Enumerate the travellers by positive integers $1, 2, \ldots, n$. Let the ith traveller's total debt be x_i; by the above, $x_1 + x_2 + \ldots + x_n = 0$. Assume w.l.o.g. that $x_1 \geqslant x_2 \geqslant \ldots \geqslant x_n$. Let the first traveller pay x_1 to the second traveller, then the second traveller pay $x_1 + x_2$ to the third one etc., until the traveller No $n-1$ pay $x_1 + x_2 + \ldots + x_{n-1}$ to the traveller No n. Since $x_1 + x_2 + \ldots + x_n = 0$ and $x_1 \geqslant x_2 \geqslant \ldots \geqslant x_n$, all the payments are non-negative. Altogether, the traveller No i has received $x_1 + \ldots + x_{i-1}$ and paid $x_1 + \ldots + x_i$ which means that the account balance has decreased by x_i. Hence all debts are paid.

We show now that $n-1$ bank transfers can be necessary. Every transfer can increase only one traveller's account balance. Hence if $n-1$ traveller's total debt is negative (they all have covered some expenses done by the last traveller) then $n-1$ transfers are needed to make these $n-1$ traveller's total debt zero.

Solution 3: We show at first that $n-1$ transfers is sufficient for paying all debts. Let all debts be written into some list. Fix a traveller A arbitrarily. If the list contains a debt x of some traveller B to another traveller C, where $B \neq A$ and $C \neq A$, then replace this debt with a debt x of the traveller B to the traveller A and a debt x of the traveller A to the traveller C. This way, we transform the initial list of debts to an equivalent list of debts which are all either those of the traveller A or those to the traveller A. Let any other traveller X add up all their debts to A and subtract the debts of the traveller A to them. All travellers who got a positive difference must pay this difference to A, after which the traveller A must pay all other travellers the absolute value of the difference. This way, all debts will be paid by at most $n-1$ bank transfers.

The necessity of $n-1$ transfers can be proved as in the previous solutions.

Solution 4: We show at first that $n-1$ transfers is sufficient for paying all debts. Consider the situation as a (non-directed) graph where vertices are the travellers and an edge between two vertices exists if and only if one of them has to pay a debt to the other. Suppose that the graph contains a cycle. Let a traveller on this cycle have a debt x to pay to the next traveller on the cycle. By decreasing each traveller's debt to the next on this cycle traveller by x, we obtain an equivalent situation with a smaller number of edges. Continuing this way, we reach a situation corresponding to an acyclic graph. It is known that an acyclic graph with n vertices contains at most $n-1$ edges. Define directions to these edges according to which traveller must pay a positive amount of money to the other one. As there are n vertices and $n-1$ edges, the graph must contain a vertex with no incoming edges. This traveller can pay all debts. After removing this vertex from the graph, there must again be a vertex with no incoming edges, etc.. Hence $n-1$ bank transfers is always sufficient.

The necessity of $n-1$ transfers can be proved as in the previous solutions.

O6 *(Juniors.)* The least 100 positive integers are written in the increasing order in a row to form one large number 12345678910111213 . . . . . . 9899100. Then 100 digits of this number are removed. Which digits should be chosen for removal in order to leave the greatest possible number there?

Answer: All different from 9 digits until the five of the number 57 and also the five in the number 58.

Solution: The number of digits in the remaining number does not depend on the particular digits removed. Hence the larger the initial digits of the remaining number, the larger the number itself is. It is possible to get at most 5 nines to the beginning (from numbers 9, 19, 29, 39 and 49). For that, one must remove all different from 9 digits until the four in the number 49, which is 84 removed digits. Now 16 more digits are to be removed, which does not allow one to obtain nine or eight as the next remaining digit. In order to obtain a seven, one has to remove both digits of numbers 50 through 56 and the five of the number 57. Then one can remove one more digit. Removing the five of the number 58 leaves an eight after the remaining seven, which is the largest option. All in all, the largest number that can remain is 9999978596061 . . . . . . 9899100.

O7 *(Juniors.)* Pippi, Tommy and Annika landed on a desolate islet where cannibals immediately served them one after another for lunch. Marine police arriving in the evening found out the following facts:

(1) Pippi was not served right after Tommy;
(2) Tommy was not served right before Annika;
(3) Annika was not served first.

Can policemen determine based on these facts in which order the cannibals served children for lunch?

Answer: Yes.

Solution: Condition 1 is equivalent to claiming that Tommy was not served right before Pippi. Along with condition 2, it implies that Tommy was not served right before either of the other children. Hence Tommy was served last. By condition 3, the only possibility is that Annika was served second and Pippi was served first. Thus policemen can determine uniquely in which order the cannibals devoured children.

O8 *(Juniors.)* Let a, b and x be positive integers. Prove that $x \cdot \gcd(a, b)$ is divisible by $\gcd(x, a) \cdot \gcd(x, b)$.

Solution 1: We have $\gcd(x, a) \gcd(x, b) \mid xa$ as $\gcd(x, b) \mid x$ and $\gcd(x, a) \mid a$. Similarly, we obtain $\gcd(x, a) \gcd(x, b) \mid xb$. Hence $\gcd(x, a) \gcd(x, b)$ is a common divisor of xa and xb. Therefore $\gcd(x, a) \gcd(x, b) \mid \gcd(xa, xb)$. The desired claim follows as $\gcd(xa, xb) = x \gcd(a, b)$.

Solution 2: Denote by $v_p(n)$ the exponent of prime p in the canonical representation of n. Let p be any prime number and let $\alpha = v_p(a)$, $\beta = v_p(b)$ and $\xi = v_p(x)$. W.l.o.g., $\alpha \geqslant \beta$. Then $v_p(\gcd(a, b)) = \beta$ and, consequently, $v_p(x \cdot \gcd(a, b)) = \xi + \beta$. Consider three cases:

- If $\xi \geqslant \alpha$ then $v_p(\gcd(x,a)) = \alpha$ and $v_p(\gcd(x,b)) = \beta$ which imply $v_p(\gcd(x,a)\cdot\gcd(x,b)) = \alpha+\beta$. As $\xi \geqslant \alpha$, we have $\xi+\beta \geqslant \alpha+\beta$, meaning that $v_p(x\cdot\gcd(a,b)) \geqslant v_p(\gcd(x,a)\cdot\gcd(x,b))$.
- If $\alpha > \xi > \beta$ then $v_p(\gcd(x,a)) = \xi$ and $v_p(\gcd(x,b)) = \beta$ which imply $v_p(\gcd(x,a)\cdot\gcd(x,b)) = \xi+\beta$. Therefore this case implies $v_p(x\cdot\gcd(a,b)) = v_p(\gcd(x,a)\cdot\gcd(x,b))$.
- If $\beta \geqslant \xi$ then $v_p(\gcd(x,a)) = v_p(\gcd(x,b)) = \xi$. These equalities imply $v_p(\gcd(x,a)\cdot\gcd(x,b)) = \xi+\xi$. We have $\xi+\beta \geqslant \xi+\xi$ since $\beta \geqslant \xi$, meaning that $v_p(x\cdot\gcd(a,b)) \geqslant v_p(\gcd(x,a)\cdot\gcd(x,b))$.

As $v_p(x\cdot\gcd(a,b)) \geqslant v_p(\gcd(x,a)\cdot\gcd(x,b))$ in all cases and p is arbitrary, $x\cdot\gcd(a,b)$ is divisible by $\gcd(x,a)\cdot\gcd(x,b)$.

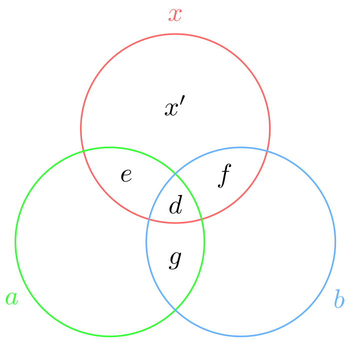


Fig. 6

Solution 3: Let $d = \gcd(x,a,b)$ and $e = \frac{\gcd(x,a)}{d}$, $f = \frac{\gcd(x,b)}{d}$, $g = \frac{\gcd(a,b)}{d}$, $x' = \frac{x}{def}$ (Fig. 6). As d is a common divisor of x and a, a common divisor of x and b, as well as a common divisor of a and b, the numbers e, f and g are integers. Note also that

$$\begin{aligned}\operatorname{lcm}(\gcd(x,a),\gcd(x,b)) &= \frac{\gcd(x,a)\gcd(x,b)}{\gcd(\gcd(x,a)\gcd(x,b))}\\ &= \frac{\gcd(x,a)\gcd(x,b)}{\gcd(x,a,b)} = \frac{de\cdot df}{d} = def.\end{aligned}$$

As x is a common multiple of $\gcd(x,a)$ and $\gcd(x,b)$, this implies that x' is an integer, too. Thus the equalities $x\gcd(a,b) = x'\cdot def\cdot dg = x'g\cdot d^2ef$ and $\gcd(x,a)\gcd(x,b) = de\cdot df = d^2ef$ imply $\gcd(x,a)\gcd(x,b) \mid x\gcd(a,b)$.

O9 *(Juniors.)* Solve the system of equations

$$\begin{cases} x+y = z^2,\\ y+z = x^2,\\ z+x = y^2.\end{cases}$$

Reckon with the possibility that the variables might not have integer values.

Answer: $x=y=z=0$ and $x=y=z=2$.

Solution: Subtract the second equation from the first one. After collecting similar terms in the l.h.s. and factorizing in the r.h.s., we obtain

$$x - z = (z - x)(z + x).$$

This is equivalent to $(x - z)(x + z + 1) = 0$. Thus either $x - z = 0$ or $x + z + 1 = 0$. The latter is impossible as it would imply $x + z = -1 < 0$, but $x + z \geqslant 0$ by the third equation of the original system. Hence $x - z = 0$, i.e., $x = z$. Analogously, we obtain $y = z$. Hence $x = y = z$. Substituting x for both y and z in the original system gives $2x = x^2$. This is equivalent to the equation $x(x - 2) = 0$ whose solutions are $x = 0$ and $x = 2$. Consequently, $(x, y, z) = (0, 0, 0)$ or $(x, y, z) = (2, 2, 2)$.

O10 *(Juniors.)* A point D on the bisector of the internal angle of a triangle ABC at its vertex C and a point E on the side AB of the triangle ABC satisfy $\angle BED < 90°$. The circumcircle of the triangle ADE intersects the side AC of the triangle ABC at point F ($F \neq A$, $F \neq C$). The circumcircle of the triangle BDE intersects the side BC of the triangle ABC at point G ($G \neq B$, $G \neq C$). Prove that $DF = DG$.

Solution 1: By conditions, $AEDF$ is an inscribed quadrilateral (Fig. 7). Thus $\angle AED = 180° - \angle AFD = \angle CFD$. Similarly, we obtain $\angle BED = \angle CGD$. Hence $\angle CFD + \angle CGD = \angle AED + \angle BED = 180°$. Consequently, $CFDG$ is an inscribed quadrilateral, whence $\angle DFG = \angle DCG = \angle DCF = \angle DGF$. Hence $DF = DG$.

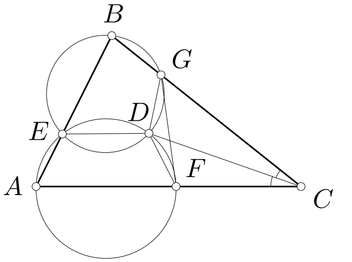


Fig. 7

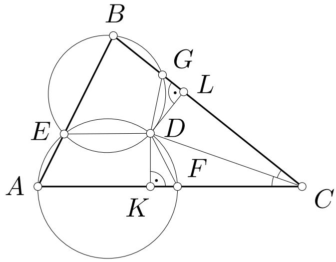


Fig. 8

Solution 2: Let K and L be the projections of point D to sides AC and BC, respectively (Fig. 8). As D is on the bisector, $DK = DL$. Hence the desired claim is equivalent to the claim that the right triangles DKF and DLG are equal. The latter in turn is equivalent to $\angle DFK = \angle DGL$. But the latter is indeed true since, from inscribed quadrilaterals $AEDF$ and $BEDG$,

$$\angle DFK = \angle DFA = 180° - \angle AED = \angle BED = 180° - \angle BGD = \angle DGL.$$

O11 *(Juniors.)* Let n be a positive integer. How many n-digit palindromes are there? A positive integer is called a *palindrome* if reversing the order of its digits gives the same number.

Answer: $9 \cdot 10^{\lfloor \frac{n-1}{2} \rfloor}$, i.e., $9 \cdot 10^{\frac{n}{2}-1}$ for even n and $9 \cdot 10^{\frac{n-1}{2}}$ for odd n.

Solution: If n is even then the first $\frac{n}{2}$ digits can be chosen arbitrarily. The first digit can be chosen in 9 ways (it cannot be zero) and each of the following $\frac{n}{2}-1$ digits can be chosen in 10 ways. This determines the palindrome. Hence the total number of palindromes is $9 \cdot 10^{\frac{n}{2}-1}$.
If n is odd then the first $\frac{n+1}{2}$ digits can be chosen arbitrarily. This determines the palindrome. Similarly to the previous case, the total number of palindromes is $9 \cdot 10^{\frac{n+1}{2}-1}$ which is equivalent to $9 \cdot 10^{\frac{n-1}{2}}$.
Regardless of the parity of n, there are $9 \cdot 10^{\left\lfloor \frac{n-1}{2} \right\rfloor}$ palindromes in total.

O12 *(Seniors.)* Let x and y be positive real numbers that satisfy the equality $x+y=x^2+y^2$. May one be sure that $x=y$?
Answer: No.
Solution 1: Substituting $x=\frac{1}{2}$ leads to the quadratic equation

$$y^2-y-\frac{1}{4}=0,$$

whose positive solution is $y=\frac{1}{2}+\sqrt{\frac{1}{2}}$. In this case $y>x$, whence $x=y$ does not necessarily hold.
Solution 2: By bringing all terms to the l.h.s., we obtain

$$x^2-x+(y^2-y)=0. \tag{1}$$

Treat (1) as a quadratic equation w.r.t. x. Its discriminant is $1-4\left(y^2-y\right)$. If $0<y<1$ then $y^2-y<0$, whence the discriminant is positive and the equation (1) has real solutions. The solution $x=\frac{1+\sqrt{1-4(y^2-y)}}{2}$ is positive. In the case $x=y$, the equation would reduce to $2x=2x^2$ which is equivalent to $2x(x-1)=0$. Because of positivity, the only option would be $x=y=1$. Thus if $0<y<1$ then $x\neq y$.

O13 *(Seniors.)* Find all positive integers n for which there exists a quadrilateral on the coordinate plane whose all vertices have integral coordinates, area is n and two shortest side lengths sum up to n, too.
Answer: All integers greater than 1.
Solution 1: All sides of a polygon whose all vertices have integral coordinates must have length at least 1. Hence the sum of two shortest side length is at least 2, which implies $n \geqslant 2$.
For any $n \geqslant 2$, the trapezoid determined by vertices $O=(0,0)$, $P=(1,0)$, $Q=(2n-1,2)$, $R=(n,2)$ meets the conditions (Fig. 9 depicts the trapezoid in the case $n=5$). Indeed, the sum of lengths of two bases is n and the height is 2, whence the area is n. Bases OP and QR are the two shortest sides of the trapezoid since $OR>n>n-1$ and $PQ>2n-2>n-1$.
Solution 2: As in Solution 1, we show that $n \geqslant 2$.
For any $n \geqslant 2$, the concave quadrilateral determined by vertices $A=(0,0)$, $B=(0,\left\lfloor \frac{n}{2} \right\rfloor)$, $C=(-2,-2)$ and $D=(n-\left\lfloor \frac{n}{2} \right\rfloor,0)$ meets the conditions

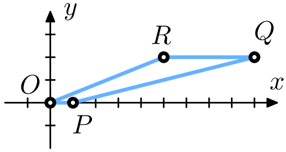


Fig. 9

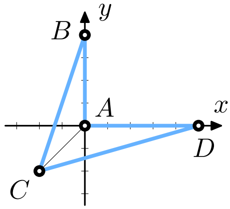


Fig. 10

(Fig. 10 depicts the quadrilateral in the case $n = 9$). Indeed, the areas of the triangles ABC and ADC are $\lfloor \frac{n}{2} \rfloor$ and $n - \lfloor \frac{n}{2} \rfloor$, respectively, whence the area of the quadrilateral $ABCD$ is n. In addition, $AB + AD = n$. The sides AB and AD are two shortest ones since $BC > \lfloor \frac{n}{2} \rfloor - (-2) > n - \lfloor \frac{n}{2} \rfloor \geqslant \lfloor \frac{n}{2} \rfloor$ and $CD > \left(n - \lfloor \frac{n}{2} \rfloor\right) - (-2) \geqslant \lfloor \frac{n}{2} \rfloor - (-2) > n - \lfloor \frac{n}{2} \rfloor \geqslant \lfloor \frac{n}{2} \rfloor$.

O14 *(Seniors.)* Let m, n, a, b be distinct positive integers such that $ab = mn$. Into every cell of the 2×2 table in the figure, one writes the greatest common divisor of the number in front of the corresponding row and the number on the top of the corresponding column.

	m	n
a		
b		

(a) Can it happen that neither of the row products equals the number in front of the row and neither of the column products equals the number on the top of the column?

(b) May one be sure that, whenever $\gcd(m, n) = 1$, both row products equal the number in front of the row and both column products equal the number on the top of the column?

Answer: a) Yes; b) Yes.

Solution 1:

(a) Take $a = p$, $b = p^4$, $m = p^2$ and $n = p^3$ where p is any prime number. Then a, b, m and n are distinct positive integers such that $ab = mn$. Each row product is greater than the number in front of the row and each column product is greater than the number on the top of the column (Fig. 11).

(b) Assume that $\gcd(m, n) = 1$. Let p be an arbitrary prime number and let its exponents in the canonical representations of a, b, m and n be α, β, μ and ν, respectively. The condition $\gcd(m, n) = 1$ implies that either $\mu = 0$ or $\nu = 0$; w.l.o.g., $\nu = 0$. The condition $ab = mn$ now implies $\alpha + \beta = \mu$, whence $\mu \geqslant \alpha$ and $\mu \geqslant \beta$. Thus the exponents of p in the canonical representations of the numbers in the table are those shown in Fig. 12; the numbers in front of the rows and on the top of the columns are also replaced with the corresponding exponents in their canonical representations. Obviously,

	p^2	p^3
p	p	p
p^4	p^2	p^3

Fig. 11

	μ	ν
α	α	0
β	β	0

Fig. 12

each row sum equals the number in front of the row and each column sum equals the number on the top of the column. As this holds for any prime, each row product equals the number in front of the row and each column product equals the number on the top of the column.
Solution 2: We use the distributivity law $k \gcd(x, y) = \gcd(kx, ky)$ repeatedly. Taking the condition $ab = mn$ into account, we obtain

$$\begin{aligned}
\gcd(a,m)\gcd(a,n) &= \gcd(a^2, an, am, mn) \\
&= \gcd(a^2, an, am, ab) = a\gcd(a,n,m,b), \\
\gcd(b,m)\gcd(b,n) &= \gcd(b^2, bn, bm, mn) \\
&= \gcd(b^2, bn, bm, ab) = b\gcd(b,n,m,a), \\
\gcd(a,m)\gcd(b,m) &= \gcd(ab, am, bm, m^2) \\
&= \gcd(mn, am, bm, m^2) = m\gcd(n,a,b,m), \\
\gcd(a,n)\gcd(b,n) &= \gcd(ab, an, bn, n^2) \\
&= \gcd(mn, an, bn, n^2) = n\gcd(m,a,b,n).
\end{aligned}$$

Thus each row product equals $\gcd(a, b, m, n)$ times the number in front of the row and each column product equals $\gcd(a, b, m, n)$ times the number on the top of the column. If $\gcd(m, n) = 1$ then also $\gcd(a, b, m, n) = 1$ which solves part b) of the problem. For part a), choose distinct positive integers a, b, m, n such that $ab = mn$ and $\gcd(a, b, m, n) > 1$; for instance, $a = pqr$, $b = ps$, $m = pq$, $n = prs$, where p, q, r, s are distinct primes.
Remark: Solution 2 demonstrates that the statement of part b) is still true after replacing the assumption $\gcd(m, n) = 1$ with the weaker assumption $\gcd(a, b, m, n) = 1$.

O15 *(Seniors.)* Find all pairs (x, y) of real numbers such that

$$x^9 + 4x^6y + 6x^3y^2 + 4y^3 = 0.$$

Answer: All pairs $(x, -\frac{x^3}{2})$ where x is any real number.
Solution 1: By multiplying both sides by x^3 and adding y^4 to both sides, we obtain $\left(x^3 + y\right)^4 = y^4$. This equality holds if and only if either $x^3 + y = y$ or $x^3 + y = -y$. The first case implies $x = 0$ which, being substituted to the original equation, gives $y = 0$. The second case implies $y = -\dfrac{x^3}{2}$ that entails the solution obtained from the first case. An easy check shows that all the found solutions satisfy the original equation.
Solution 2: Note that $x^9 + 4x^6y + 6x^3y^2 + 4y^3 = \left(x^3 + 2y\right)\left(\left(x^3 + y\right)^2 + y^2\right)$. Thus $x^9 + 4x^6y + 6x^3y^2 + 4y^3 = 0$ holds if and only if either $x^3 + 2y = 0$ or $x^3 + y = y = 0$. The first case is equivalent to $y = -\frac{x^3}{2}$, whence all pairs of the form $\left(x, -\frac{x^3}{2}\right)$ satisfy the equation. The second case implies $x = y = 0$. This solution is included in the family of solutions found in the first case.
Solution 3: Denoting $z = x^3$, we obtain $z^3 + 4yz^2 + 6y^2z + 4y^3 = 0$. Consider

this as a cubic equation w.r.t. z. Factorizing in the l.h.s. gives

$$(z+2y)\left(z^2+2yz+2y^2\right)=0.$$

Thus either $z+2y=0$ or $z^2+2yz+2y^2=0$. The first of these cases implies $y=-\frac{z}{2}=-\frac{x^3}{2}$. Hence the original equation is satisfied by all pairs of the form $\left(x,-\frac{x^3}{2}\right)$. Consider the second case now. The discriminant of the quadratic equation $z^2+2yz+2y^2=0$ is $(2y)^2-4\cdot 2y^2$ which equals $-4y^2$. It is non-negative only if $y=0$. In this case also $z=0$ and $x=0$. The pair $(0,0)$ belongs to the family of solutions found before.

O16 *(Seniors.)* A convex quadrilateral $ABCD$ has $\angle ABC=\angle ADC$ and $\angle BAD>\angle DCB$. The bisector of the angle ADC and the perpendicular bisector of the line segment AC meet at point P. Point E on the line segment AC satisfies $\frac{AB}{BC}=\frac{CE}{EA}$. Prove that $\angle EPC=\angle ACB$.

Solution 1: Point P lies on the circumcircle of the triangle ADC. Take point F such that $ABCF$ is a parallelogram (Fig. 13). As $\angle AFC=\angle ABC=\angle ADC$, the point F lies on the circumcircle of the triangle ADC. By P bisecting the arc AC, we obtain $\angle AFP=\angle PFC$. Hence FP bisects the angle AFC. Let E' be the intersection of lines FP and AC. Then $\frac{CE'}{E'A}=\frac{CF}{FA}$ by the angle bisector theorem. By properties of parallelogram and the conditions of the problem, $\frac{CF}{FA}=\frac{AB}{BC}=\frac{CE}{EA}$. Hence $\frac{CE'}{E'A}=\frac{CE}{EA}$, implying $E=E'$, i.e., E lies on the line FP (Fig. 14). Consequently, $\angle EPC=\angle FPC=\angle FAC=\angle ACB$.

Solution 2: Point P lies on the circumcircle of the triangle ADC. Let G be the point symmetric to B w.r.t. the line AC (Fig. 15). Then G lies on the circumcircle of the triangle ADC since $\angle AGC=\angle ABC=\angle ADC$. Thus $\angle AGP=\angle PGC$, implying that GP bisects the angle AGC.

Let Q be the point symmetric to E w.r.t. the perpendicular bisector of the line segment AC (Fig. 16). As points A and C are symmetric w.r.t. the same line, we have $\frac{AQ}{QC}=\frac{CE}{EA}=\frac{AB}{BC}=\frac{AG}{GC}$. Hence Q lies on the bisector of the angle AGC and $\angle EPC=\angle APQ=\angle APG=\angle ACG=\angle ACB$.

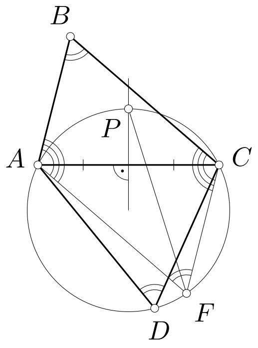


Fig. 13

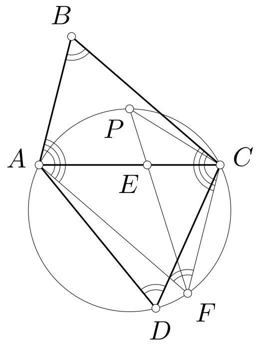


Fig. 14

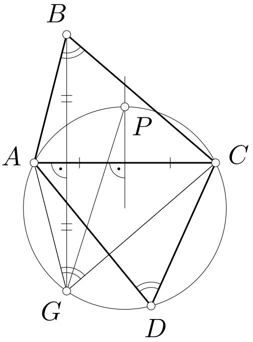


Fig. 15

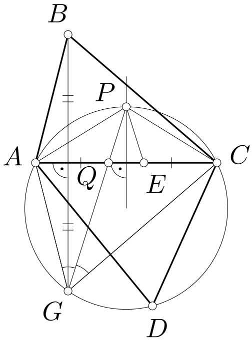


Fig. 16

O17 *(Seniors.)* Players M_1 and M_2 play the following game. On the first move, M_1 chooses a positive integer k. Then M_2 chooses integers r_1 and r_2. After that, M_1 and M_2 alternately move a piece on a strip of width 1 that is infinite in both directions and divided into unit squares. The player M_1 starts by moving the piece by 1 unit in one or another direction. On every subsequent move, one must move the piece in one or another direction by 1 more units than the opponent just did. The game ends when one player, say M_i, moves the piece on a square that lies at distance more than k units from the original location of the piece. Thereby, M_i wins if the sum of r_i and the number of the last move (only that player's moves are counted) is even, otherwise the opponent wins. Which player has a winning strategy?
Answer: M_2.
Solution: Enumerate the squares by integers in such a way that the original location of the piece is on square 0. Then the numbers of moves (only one player's moves are counted) have the same parity as the target squares of the moves. Let M_2 choose r_1 and r_2 such that $k + r_1$ is even and $k + r_2$ is odd. In the rest of the game, let M_2 make moves in such a way that the absolute value of the number of the target square were as small as possible, except if M_2 can win in one move. In order to show that this is a winning strategy, note that the number i of the square where the piece can be after both players have made n moves, assuming that the game has not yet ended, satisfies $-n \leqslant i \leqslant n$. Indeed, this condition holds for $n = 1$ (the piece moves either $0 \to 1 \to -1$ or $0 \to -1 \to 1$) and if the piece is on the square i after both players having made n moves, where $-n \leqslant i \leqslant n$, then the possible continuations are $i \to i - (2n+1) \to i+1$ and $i \to i + (2n+1) \to i-1$, whereby $-(n+1) \leqslant i-1 < i+1 \leqslant n+1$. Suppose now that M_1 wins the game. Consider two cases:

- Suppose that M_1 wins on their own move. W.l.o.g., let the last move be from the square i to the square $i + 2n + 1 > k$. As M_1 cannot win on the first move done with piece, the piece must have arrived to the square i as the result of the previous move of M_2. By the above, $i \leqslant n$. If the previous move was from the square $i - 2n$ to the square i then $i - 4n \geqslant -k$, otherwise M_2 would have moved to the square $i - 4n$ and won. Summing up the inequalities $i + 2n + 1 > k$ and $i - 4n \geqslant -k$ gives $2i - 2n + 1 > 0$ which is equivalent to $i \geqslant n$. As $i \leqslant n$, this implies $i = n$. Thus $k < i + 2n + 1 = 3n + 1$ and $-k \leqslant i - 4n = -3n$ which together imply $k = 3n$. But then k and n have the same parity. By the strategy of M_2, the numbers $n + r_2$ and $n + 1 + r_1$ are odd, whence M_1 could not win on their next move.

If the previous move was from the square $i + 2n$ to the square i then $i + 2n \leqslant k$. As $i + 2n + 1 > k$, we have $i + 2n = k$. But this contradicts the strategy of M_2 since M_2 could move from the square k to the square $k + 2n$ and win.

- Suppose that M_1 wins on the opponent's move. W.l.o.g., let the last move be from the square i to the square $i + 2n$, where $i + 2n > k$. This means that $i \leqslant 0$, otherwise M_2 would have moved to the square $i - 2n$. If the previous move by M_1 was from the square $i - (2n - 1)$ to the square i then $i - (2n - 1) \geqslant -k$. Summing up the inequalities $i + 2n > k$ and $i - (2n - 1) \geqslant -k$ gives $2i + 1 > 0$ which implies $i \geqslant 0$. Consequently, $i = 0$ and $k = 2n - 1$. As the move number n done by M_2 ends on the square with the even number $2n$, we know that n must be even. Thus k and n have different parities. If the previous move by M_1 was from the square $i + (2n - 1)$ to the square i then $i + (2n - 1) \leqslant k$ which implies $k = i + (2n - 1)$. As $k \neq 0$, it is not the first move with piece in the game. The player M_2 must have moved the piece to the square $i + (2n - 1)$ from the square $i + 1$, implying that $i + 1 \leqslant 0$. We must also have $(i + 1) - (2n - 2) \geqslant -k$, otherwise M_2 could have won on the previous move. This implies $i + 1 = 0$ and $k = 2n - 2$. As the move number n by M_2 targets on the square with the odd number $2n - 1$, also n must be odd. Thus, again, k and n have different parities. By the strategy of M_2, this means that $n + r_2$ is even, whence M_2 cannot lose on the move number n.

Since all cases led to contradiction, M_2 must have a winning strategy.

O18 *(Seniors.)* Find all triples (a, b, c) of positive integers satisfying the equality $a! + b! = c!$.

Answer: $(1, 1, 2)$.

Solution 1: Assume w.l.o.g. that $a \leqslant b < c$.

If $a = b$ then $2 \cdot a! = c!$ which implies $(a + 1)(a + 2) \ldots c = 2$. This is possible only if $c = 2$ and $a = 1$. Hence $(a, b, c) = (1, 1, 2)$.

If $a < b$ then $a! + b! = a! \cdot (1 + (a + 1)(a + 2) \ldots b)$. Dividing the original equation by $a!$ gives the equivalent equation

$$1 + (a + 1)(a + 2) \ldots b = (a + 1)(a + 2) \ldots c.$$

Here the r.h.s. is divisible by $a+1$, but the l.h.s. differs from a multiple of $a+1$ by 1. Hence the l.h.s. is relatively prime to $a+1$ and can be divisible by $a+1$ only if $a+1=1$, i.e., $a=0$. Thus no more solutions exist.

Solution 2: Assume w.l.o.g. that $a \leqslant b < c$.

If $a = b$ then $2 \cdot a! = c!$ which implies $(a+1)(a+2)\dots c = 2$. This is possible only if $c=2$ and $a=1$. Hence $(a,b,c)=(1,1,2)$.

If $a < b$ then $a! \leqslant \frac{1}{2} \cdot b!$, whence $a!+b! \leqslant \frac{1}{2} \cdot b! + b! = \frac{3}{2} \cdot b! < 2 \cdot b! \leqslant c!$. Thus no more solutions exist.

O19 *(Seniors.)* Joonatan got 9 years old. His uncle is 36 years old and grandmother is 64 years old. Uncle claims that such a situation where the age in full years of all three are squares of integers would happen never more even if they all lived infinitely.

(a) May one be sure based on these data that uncle's claim is true?

(b) May one be sure based on these data that uncle's claim is false?

Answer: a) No; b) No.

Solution:

(a) Suppose that grandmother's birthday will become next. On grandmother's birthday 160 years later, Joonatan will be 169 years old and uncle will be 196 years old. Grandmother will get $64+160+1$, i.e., 225 years old. Thus the given data allow situations where uncle's claim is not true.

(b) Suppose that Joonatan, uncle and grandmother were all born on the same date. Then the differences of their ages are constant. Thus if, on any time moment, grandmother will be a^2 years old while uncle will be b^2 years old, then $a^2-b^2=28$. As $a^2-b^2=(a-b)(a+b)$ whereas $a-b$ and $a+b$ have the same parity, we must have $a-b=2$ and $a+b=14$, because $2 \cdot 14$ is the only way to represent 28 as the product of two positive integers of the same parity. This implies $a=8$ and $b=6$ which is the current situation. Thus the given data also allow situations where uncle's claim is not false.

O20 *(Seniors.)*

(a) Has the equation $x^3+y^3+z^3=2022$ solutions in positive integers?

(b) Has the equation $x^3+y^3+z^3-3xyz=2022$ solutions in integers?

Answer: a) No; b) No.

Solution 1: Cubes of integers are congruent to 1, 0 and -1 modulo 9. The common r.h.s. of both equations is 2022 which is congruent to 6 modulo 9.

(a) As $x^3+y^3+z^3 \equiv 6 \equiv -3 \pmod 9$, we have $x^3 \equiv y^3 \equiv z^3 \equiv -1 \pmod 9$. This is possible only if $x \equiv y \equiv z \equiv -1 \pmod 3$. As 2022 is divisible by 2 but not by 8, exactly one of x, y, z is even. Since $14^3 > 2022$, even values can be 2 and 8 and odd values can be 5 and 11. But $2 \cdot 11^3$ is too large while $11^3+5^3+8^3$ is too small.

(b) Suppose that $3 \mid x$. Then $9 \mid x^3$ and $9 \mid 3xyz$, whence $y^3+z^3 \equiv 6 \equiv -3 \pmod 9$. This is impossible. A similar contradiction arises if $3 \mid y$ or $3 \mid z$. Hence none of x, y, z is divisible by 3. On the other hand, as $3 \mid 3xyz$ and

$3 \mid 2022$, we must also have $3 \mid x^3+y^3+z^3$. This is possible only if either $x^3 \equiv y^3 \equiv z^3 \equiv 1 \pmod 9$ or $x^3 \equiv y^3 \equiv z^3 \equiv -1 \pmod 9$. In the case $x^3 \equiv y^3 \equiv z^3 \equiv 1 \pmod 9$, the only option is $x \equiv y \equiv z \equiv 1 \pmod 3$. Thus $xyz \equiv 1 \pmod 3$ and $3xyz \equiv 3 \pmod 9$, leading to contradiction as $x^3+y^3+z^3-3xyz \equiv 0 \pmod 9$. If $x^3 \equiv y^3 \equiv z^3 \equiv -1 \pmod 9$ then $x^3+y^3+z^3 \equiv -3 \equiv 6 \pmod 9$, implying $3xyz \equiv 0 \pmod 9$. This is also not possible if x, y, z are not divisible by 3.

Solution 2:

(a) Assume w.l.o.g. that $x \geqslant y \geqslant z$. As $x^3 < 2022$, we have $x \leqslant 12$. On the other hand, $3x^3 \geqslant 2022$ implies $x \geqslant 9$. Hence $9 \leqslant x \leqslant 12$.

If $x=9$ then $y^3+z^3 = 2022-729 = 1293$. As $2 \cdot 8^3 < 1293$, the only option is $y=9$. Thus $z^3 = 1293-729 = 564$, but 564 is not a cube of integer.

If $x=10$ then $y^3+z^3 = 2022-1000 = 1022$. As $2 \cdot 7^3 < 1022$, we get $y \geqslant 8$. An easy check shows that $8 \leqslant y \leqslant 10$ does not give integral solutions.

If $x=11$ then $y^3+z^3 = 2022-1331 = 691$. As $2 \cdot 7^3 < 691$, we must have $y \geqslant 8$. But $y=8$ does not give an integral solution and $9^3 > 691$.

Finally, if $x=12$ then $y^3+z^3 = 2022-1728 = 294$. As $2 \cdot 5^3 < 294$, we must have $y \geqslant 6$. But $y=6$ does not give an integral solution and $7^3 > 294$.

(b) We can factorize the l.h.s. of the equation as

$$\begin{aligned} x^3+y^3+z^3-3xyz &= (x+y+z)\left(x^2+y^2+z^2-xy-yz-zx\right) \\ &= (x+y+z)\left((x+y+z)^2-3(xy+yz+zx)\right). \end{aligned}$$

Thus $x+y+z \mid 2022$. If $3 \mid x+y+z$ then both factors would be divisible by 3, whence $9 \mid x^3+y^3+z^3-3xyz$. But $9 \nmid 2022$. On the other hand, if $3 \nmid x+y+z$ then neither factor would be divisible by 3, implying that $3 \nmid x^3+y^3+z^3-3xyz$. But $3 \mid 2022$. Hence no solutions in integers exist.

Remark: Dropping positivity condition in part a) would enable the solution $\{x=38, y=-13, z=-37\}$ and all its permutations.

O21 *(Seniors.)* Prove for all positive real numbers x and y the inequality

$$\frac{x^5+y^5}{x+y}+1 \geqslant x^2+y^2.$$

Solution 1: It is enough to show that

$$\frac{x^5+y^5}{x+y}+1-\left(x^2+y^2\right) \geqslant 0. \tag{2}$$

The l.h.s. of (2) can be represented as $\frac{x^5+y^5+x+y-x^3-x^2y-xy^2-y^3}{x+y}$. As the denominator of this fraction is positive, it suffices to show that the numerator of the fraction is non-negative. Note that

$$\begin{aligned} & x^5+y^5+x+y-x^3-x^2y-xy^2-y^3 \\ &= x^5+y^5+x+y-2x^3+x^3-x^2y-xy^2-2y^3+y^3 \\ &= x^5-2x^3+x+y^5-2y^3+y+x^3-x^2y-xy^2+y^3 \\ &= x\left(x^4-2x^2+1\right)+y\left(y^4-2y^2+1\right)+\left(x^2-y^2\right)(x-y) \\ &= x\left(x^2-1\right)^2+y\left(y^2-1\right)^2+(x-y)^2(x+y). \end{aligned}$$

All summands in the obtained expression are non-negative since x and y are non-negative and squares of real numbers are non-negative. Hence the whole expression is non-negative.

Solution 2: Multiplying the equation by $x + y$ gives the equivalent equation

$$x^5 + y^5 + x + y \geqslant x^3 + x^2y + xy^2 + y^3. \qquad (3)$$

By AM-GM,

$$\frac{x^5 + x}{2} \geqslant \sqrt{x^6} = x^3, \qquad (4)$$

$$\frac{y^5 + y}{2} \geqslant \sqrt{y^6} = y^3, \qquad (5)$$

$$\frac{x^3 + xy^2}{2} \geqslant \sqrt{x^4y^2} = x^2y, \qquad (6)$$

$$\frac{y^3 + x^2y}{2} \geqslant \sqrt{x^2y^4} = xy^2. \qquad (7)$$

By adding up inequalities (4), (5), (6) and (7), we obtain

$$\frac{1}{2}x^5 + \frac{1}{2}x + \frac{1}{2}y^5 + \frac{1}{2}y + \frac{1}{2}x^3 + \frac{1}{2}xy^2 + \frac{1}{2}y^3 + \frac{1}{2}x^2y \geqslant x^3 + y^3 + x^2y + xy^2$$

which is equivalent to

$$\frac{1}{2}x^5 + \frac{1}{2}y^5 + \frac{1}{2}x + \frac{1}{2}y \geqslant \frac{1}{2}x^3 + \frac{1}{2}x^2y + \frac{1}{2}xy^2 + \frac{1}{2}y^3.$$

Multiplying the last inequality by 2 gives the desired inequality (3).

Solution 3: We prove that

$$\frac{x^5 + y^5}{x + y} + 1 \geqslant \frac{x^4 + y^4}{2} + 1 \geqslant x^2 + y^2. \qquad (8)$$

For the first inequality, it is sufficient to show $\frac{x^5+y^5}{x+y} \geqslant \frac{x^4+y^4}{2}$ which is equivalent to $x^5 + y^5 \geqslant x^4y + xy^4$. For proving this inequality, we write the l.h.s. in the form $\frac{4x^5+y^5}{5} + \frac{x^5+4y^5}{5}$. Applying AM-GM to x^5, x^5, x^5, x^5, y^5 and then also to x^5, y^5, y^5, y^5, y^5 gives us $\frac{4x^5+y^5}{5} \geqslant \sqrt[5]{(x^5)^4 y^5} = x^4y$ and $\frac{x^5+4y^5}{5} \geqslant \sqrt[5]{x^5 (y^5)^4} = xy^4$, respectively. The desired inequality follows.

For the second inequality of (8), we apply AM-GM to x^4, 1 and then to y^4, 1. We obtain $\frac{x^4+1}{2} \geqslant \sqrt{x^4} = x^2$ and $\frac{y^4+1}{2} \geqslant \sqrt{y^4} = y^2$, respectively. Thus

$$\frac{x^4 + y^4}{2} + 1 = \frac{x^4 + 1}{2} + \frac{y^4 + 1}{2} \geqslant x^2 + y^2.$$

O22 *(Seniors.)* In a triangle ABC, the midpoint of side BC is M and the orthocenter is H. Points P and Q are chosen on line BC such that points P, B, M, C, Q occur in this order and $PB = CQ$. The line passing through B perpendicular to PH and the line passing through C perpendicular to QH

intersect the line AH at points D and E, respectively. Prove that the lines AM, PD and QE meet in one point.

Solution: Let the lines PD and QE intersect in F (Fig. 17). Since $HD \perp BP$ and $BD \perp HP$, point D is the orthocenter of the triangle BHP, implying that $PD \perp BH$. Together with $AC \perp BH$, this implies $PD \parallel AC$. Analogously, we get $QE \parallel AB$. Thus the triangles ABC and FQP are similar. Note that $PM = PB + BM = MC + CQ = MQ$ which shows that M is the midpoint of PQ. Thus the triangles ABM and FQM are similar. Consequently, $\angle AMB = \angle FMQ$, implying that AMB and FMQ are vertical angles and the line AM passes through F.

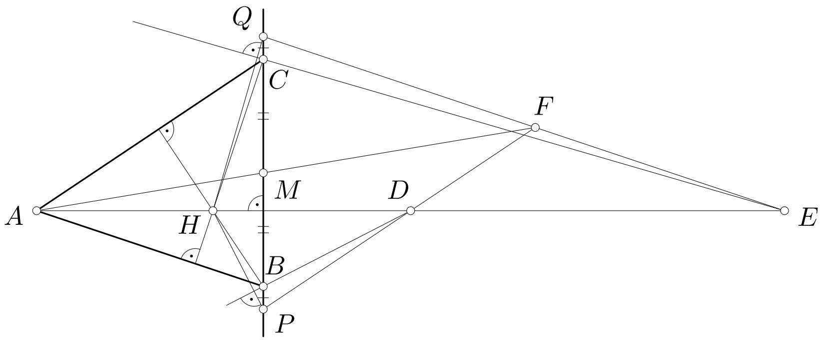


Fig. 17

O23 *(Seniors.)* Andres, Bearu and Crõõt jointly count natural numbers 1 through 2022 in increasing order. Each of them must say exactly 674 numbers, but none of them may say two consecutive numbers. Prove that they have at least 10^{400} possibilities to perform counting according to the rules.

Solution 1: If we split the numbers into groups of three consecutive numbers and numbers in every group are said by different parties then Andres, Bearu and Crõõt will say equally many numbers. In addition, they must assure that the last number of any group and the first number of the next group are said by different parties. Thus there are 6 ways to say the numbers in the first group and 4 ways to say the number in each following group. Hence there are $6 \cdot 4^{673}$ ways to say all numbers. It remains to note that $6 \cdot 4^{673} > 4^{670} = (4^5)^{134} = 1024^{134} > (10^3)^{134} = 10^{402} > 10^{400}$.

Solution 2: We begin with studying how many ways there are to choose the numbers Andres must say. The tuple $(a_1, a_2, \ldots, a_{674})$ of numbers said by Andres is determined by the tuple $(x_1, x_2, \ldots, x_{674})$ of differences where $a_1 = x_1$ and $a_i = a_{i-1} + x_i$ for every $i > 1$, if we require that $x_i \geqslant 2$ for every $i > 1$ and $x_1 + x_2 + \ldots + x_{674} \leqslant 2022$. The tuple $(x_1, x_2, \ldots, x_{674})$ is determined by the tuple $(y_1, y_2, \ldots, y_{674})$ where $x_1 = y_1$ and $x_i = y_i + 1$ for every $i > 1$, but now the side condition is $y_1 + y_2 + \ldots + y_{674} \leqslant 1349$. Such tuple $(y_1, y_2, \ldots, y_{674})$ is determined by a subset $\{z_1, z_2, \ldots, z_{674}\}$ of

$\{1, 2, \ldots, 1349\}$ (assuming that $z_1 < z_2 < \ldots < z_{674}$) where $y_1 = z_1$ and $y_i = z_i - z_{i-1}$ for every $i > 1$. There are C_{1349}^{674} ways to choose such subset. Note that $C_{1349}^{674} = C_{1349}^{1349-674} = C_{1349}^{675}$, while $C_{1349}^{674} + C_{1349}^{675} = C_{1350}^{675} = C_{2\cdot 675}^{675}$ by Pascal's rule. Thus $C_{1349}^{674} = \frac{1}{2} C_{2\cdot 675}^{675}$.

Having chosen the numbers Andres must say, all other numbers must be distributed between Bearu and Crõõt in such a way that neither of them gets any two consecutive numbers. Consider these numbers as segments of consecutive numbers disrupted by Andres' numbers only. Let there by m such intervals; obviously $673 \leqslant m \leqslant 675$. Bearu and Crõõt must say the numbers in each interval alternately, whence the sayer of the first number of each interval determines the sayers of all numbers in the interval. If there are even number of numbers in every interval then there are 2 ways to choose the sayer of the first number in each interval; this makes altogether 2^m ways to distribute numbers between Bearu and Crõõt. If $2k$ intervals contain an odd number of numbers and l intervals contain an even number of numbers then there are $C_{2k}^{k} \cdot 2^l$ ways to distribute numbers between Bearu and Crõõt, because in exactly k intervals with an odd number of numbers, Bearu must say the first number.

It is a well-known fact that $C_{2k}^{k} \geqslant \frac{2^{2k}}{2\sqrt{k}}$ for every $k \geqslant 1$. Hence there are at least $\frac{1}{2} \cdot \frac{2^{1350}}{2\sqrt{675}}$ ways to choose Andres' numbers. Using the notation of the previous paragraph, distributing other numbers between Bearu and Crõõt can be done in at least $\frac{2^{2k+l}}{2\sqrt{k}}$ ways whenever $k > 0$. Since $2k + l = m \geqslant 673$ and $k \leqslant \frac{m}{2} \leqslant 337$, distributing other numbers can be done in at least $\frac{2^{673}}{2\sqrt{337}}$ ways. The same estimation holds for $k = 0$. So the number of ways to say all numbers is at least $\frac{1}{2} \cdot \frac{2^{1350}}{2\sqrt{675}} \cdot \frac{2^{673}}{2\sqrt{337}}$, i.e., at least $\frac{2^{2020}}{\sqrt{675\cdot 337}}$. As $675 < 2^{10}$ and $337 < 2^9$, it holds that $\sqrt{673 \cdot 337} < 2^{\frac{10+9}{2}} < 2^{10}$. Thus the number of ways of correct counting is larger than 2^{2010}.

Finally, note that $2^{2010} > 2^{2000} = \left(2^5\right)^{400} = 32^{400} > 10^{400}$.

Remark 1: The inequality $C_{2k}^{k} \geqslant \frac{2^{2k}}{2\sqrt{k}}$ used in Solution 2 can be easily proven as follows. We know that

$$C_{2k}^{k} = \frac{(2k)!}{k!\cdot k!} = \frac{(1\cdot 3\cdot \ldots \cdot (2k-1))\cdot(2\cdot 4\cdot \ldots \cdot 2k)}{(1\cdot 2\cdot \ldots \cdot k)\cdot(1\cdot 2\cdot \ldots \cdot k)}$$

$$= \frac{(1\cdot 3\cdot \ldots \cdot (2k-1))\cdot 2^k}{1\cdot 2\cdot \ldots \cdot k} = \frac{(1\cdot 3\cdot \ldots \cdot (2k-1))\cdot 2^{2k}}{2\cdot 4\cdot \ldots \cdot 2k} = \left(\frac{3}{2} \cdot \frac{5}{4} \cdot \ldots \cdot \frac{2k-1}{2k-2}\right) \cdot \frac{2^{2k}}{2k}.$$

As $\left(\frac{2j-1}{2j-2}\right)^2 = \frac{(2j-1)^2}{(2j-2)^2} = \frac{4j^2-4j+1}{4(j-1)^2} > \frac{4j(j-1)}{4(j-1)^2} = \frac{j}{j-1}$,

$$C_{2k}^{k} \geqslant \left(\sqrt{\frac{2}{1}} \cdot \sqrt{\frac{3}{2}} \cdot \ldots \cdot \sqrt{\frac{k}{k-1}}\right) \cdot \frac{2^{2k}}{2k} = \sqrt{k} \cdot \frac{2^{2k}}{2k} = \frac{2^{2k}}{2\sqrt{k}}.$$

Remark 2: A special prize was awarded to contestants who prove the claim for the largest $n > 10^{400}$. Solution 2 could be shortened if the aim were proving the inequality given in the problem rather than finding as sharp inequality as possible. Namely, it turns out that the lower bound for the number of ways to choose numbers Andres must say, $\frac{1}{2} \cdot \frac{2^{1350}}{2\sqrt{675}}$, is greater than 10^{400}. Indeed, as $\sqrt{675} < \sqrt{10000} = 100 < 256 = 2^8$, we obtain

$$\frac{1}{2} \cdot \frac{2^{1350}}{2\sqrt{675}} > \frac{2^{1350}}{2^{10}} = 2^{1340} = \left(2^{10}\right)^{134} > \left(10^3\right)^{134} = 10^{402} > 10^{400}.$$

Hence it is unnecessary to count ways to distribute other numbers between Bearu and Crõõt (since it is clear that there is at least one distribution for every choice of Andres' numbers).

Selected Problems from the Final Round of National Olympiad

F1 *(Grade 7.)* On the table there are 24 unopened decks of cards containing the same number of cards. Janek opens some of the decks and takes $\frac{1}{4}$ of the cards from each opened deck. Then Kaarel opens some decks and takes $\frac{1}{5}$ of the cards from the decks he opened. Together they have now $\frac{1}{6}$ of the cards originally on the table. Find all possibilities of the number of unopened decks still on the table.

Answer: 5, 6, 7.

Solution: Denote the number of cards in each deck by n. Suppose Janek opens a decks and Kaarel opens b decks. So Janek takes $\frac{1}{4}an$ and Kaarel $\frac{1}{5}bn$ cards. Then $\frac{1}{4}an + \frac{1}{5}bn = \frac{1}{6} \cdot 24n = 4n$, hence $\frac{1}{4}a + \frac{1}{5}b = 4$, or $5a + 4b = 80$. Since 80 and $5a$ are divisible by 5, their difference $4b$ is also divisible by 5. Since $\gcd(4, 5) = 1$, b is divisible by 5. Hence $b = 5$, $b = 10$ or $b = 15$; in other cases a is nonpositive. Then correspondingly $a = 12$, $a = 8$ or $a = 4$. The number of open decks is in these cases 17, 18 or 19, the number of unopened decks is correspondingly 7, 6 or 5.

F2 *(Grade 7.)* Anna and Paul together have 3 times more candies than Kati. Among the three kids there are two who have together exactly 80 candies. If Anna gave 10 candies to Kati then Anna would have as many candies as Paul and Kati together. How many candies does each kid have?

Answer: Anna has 100, Paul 35 and Kati 45 candies.

Solution: Let all of them together have s candies. The first condition implies that Kati has $\frac{s}{4}$ candies. If Anna would give 10 candies to Kati then Anna would have half of the candies, so Anna has $\frac{s}{2} + 10$ candies. Hence Paul has $s - \frac{s}{4} - \left(\frac{s}{2} + 10\right) = \frac{s}{4} - 10$ candies.

If Anna and Paul together had 80 candies then Kati would have $s - 80$ candies, so $\frac{s}{4} = s - 80$, whence $s = \frac{320}{3}$ which is not an integer. If Anna and Kati together had 80 candies then Paul would have $s-80$ candies, implying $\frac{s}{4}-10=s-80$, whence $s=\frac{280}{3}$ which is not an integer. Hence Kati and Paul together have 80 candies and Anna has $s-80$ candies, so $\frac{s}{2}+10=s-80$, whence $s=180$. Consequently Anna has 100, Kati 45 and Paul 35 candies.

F3 *(Grade 7.)* On the side AB of a triangle ABC a point D is chosen so that $\angle BCD=3\angle ACD$, and on the side AC a point E satisfying $CE=DE$ is chosen. Find the size of the angle ACD if CD ja BE are perpendicular and the sum of angles CAD, CDB and CBE is $147°$.

Answer: $28.5°$.

Solution: Denote $\angle ACD=\alpha$. Then $\angle EDC=\angle ECD=\alpha$ since $CE=DE$.

Let the intersection point of lines CD and BE be K (Fig. 18). Since $CD\perp BE$, EK is the height of the triangle ECD and BK is the height of the triangle BCD. Since the height of the isosceles triangle ECD bisects the base, we have $CK=DK$. Hence the height BK of the triangle BCD bisects the base, which implies that the triangle BCD is isosceles with apex B. Consequently $\angle BDC=\angle BCD$.

From the first condition of the problem $\angle BCD=3\alpha$, hence $\angle BDC=3\alpha$. Considering the triangle ACD we get $\angle CAD=\angle CDB-\angle ACD=2\alpha$. On the other hand $\angle CDB+\angle CBE=3\alpha+(90°-3\alpha)=90°$. Consequently $\angle CAD+\angle CDB+\angle CBE=2\alpha+90°$. Since from the conditions of the problem $2\alpha+90°=147°$ we have $\alpha=28.5°$.

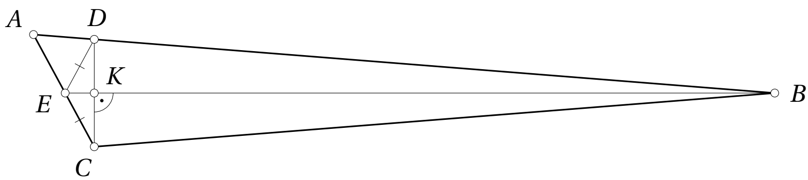


Fig. 18

F4 *(Grade 7.)* A rectangular board with positive integer side lengths is divided into unit squares. Two different unit squares are called *neighboring squares* if they share a common side.

Initially, all unit squares are empty. Mati arbitrarily chooses one unit square and writes down the number of its neighboring squares. Then Mati selects another empty unit square and writes down the number of its empty neighboring squares. Mati repeats this step until there are no more empty unit squares left. The sum of all the numbers written by Mati is 22. Find all possible dimensions of the game board.

Answer: 1×23, 2×8, 3×5, 5×3, 8×2, 23×1.

Solution: For each unit square where Mati writes the number of empty neighbors, color the sides that have empty neighboring squares blue. Thus,

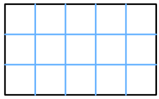


Fig. 19

the number written by Mati at each step is always equal to the number of blue-colored sides in the corresponding step. In each pair of neighboring squares, their common side is colored when Mati chooses the first of the two squares because the second square is still empty. When choosing the second square, Mati does not color that side because the neighboring square beyond that side is no longer empty. Therefore, Mati colors each segment that separates any two neighboring squares exactly once, which means that the sum of all the numbers written by Mati is equal to the total length of the lines within the game board (Figure 19 illustrates the coloring of segments in the final state for a 3×5 board).
If the board has dimensions $n \times m$, then the total length of lines parallel to one side within the game board is $n \cdot (m-1)$ and the total length of lines parallel to the perpendicular side is $m \cdot (n-1)$. From the condition of the problem, we can now form the equation:

$$n(m-1) + m(n-1) = 22. \tag{9}$$

Note that

$$\begin{aligned} n(m-1) + m(n-1) &= 2nm - n - m \\ &= (2n-1)\left(m - \tfrac{1}{2}\right) - \tfrac{1}{2} = \tfrac{(2n-1)(2m-1)-1}{2}. \end{aligned}$$

Hence, we can transform the equation (9) into the form

$$(2n-1)(2m-1) = 45.$$

The positive factors of 45 are $1, 3, 5, 9, 15, 45$. We have the following options:

- $2n-1 = 1, 2m-1 = 45$, which gives $n = 1, m = 23$;
- $2n-1 = 3, 2m-1 = 15$, which gives $n = 2, m = 8$;
- $2n-1 = 5, 2m-1 = 9$, which gives $n = 3, m = 5$.

Additionally, there are symmetric variations of these possibilities.

F5 *(Grade 8.)* Let a, b and c be positive integers satisfying $a + b = 2c$, $2a - 2b = 3c$ and $ab > 2023$. Find the smallest possible c.
Answer: 72.
Solution 1: Multiplying the first equation of the system of equations

$$\begin{cases} a + b = 2c, \\ 2a - 2b = 3c \end{cases}$$

by 3 and the second equation by 2, the right-hand sides of the equations become equal. So the left-hand sides are also equal, i.e. $3a + 3b = 4a - 4b$.

Collecting terms we get $a = 7b$, so $ab = 7b^2$. As $ab > 2023 = 7 \cdot 17^2$, we have $7b^2 > 7 \cdot 17^2$, hence $b > 17$. The least possibility is $b = 18$. Correspondingly the least a is $7 \cdot 18 = 126$ and $c = \frac{18+126}{2} = 72$.

Solution 2: Multiplying the first equation of the system of equations

$$\begin{cases} a + b = 2c, \\ 2a - 2b = 3c \end{cases}$$

by 2 and then adding and subtracting the two equations we get $4a = 7c$ and $4b = c$, whence $a = \frac{7}{4}c$ and $b = \frac{1}{4}c$. Multiplying these two equations gives $ab = \frac{7}{16}c^2 = 7 \cdot \left(\frac{c}{4}\right)^2$. Since $ab > 2023 = 7 \cdot 17^2$ we have

$$7 \cdot \left(\frac{c}{4}\right)^2 > 7 \cdot 17^2.$$

Thus $\left(\frac{c}{4}\right)^2 > 17^2$, so $\frac{c}{4} > 17$, hence $c > 4 \cdot 17 = 68$. Since $\frac{c}{4} = b$ and b is an integer, c must be a multiple of 4. Therefore the least possibility is $c = 72$.

F6 *(Grade 8.)* Scooter A travels the first third of the road between Kükametsa and Kapa-Kohila at a speed of 60 km/h. For the rest of the road, scooter A travels at a slower constant speed. Scooter B covers the first third of the same road at a speed of 100 km/h and then continues on the remaining part of the road at a speed higher than 50 km/h. Can scooters A and B reach Kapa-Kohila from Kükametsa in the same amount of time?

Answer: No.

Solution: Let the distance between Kükametsa and Kapa-Kohila be s kilometers. Scooter B covers the first third of the road in $\frac{s}{300}$ hours. If scooter B were to travel at a speed of 50 km/h for the remaining part of the road, it would take $\frac{2s}{3 \cdot 50} = \frac{s}{75}$ hours for that portion, and the total time for the entire journey would be $\frac{s}{300} + \frac{s}{75} = \frac{5s}{300} = \frac{s}{60}$ hours. Therefore, scooter B would cover the entire road with an average speed of 60 km/h. However, since scooter B actually travels the last two-thirds of the road faster than 50 km/h, its average speed for the entire journey is greater than 60 km/h. On the other hand, scooter A travels the first third of the road at a speed of 60 km/h and the remaining two-thirds at a slower pace than 60 km/h, resulting in an average speed for the entire journey that is less than 60 km/h. Therefore, scooter A and scooter B cannot reach Kapa-Kohila from Kükametsa in the same amount of time.

F7 *(Grade 8.)* Consider a rectangle $ABCD$. Let E be the midpoint of AB and let F be a point on BC. Denote the point of intersection of CE and DF by G. Find the size of the angle CEF if $CF = CG$ and $DE = DG$.

Answer: 18°.

Solution 1: Denote $\angle CGF = \angle DGE = \alpha$ (Fig. 20). The given conditions $CF = CG$ and $DE = DG$ also imply $\angle CFG = \angle DEG = \alpha$. Therefore $\angle EDF = \angle EDG = 180^\circ - 2\alpha$ and $\angle ECF = \angle GCF = 180^\circ - 2\alpha$. From the right triangle CDF we get $\angle FDC = 90^\circ - \angle DFC = 90^\circ - \alpha$. Hence

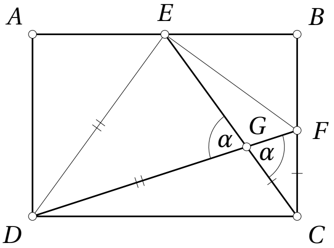


Fig. 20

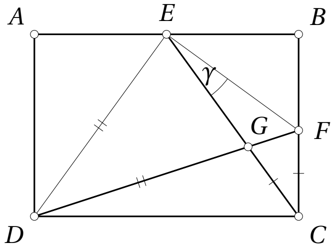


Fig. 21

$\angle EDC = \angle EDF + \angle FDC = (180° - 2\alpha) + (90° - \alpha) = 270° - 3\alpha$ and $\angle ECD = 90° - \angle ECF = 90° - (180° - 2\alpha) = 2\alpha - 90°$.

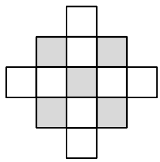
Since E bisects AB, triangles ADE and BCE are equal. Thus $ED = EC$, which implies $\angle EDC = \angle ECD$. Consequently $270° - 3\alpha = 2\alpha - 90°$, whence $\alpha = 72°$.

Since isosceles triangles CFG and DEG are similar, we have $\frac{FG}{EG} = \frac{GC}{GD}$. Since $\angle FGE = \angle CGD$, the last equality implies that triangles CDG and FEG are similar, too. Therefore $\angle CEF = \angle GEF = \angle GDC = 90° - \alpha = 18°$.

Solution 2: Let $\angle CEF = \gamma$ (Fig. 21). The given conditions $CF = CG$ and $DE = DG$ imply $\angle DEG = \angle DGE$ and $\angle CGF = \angle CFG$. Because of $\angle DGE = \angle CGF$ we get $\angle DEG = \angle CFG$ and $\angle CED = \angle CFD$. Hence the points C, D, E, F lie on a circle, which implies $\angle CDF = \angle CEF = \gamma$ and $\angle CED = \angle CFD = 90° - \gamma$. From isosceles triangle CFG we get $\angle FCG = 180° - 2(90° - \gamma) = 2\gamma$. Similarly from isosceles triangle DEG we get $\angle EDG = 2\gamma$. Hence $\angle ECD = 90° - 2\gamma$ ja $\angle EDC = 2\gamma + \gamma = 3\gamma$.

Since E bisects AB, triangles ADE and BCE are equal. Thus $ED = EC$, implying $\angle EDC = \angle ECD$. Consequently $90° - 2\gamma = 3\gamma$, whence $\gamma = 18°$.

F8 *(Grade 8.)* Natural numbers from 1 to 13 are written in the squares of a diamond-shaped grid, such that each square contains exactly one number. The sum of numbers in the squares of the longest row, the sum of numbers in the squares of the longest column, and the sum of numbers in the five gray-colored squares are equal. Determine all possible values for the number in the middle square.



Answer: 1, 4, 7, 10, 13.

Solution: Let k be the number in the middle square, and let s be the sum of numbers in the squares of the longest row, the sum of numbers in the squares of the longest column, and the sum of numbers in the five gray-colored squares. Only the number in the middle square appears in all these sums, while the remaining numbers appear exactly once in one of the sums. Thus, we have:

$$(1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12 + 13) + 2k = 3s$$

which simplifies to $91 + 2k = 3s$. Therefore, $91 + 2k$ must be divisible by 3.

By testing, we can see that this condition is satisfied for $k = 1$, $k = 4$, $k = 7$, $k = 10$, and $k = 13$. To show that there exist corresponding arrangements of numbers, it suffices to demonstrate that the remaining numbers can be divided into three groups with equal numbers of elements and equal sums. Indeed, such distributions can be found:

$k = 1:$	$k = 4:$	$k = 7:$	$k = 10:$	$k = 13:$
$2, 7, 8, 13$	$1, 7, 8, 13$	$1, 6, 8, 13$	$1, 6, 7, 13$	$1, 6, 7, 12$
$3, 6, 9, 12$	$2, 6, 9, 12$	$2, 5, 9, 12$	$2, 5, 8, 12$	$2, 5, 8, 11$
$4, 5, 10, 11$	$3, 5, 10, 11$	$3, 4, 10, 11$	$3, 4, 9, 11$	$3, 4, 9, 10$

F9 *(Grade 8.)* In the figure, there is a grid of dimensions 5×5 filled with numbers 0, 2, and 3. How many different possibilities are there to choose 4 distinct unit squares such that the selected squares can be traversed in a sequence where the numbers in the squares combine to form the number 2023? When traversing the squares, one can move from one square to another only if they share a common side or vertex.

3	3	3	3	3
3	2	2	2	3
3	2	0	2	3
3	2	2	2	3
3	3	3	3	3

Answer: 204.

Solution: There are 4 squares with the number 2 from which one can move to 5 different squares with the number 3, and 4 squares with the number 2 from which one can move to 3 different squares with the number 3. Therefore, there are $4 \cdot 5 + 4 \cdot 3 = 32$ possibilities for choosing the last two squares. The only square with the number 0 is the middle one, and one can move to any of the squares with the number 2 from there. Hence, there are also 32 possibilities for the last three squares.

Any square with the number 2 that has not been selected already can be chosen as the first square. There are 7 possible choices for this.

Hence there are $7 \cdot 32 = 224$ ways for selecting the required path.

In the problem we are asked about the number of possibilities to choose 4 squares that can be traversed in the required sequence. The number of paths counts twice the choices that allow counting the number 2023 in two ways (starting from two different squares with the number 2). These are precisely the choices where one can move from both squares with the number 2 to the square with the number 3 that is included in the selection. Such two squares with the number 2 can appear in two ways.

- *Two squares with a common side* (Fig. 22). There are exactly 8 possibilities for choosing two squares from those with the number 2. Each of these possibilities occurs twice because there are two choices for the square with the number 3. Therefore, there are a total of 16 such possibilities.
- *Two squares that are in the same row or column with exactly one square between them* (Fig. 23). There are four possibilities for choosing two such squares. All of these occur once because there is only one choice for the square with the number 3.

So there are $16+4=20$ choices counted twice. Thus, the number of choices for selecting four squares that meet the given condition is $224-20=204$.

3	3	3	3	3
3	2	2	2	3
3	2	0	2	3
3	2	2	2	3
3	3	3	3	3

Fig. 22

3	3	3	3	3
3	2	2	2	3
3	2	0	2	3
3	2	2	2	3
3	3	3	3	3

Fig. 23

F10 *(Grade 9.)* Let a, b and c be integers which satisfy the conditions $ab+bc+ca=1$ and $a+b=c$. Prove that the product abc is divisible by 3.

Solution 1: As $ab=1-bc-ca=1-c(a+b)=1-c^2=(1-c)(1+c)$, we have $abc=(1-c)(1+c)c=-(c-1)c(c+1)$. One of the three consecutive integers $c-1, c, c+1$ must be divisible by 3, so their product must be divisible by 3 as well. Therefore abc is also divisible by 3.

Solution 2: Assume the contrary, i.e. that abc is not divisible by 3. Then none of the numbers a, b, c is divisible by 3. If $a \not\equiv b \pmod 3$, the remainders must be 1 and 2 in some order, but then $c=a+b\equiv 0 \pmod 3$, a contradiction. So there remain two options. If $a\equiv b\equiv 1 \pmod 3$ and $c\equiv 2 \pmod 3$, then $ab+bc+ca\equiv 2 \pmod 3$. If $a\equiv b\equiv 2 \pmod 3$ and $c\equiv 1 \pmod 3$, then analogously $ab+bc+ca\equiv 2 \pmod 3$. Thus in both cases $ab+bc+ca\equiv 2 \pmod 3$, contradicting $ab+bc+ca=1$. Therefore $3 \mid abc$.

Solution 3: We first notice that

$$ab+bc+ca=ab+b(a+b)+(a+b)a=a^2+b^2+3ab.$$

Since $ab+bc+ca=1$ and $3 \mid 3ab$, we see that $a^2+b^2\equiv 1 \pmod 3$. Since a^2 and b^2 can only have remainders of 0 or 1, one of the numbers a^2 and b^2 must yield a remainder of 0 upon division by 3 (and the other one must yield a remainder of 1). But then either a or b will be divisible by 3. Therefore abc is also divisible by 3.

F11 *(Grade 9.)* Given numbers x, y, z such that $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{x+y+z}$, prove that $\frac{1}{x^{2023}}+\frac{1}{y^{2023}}+\frac{1}{z^{2023}}=\frac{1}{x^{2023}+y^{2023}+z^{2023}}$.

Solution: Taking all the fractions in $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{x+y+z}$ to a common denominator and factoring the numerator yields $\frac{(x+y)(y+z)(z+x)}{xyz(x+y+z)}=0$. Similarly the equation $\frac{1}{x^{2023}}+\frac{1}{y^{2023}}+\frac{1}{z^{2023}}=\frac{1}{x^{2023}+y^{2023}+z^{2023}}$ is equivalent to $\frac{\left(x^{2023}+y^{2023}\right)\left(y^{2023}+z^{2023}\right)\left(z^{2023}+x^{2023}\right)}{x^{2023}y^{2023}z^{2023}\left(x^{2023}+y^{2023}+z^{2023}\right)}=0$. So the given condition holds if and only if at least one of the numbers $x+y$, $y+z$ and $z+x$ is equal to zero. But if $x+y=0$, then $y=-x$, which yields $y^{2023}=-x^{2023}$ and $x^{2023}+y^{2023}=0$, which yields the desired result. An analogous argument works when $y+z=0$ or $z+x=0$.

F12 *(Grade 9.)* Let c be a circle, O its center and AB its diameter. Let M be the midpoint of the segment AO and CD be a chord of the circle c passing through the point M. Let $H \neq D$ be a point on the chord CD, such that $DM = MH$. Find the angle ABC, given that $BH \perp CD$.
Answer: 45°.
Solution: From equal inscribed angles we get $\angle ABC = \angle ADC = \angle ADH$ (Fig. 24). As the midpoints of the diagonals of $ADOH$ coincide, the quadrilateral $ADOH$ must be a parallelogram, meaning its opposite sides are parallel. Thus $\angle ADH = \angle OHD$. In conclusion, we have $\angle ABC = \angle OHD$.
Let K be the intersection of the lines HO and BD (Fig. 25). As M is the midpoint of DH, BM is a median in the triangle BDH. But then O is the centroid of BDH, since $OM = \frac{1}{2}OA = \frac{1}{2}OB$. Therefore HK is also a median in BDH, meaning that K is the midpoint of BD. Therefore the triangles BKO and DKO are equal, as all of their corresponding sides have equal lengths. Thus $\angle KOB = \angle KOD$ and also $\angle HOB = \angle HOD$, which yields the equality of the triangles HOB and HOD due to them having two equal sides and an equal angle between them. Therefore $\angle OHB = \angle OHD = \frac{1}{2}\angle BHD = 45^\circ$ and thus also $\angle ABC = 45^\circ$.

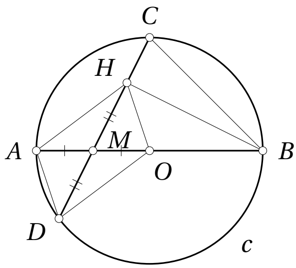


Fig. 24

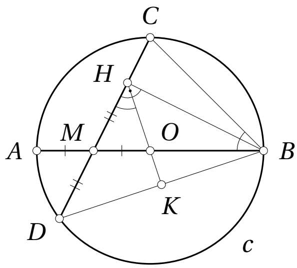


Fig. 25

F13 *(Grade 9.)* A strip of width 1 is divided into unit squares. Juku and Miku take turns placing pieces one by one on the squares of the strip. A piece can be placed on any square except those already containing a piece and those adjacent to such squares. Juku places the first piece and the player who cannot place a piece loses. Prove that whenever Miku has a winning strategy for a strip of length k, Juku has a winning strategy for strips of lengths $k+1$, $k+2$ and $k+3$.
Solution: For a strip of odd length, Juku can win by placing the first piece in the central square and mirroring each of Miku's moves with respect to the

center of the strip, making sure that he will never run out of moves first. So if Miku has a winning strategy for a strip of length k, then k is even. But then $k+1$ and $k+3$ are odd, meaning that for strips of these lengths, Juku will have a winning strategy. For a strip of length $k+2$, Juku can win by placing the first piece in the first square, eliminating the first two squares of the strip, and then using the strategy with which the second player can win for a strip of length k (Fig. 26).

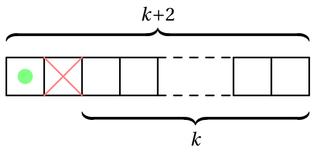


Fig. 26

F14 *(Grade 9.)* Juku draws two isosceles triangles. In both triangles, the altitude drawn onto a leg divides the angle at the base into two parts, which have a ratio of $1 : x$, where x is the same positive number for both triangles (it is not known if x is an integer). In both triangles, a vertex can be chosen such that the angles at the chosen vertices are equal. Can we be certain that the two triangles Juku drew are similar?

Answer: Yes.

Solution: Since the vertex angle uniquely defines the base angle and vice versa, the triangles are clearly similar if the equal angle α is either the base angle in both triangles or the vertex angle in both triangles. So it remains only to consider the case where the vertex angle of one triangle is equal to the base angle of the other triangle.

In a triangle with vertex angle α (Fig. 27), the altitude drawn onto a leg divides the angle at the base into angles $90^\circ - \alpha$ and $90^\circ - \frac{180^\circ - \alpha}{2} = \frac{\alpha}{2}$. Since their ratio in some order is $1 : x$, we have either $90^\circ - \alpha = x \cdot \frac{\alpha}{2}$ or $\frac{\alpha}{2} = x \cdot (90^\circ - \alpha)$. In a triangle with base angle α (Fig. 28), the altitude drawn onto a leg divides the angle at the base into angles of sizes $90^\circ - \alpha$ and $90^\circ - (180^\circ - 2\alpha) = 2\alpha - 90^\circ$. Since their ratio in some order is also $1 : x$, we have either $90^\circ - \alpha = x \cdot (2\alpha - 90^\circ)$ or $2\alpha - 90^\circ = x \cdot (90^\circ - \alpha)$.

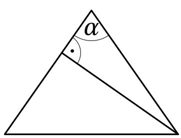


Fig. 27

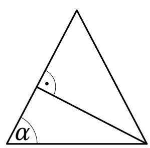


Fig. 28

So it remains to consider the four possible combinations.

- If $90^\circ - \alpha = x \cdot \frac{\alpha}{2}$ and $90^\circ - \alpha = x \cdot (2\alpha - 90^\circ)$, then $\frac{\alpha}{2} = 2\alpha - 90^\circ$, which yields $\alpha = 60^\circ$.
- If $90^\circ - \alpha = x \cdot \frac{\alpha}{2}$ and $2\alpha - 90^\circ = x \cdot (90^\circ - \alpha)$, then the equations rearrange to $\alpha = \frac{2 \cdot 90^\circ}{2+x}$ and $\alpha = \frac{(1+x)\cdot 90^\circ}{2+x}$. Therefore $2 = 1 + x$, which means $x = 1$. Substituting this into the previous conditions easily yields $\alpha = 60^\circ$.
- If $\frac{\alpha}{2} = x \cdot (90^\circ - \alpha)$ and $90^\circ - \alpha = x \cdot (2\alpha - 90^\circ)$, then the equations rearrange to $\alpha = \frac{2x \cdot 90^\circ}{1+2x}$ and $\alpha = \frac{(1+x)\cdot 90^\circ}{1+2x}$. Therefore $2x = 1 + x$, which means $x = 1$. Substituting this into the previous conditions easily yields $\alpha = 60^\circ$.
- If $\frac{\alpha}{2} = x \cdot (90^\circ - \alpha)$ and $2\alpha - 90^\circ = x \cdot (90^\circ - \alpha)$, then $\frac{\alpha}{2} = 2\alpha - 90^\circ$, which yields $\alpha = 60^\circ$.

So in all cases $\alpha = 60^\circ$. But if any of the angles of an isosceles triangle is 60°, the triangle is also equilateral. Therefore Juku draws two equilateral triangles, which are clearly similar.

F15 *(Grade 10.)* A positive integer n is written as a sum of at least two positive integers in such a way that the sum of the squares of the addends is a perfect square. Prove that $(n - x)^2 > 2x$ holds for every addend x.

Solution: Let $k \geqslant 2$ be the number of addends. Let x be any of the addends and let $z_2, \ldots, z_k$ be the other addends. By the conditions of the problem $x + z_2 + \ldots + z_k = n$ and $x^2 + z_2^2 + \ldots + z_k^2$ is a perfect square. Therefore $x^2 + z_2^2 + \ldots + z_k^2 \geqslant (x+1)^2$, as the addends are all positive. Thus

$$z_2^2 + \ldots + z_k^2 \geqslant (x+1)^2 - x^2 = 2x + 1.$$

Now from the first condition of the problem we get

$$(n - x)^2 = (z_2 + \ldots + z_k)^2 \geqslant z_2^2 + \ldots + z_k^2,$$

as multiplying out $(z_2 + \ldots + z_k)^2$ yields the squares $z_2^2, \ldots, z_k^2$ and in the case $k > 2$ other positive terms. In conclusion $(n - x)^2 \geqslant 2x + 1$, which yields $(n - x)^2 > 2x$.

F16 *(Grade 10.)* Let O be the circumcenter of an acute triangle ABC. Points D and E are chosen on the lines AB and AO respectively, such that the points A, B, D and A, O, E lie on the lines in this order and E lies inside both the triangle ABC and the circumcircle of BCD. Let F be the intersection of the circumcircles of ACE and BCD ($F \neq C$, $F \neq E$). Find $\angle DFE$.

Answer: 90°.

Solution 1: Denote $\angle ABC = \beta$. Then $\angle COA = 2\beta$ and the isosceles triangle ACO yields

$$\angle CAO = \angle ACO = \frac{1}{2}(180^\circ - 2\beta) = 90^\circ - \beta.$$

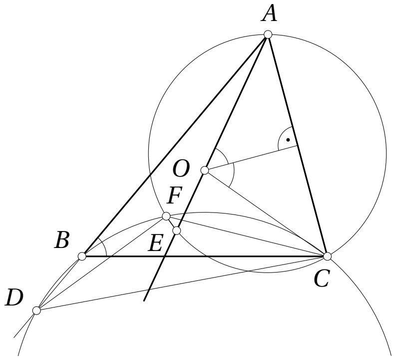


Fig. 29

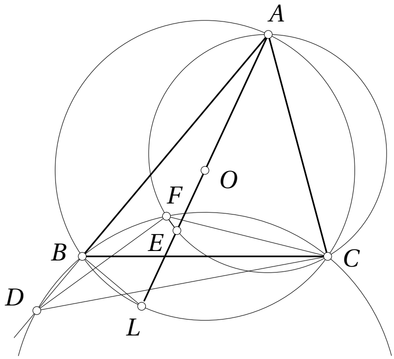


Fig. 30

Now, inscribed angles in the circumcircles of BCD and ACE (Fig. 29) yield

$$\begin{aligned}\angle DFE &= \angle DFC - \angle EFC \\ &= \angle DBC - \angle EAC = (180^\circ - \beta) - (90^\circ - \beta) = 90^\circ.\end{aligned}$$

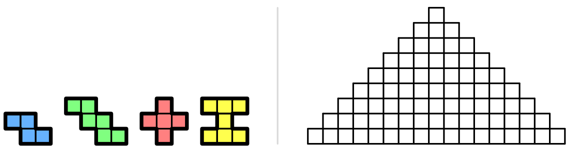
Solution 2: Let L be the second intersection of AO with the circumcircle of ABC (Fig. 30). Now, inscribed angles in the circumcircles of BCD and ABC yield

$$\begin{aligned}\angle DFE &= \angle DFC - \angle EFC \\ &= \angle DBC - \angle EAC \\ &= \angle DBC - \angle LAC = \angle DBC - \angle LBC = \angle DBL = 180^\circ - \angle ABL.\end{aligned}$$

But $\angle ABL = 90^\circ$, since AL is a diameter in the circumcircle of ABC. Thus also $\angle DFE = 90^\circ$.

F17 *(Grade 10.)*

(a) In the figure on the left are depicted four types of jigsaw pieces. Is it possible to construct a 9-level pyramid, depicted in the figure on the right, from such pieces? There is an unlimited supply of each type of piece, the pieces can be rotated and reflected, but they may not overlap.



(b) What about a 10-level pyramid?

Answer: a) Yes; b) No.

Solution:

(a) One possible arrangement of pieces is depicted in Fig. 31.

(b) We will colour the 10-level pyramid and jigsaw pieces in a checkerboard pattern, as depicted in Fig. 32. We notice that there are 10 more white

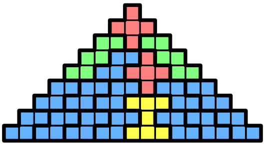


Fig. 31

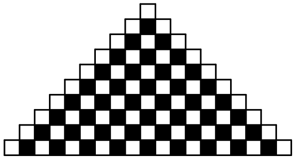


Fig. 32

squares than black squares. More specifically, the difference is not divisible by 3. On the other hand, the first and the second piece will always cover an equal amount of white and black squares, whereas the third and fourth piece will always cover 3 more of one type of squares than the other type. Thus the difference between the numbers of white and black squares covered will always be divisible by 3. Therefore it is not possible to construct a 10-level pyramid from the given pieces.

F18 *(Grade 10.)* Juku and Miku play the following game on a coordinate plane. Both players have a piece which is initially located at the origin. In their turn, a player has to move their piece by one unit in either the positive or negative direction of either the x or y-axis. Juku starts the game by taking k turns, then Juku and Miku take alternating turns in this order with both players taking n turns. Miku wins, if the final distance between the two pieces is an integer, and Juku wins otherwise. Find all pairs (n, k) of positive integers for which Miku has a winning strategy.

Answer: pairs $(n, 1)$ and $(n, 2)$ for all positive integers n.

Solution: If $k = 1$ or $k = 2$, then after Juku's first $k + 1$ moves the x- or y-coordinate of Juku's piece will have an absolute value of at most 1. Then Miku can make his first move such that the corresponding coordinates of the two pieces will be equal, meaning the distance between the pieces will be an integer. By copying Juku's following moves, Miku can win by making sure that the coordinates will also be equal at the end of the game.

If $k \geqslant 3$ and k is odd, then Juku can end his first $k + 1$ moves in the point $(2, 2)$ by using the first 4 moves to get there and moving back and forth with the rest of the moves. By copying Miku's following moves (except for Miku's last move), the vector connecting the two pieces at the end of the game will differ from $(2, 2)$ by either $(\pm 1, 0)$ or $(0, \pm 1)$. So the distance between the pieces will be either $\sqrt{1^2 + 2^2} = \sqrt{5}$ or $\sqrt{3^2 + 2^2} = \sqrt{13}$. As neither of them is an integer, Juku wins.

If $k \geqslant 3$ and k is even, then Juku can end his first $k + 1$ moves in the point $(3, 2)$ by using the first 5 moves to get there and moving back and forth with the rest of the moves. By copying Miku's following moves except the last one, the vector connecting the two pieces at the end of the game will differ from $(3, 2)$ by either $(\pm 1, 0)$ or $(0, \pm 1)$. So the distance between the pieces will be $\sqrt{2^2 + 2^2} = \sqrt{8}$ or $\sqrt{4^2 + 2^2} = \sqrt{20}$ or $\sqrt{3^2 + 1^2} = \sqrt{10}$ or $\sqrt{3^2 + 3^2} = \sqrt{18}$. As none of those is an integer, Juku wins.

F19 *(Grade 11.)* How many consecutive zeros are there at the end of $2022! + 2023!$?

Answer: 503.

Solution: Every other factor in the product $1 \cdot 2 \cdot \ldots \cdot (n-1) \cdot n$ will be divisible by 2, every fourth by 2^2, every eighth by 2^3 etc. So the exponent of 2 in $n!$ is $\left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n}{2^2} \right\rfloor + \left\lfloor \frac{n}{2^3} \right\rfloor + \ldots$. Similarly the exponent of 5 in $n!$ is $\left\lfloor \frac{n}{5} \right\rfloor + \left\lfloor \frac{n}{5^2} \right\rfloor + \left\lfloor \frac{n}{5^3} \right\rfloor + \ldots$. As every addend in the first sum is greater than or equal to the corresponding addend in the second sum, the exponent of 2 is not less than the exponent of 5. More specifically, the exponent of 5 in $2022!$ is $\left\lfloor \frac{2022}{5} \right\rfloor + \left\lfloor \frac{2022}{25} \right\rfloor + \left\lfloor \frac{2022}{125} \right\rfloor + \left\lfloor \frac{2022}{625} \right\rfloor + \left\lfloor \frac{2022}{3125} \right\rfloor + \ldots$, which equals $404 + 80 + 16 + 3 + 0 + \ldots = 503$. Thus the exponent of 2 is at least 503.

Now notice that $2022! + 2023! = 2022! \cdot (1 + 2023) = 2022! \cdot 2024$. Therefore the exponent of 5 in $2022! + 2023!$ is also 503, whereas the exponent of 2 is greater than 503. Thus $2022! + 2023!$ is divisible by 10^{503}, but not by 10^{504}. Therefore there are exactly 503 consecutive zeros at the end of $2022! + 2023!$.

F20 *(Grade 11.)* Do there exist positive real numbers a and b satisfying the system of inequalities

$$\begin{cases} \sqrt{119}a + \sqrt{17}b \leqslant 2ab, \\ a^2 + b^2 \leqslant 2\sqrt{2023}? \end{cases}$$

Answer: No.

Solution: The AM-GM inequality for the numbers $\sqrt{119}a$ and $\sqrt{17}b$ yields

$$\sqrt{119}a + \sqrt{17}b \geqslant 2\sqrt{\sqrt{119}a \cdot \sqrt{17}b} = 2\sqrt{\sqrt{2023}ab}.$$

Combining this with $2ab \geqslant \sqrt{119}a + \sqrt{17}b$ yields $2ab \geqslant 2\sqrt{\sqrt{2023}ab}$, where division by $2\sqrt{ab}$ yields $\sqrt{ab} \geqslant \sqrt{\sqrt{2023}}$ that is equivalent to $ab \geqslant \sqrt{2023}$. Combining this with $2\sqrt{2023} \geqslant a^2 + b^2$ and $a^2 + b^2 \geqslant 2ab$ yields

$$2ab \geqslant 2\sqrt{2023} \geqslant a^2 + b^2 \geqslant 2ab.$$

Thus $a^2 + b^2 = 2ab = 2\sqrt{2023}$. The first equality yields $a = b$, implying $a^2 = \sqrt{2023}$. However, squaring the sides of $\sqrt{119}a + \sqrt{17}b \leqslant 2ab$ and using $a = b$ yields $\left(119 + 17 + 2\sqrt{2023}\right) a^2 \leqslant 4a^4$, which is equivalent to $136 + 2\sqrt{2023} \leqslant 4a^2$. Since $a^2 = \sqrt{2023}$, we also have $136 \leqslant 2\sqrt{2023}$, which does not hold, as $4 \cdot 2023 < 10000 < 136^2$. Therefore there exist no positive real numbers a and b satisfying the system of inequalities.

F21 *(Grade 11.)* The internal and external angle bisectors of the angle at the vertex A of a triangle ABC intersect the circumcircle of ABC at $D \neq A$ and $E \neq A$ respectively. Let F be the intersection of the lines AD and BC and D' the reflection of D over the point F. Prove that the points B, D', E are collinear if and only if $\angle BAC = 2\angle ACB$.

Solution: Denote $\alpha = \angle BAD = \angle DAC$ (marked in Fig. 33). Then from equal inscribed angles we also get $\angle DBC = \angle DCB = \alpha$. As the internal and external angle bisectors are perpendicular, $\angle DAE = \angle DBE = 90°$. Thus B, D', E being collinear is equivalent to $\angle DBD' = \angle DBE = 90°$. This in turn is equivalent to B lying on the circle with diameter DD', which because of F being the center of DD' is equivalent to $FB = FD$ and also $\angle FDB = \angle FBD = \alpha$. But equal inscribed angles yield $\angle ACB = \angle FDB$, which means that the collinearity of B, D' and E is equivalent to $\angle ACB = \alpha$ or $\angle BAC = 2\alpha = 2\angle ACB$.

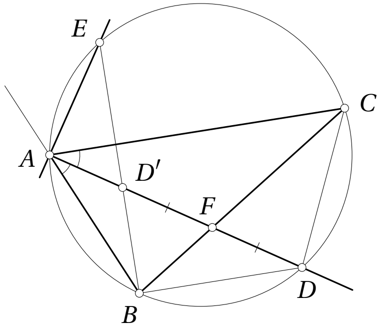


Fig. 33

F22 *(Grade 11.)* Let n and m be positive integers. There are n cards in a deck and m distinct symbols on each card, such that every two cards have at most one symbol in common. Prove that there are at least $\frac{1+\sqrt{1+4nm(m-1)}}{2}$ distinct symbols among all of the cards.

Solution 1: Let x be the number of distinct symbols among all cards. Then there are $x(x-1)$ pairs consisting of two distinct symbols. On any card, there are $m(m-1)$ such pairs of symbols. Since every two cards have at most one symbol in common, no pair of symbols can repeat, so there are $nm(m-1)$ such pairs of symbols among all cards. Therefore the inequality $x(x-1) \geqslant nm(m-1)$ holds, which yields $x^2 - x - nm(m-1) \geqslant 0$.

The quadratic polynomial $x^2 - x - nm(m-1)$ has roots $\frac{1-\sqrt{1+4nm(m-1)}}{2}$ and $\frac{1+\sqrt{1+4nm(m-1)}}{2}$. As the leading coefficient of the polynomial is positive, the inequality $x^2 - x - nm(m-1) \geqslant 0$ holds when $x \geqslant \frac{1+\sqrt{1+4nm(m-1)}}{2}$ or $x \leqslant \frac{1-\sqrt{1+4nm(m-1)}}{2}$. Since $\sqrt{1+4nm(m-1)} \geqslant \sqrt{1} = 1$, the second inequality would mean $x \leqslant 0$, which is impossible. So the inequality $x \geqslant \frac{1+\sqrt{1+4nm(m-1)}}{2}$ must always hold.

Solution 2: Let x be the number of distinct symbols among all cards. We first prove a special case of the Johnson bound from coding theory and the theory of combinatorial design: $n \leqslant \left\lfloor \frac{x}{m} \left\lfloor \frac{x-1}{m-1} \right\rfloor \right\rfloor$. Every time we use a symbol, it shares a card with $m-1$ other symbols. As there are $x-1$ other symbols, we can use any symbol at most $\left\lfloor \frac{x-1}{m-1} \right\rfloor$ times. So we can use the x symbols to fill a total of $x \left\lfloor \frac{x-1}{m-1} \right\rfloor$ slots. So there can be at most $\left\lfloor \frac{x}{m} \left\lfloor \frac{x-1}{m-1} \right\rfloor \right\rfloor$ cards.

Getting rid of the floor function, we get the inequality $n \leqslant \frac{x^2-x}{m^2-m}$, which can be transformed to the quadratic inequality in Solution 1.

F23 *(Grade 11.)* For any positive real number x, we can perform the following operations with the hands of a clock:

(1) Rotate the minute hand by x degrees in either direction, in which case the hour hand rotates by $\frac{x}{12}$ degrees in the same direction.

(2) Rotate both hands by x degrees in either direction.

Find the smallest real number α for which the clock hands can be turned from any position (not necessarily one showing a valid time) into any other position in such a way that the minute hand rotates by a total of α degrees (taking into account rotation in both directions).

Answer: 360.

Solution: Without loss of generality assume that the desired position of the clock hands (the *correct* time) is 12.00.

We will show that from any initial position we can get to 12.00 in such a way that the minute hand rotates by at most 360°. We consider three cases.

- If the hour hand points at least 30° away from 12 in either direction and the minute hand points between 12 and the hour hand (counting clockwise from 12), then we first rotate both hands clockwise until the hour hand points between 11 and 12 and the hands show a valid time. Then we rotate the minute hand until it points to 12, at which point the position of the hands will be correct. As the minute hand has only rotated clockwise and hasn't passed 12, its total rotation has been at most 360°.

- If the hour hand points at least 30° away from 12 in either direction and the minute hand points between the hour hand and 12 (counting clockwise from 12), then we proceed symmetrically to the previous case, rotating the hands counterclockwise instead. Similarly the total rotation of the minute hand will be at most 360°.

- If the hour hand points less than 30° away from 12 in either direction, then we first rotate both hands until they show a valid time and the hour hand is as close to 12 as possible. As any rotation of both hands by $\frac{360^\circ}{11}$ passes through a valid position, the hour hand will point at most $\frac{360^\circ}{22}$ away from 12 in the position we have reached. Then we can rotate the minute hand by at most $\frac{12}{22} \cdot 360^\circ$ to reach the cor-

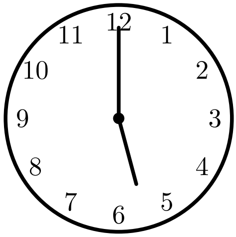


Fig. 34

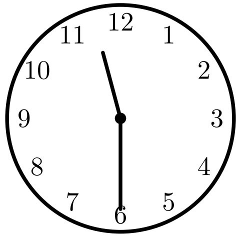


Fig. 35

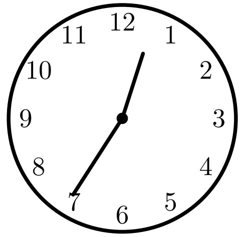


Fig. 36

rect position. The total rotation of the minute hand has been at most $\left(\frac{1}{12} + \frac{1}{22} + \frac{12}{22}\right) \cdot 360^\circ < 360^\circ$.

We now show that there exists an initial position of the clock hands for which the minute hand must be rotated by at least 360° in order to reach the correct time. Consider the position where the minute hand points to 12 and the hour hand points halfway between 5 and 6 (Fig. 34). The final result of operations does not depend on the order of them, so assume without the loss of generality that we only perform the second operation once and the first operation once, in this order.

If after the second operation the hour hand is more than 30° away from 12, then the following first operation will move the minute hand by more than 360°. So it remains to consider the case where the hour hand points between 11 and 1 after the second operation. As the first operation does not change the validity of the position of the hands, the position of the hands must already be valid before it. So the time on the clock will either be 11.30 (Fig. 35) or both hands will be $\frac{1}{11}$ of a full rotation clockwise from it (Fig. 36). In the first case the minute hand will have to move by 180° during both operations, so the total rotation will be 360°. In the second case the total rotation of the minute hand will be $\left(180^\circ - \frac{360^\circ}{11}\right) + \left(180^\circ + \frac{360^\circ}{11}\right) = 360^\circ$, too.

F24 *(Grade 12.)* Let n and k be positive integers satisfying $0 < k < n$. Prove that C_n^k is divisible by at least one of the prime factors of n.

Solution 1: Clearly all of the numbers $n \cdot n$, $n \cdot k$ and $C_n^k \cdot n$ are divisible by n. As $C_n^k \cdot k = C_{n-1}^{k-1} \cdot n$, the number $C_n^k \cdot k$ must also be divisible by n. So $\gcd(n \cdot n, n \cdot k, C_n^k \cdot n, C_n^k \cdot k)$ is divisible by n. Now, as multiplication is distributive with respect to finding the greatest common divisor, we have

$$\gcd(n \cdot n, n \cdot k, C_n^k \cdot n, C_n^k \cdot k) = \gcd(n, C_n^k) \cdot \gcd(n, k).$$

So $n \mid \gcd(n, C_n^k) \cdot \gcd(n, k)$, which yields $\gcd(n, C_n^k) \cdot \gcd(n, k) \geqslant n$. But $\gcd(n, k) < n$, which means that $\gcd(n, C_n^k) > 1$, as desired.

Solution 2: Since $0 < k < n$, there must exist a prime p whose exponent in k is less than its exponent in n. Let the latter be s. By Legendre's formula, the exponent of p in $C_n^k = \frac{n!}{k!(n-k)!}$ is

$$\left(\left\lfloor\frac{n}{p}\right\rfloor + \left\lfloor\frac{n}{p^2}\right\rfloor + \left\lfloor\frac{n}{p^3}\right\rfloor + \ldots\right) - \left(\left\lfloor\frac{k}{p}\right\rfloor + \left\lfloor\frac{k}{p^2}\right\rfloor + \left\lfloor\frac{k}{p^3}\right\rfloor + \ldots\right) - \left(\left\lfloor\frac{n-k}{p}\right\rfloor + \left\lfloor\frac{n-k}{p^2}\right\rfloor + \left\lfloor\frac{n-k}{p^3}\right\rfloor + \ldots\right).$$

Now notice that for all i we have

$$\left\lfloor\frac{k}{p^i}\right\rfloor + \left\lfloor\frac{n-k}{p^i}\right\rfloor \leqslant \left\lfloor\frac{k}{p^i} + \frac{n-k}{p^i}\right\rfloor = \left\lfloor\frac{k+(n-k)}{p^i}\right\rfloor = \left\lfloor\frac{n}{p^i}\right\rfloor.$$

Furthermore, the inequality is strict for $i = s$, as $\frac{n}{p^s}$ is an integer unlike $\frac{k}{p^s}$ and $\frac{n-k}{p^s}$. Thus the exponent of p in C_n^k is positive, i.e. C_n^k is divisible by p.

F25 *(Grade 12.)* Let $P(x)$ be a polynomial of degree 2023, each of whose coefficients is either 0 or 1. Furthermore let $P(0) = 1$. Prove that every real root of $P(x)$ is less than $\frac{1-\sqrt{5}}{2}$.

Solution 1: Let a be a real root of $P(x)$. Clearly $a < 0$, as for any nonnegative value of x each of the terms in $P(x)$ is nonnegative and the constant term (equal to $P(0)$) is positive. For $a < 0$, notice that each of the terms with an odd power of a is nonpositive and each of the terms with an even power of a is nonnegative. Therefore

$$P(a) \geqslant 1 + a + a^3 + a^5 + \ldots + a^{2023} = 1 + \frac{a - a^{2025}}{1-a^2} = \frac{1 - a^2 + a - a^{2025}}{1-a^2}.$$

Notice that $\frac{1-\sqrt{5}}{2}$ is the smaller solution of $x^2 - x - 1 = 0$. So if $a \geqslant \frac{1-\sqrt{5}}{2}$, then $a^2 - a - 1 \leqslant 0$ and $1 - a^2 + a \geqslant 0$. But as $-a^{2025} > 0$ and $1 - a^2 > 0$, we conclude that $P(a) \geqslant \frac{1-a^2+a-a^{2025}}{1-a^2} > 0$, a contradiction with $P(a) = 0$. So $a < \frac{1-\sqrt{5}}{2}$, as desired.

Solution 2: As in solution 1 we see that for any real root a of $P(x)$ we must have $a < 0$ and $P(a) \geqslant a^{2023} + a^{2021} + \cdots + a + 1$.

We will use mathematical induction to show that if $\frac{1-\sqrt{5}}{2} \leqslant a < 0$, then $a^{2n+1} + a^{2n-1} + \cdots + a + 1 > 0$ for all $n = 1, 2, \ldots$. For $n = 1$ we have

$$a^3 + a + 1 > -a^2 + a + 1 \geqslant 0.$$

Assuming the statement for $n = k$, we see that for $n = k + 1$, we have

$$a^{2k+3} + \ldots + a^3 + a + 1 = a^2(a^{2n+1} + \ldots + a + 1) + (-a^2 + a + 1) > 0,$$

as the first expression is positive and the second one is nonnegative.

F26 *(Grade 12.)* Let ABC be a triangle with $\angle ACB = 90°$. Let F be the foot of the altitude drawn from C. Let the incenters of triangles ABC, ACF and BCF be I, I_1 and I_2 respectively and let M, M_1 and M_2 be the tangency points of the incircles of ABC, ACF and BCF with the sides AB, CA and CB respectively. Prove that M is both the circumcenter of II_1I_2 and the intersection of M_1I_1 and M_2I_2.

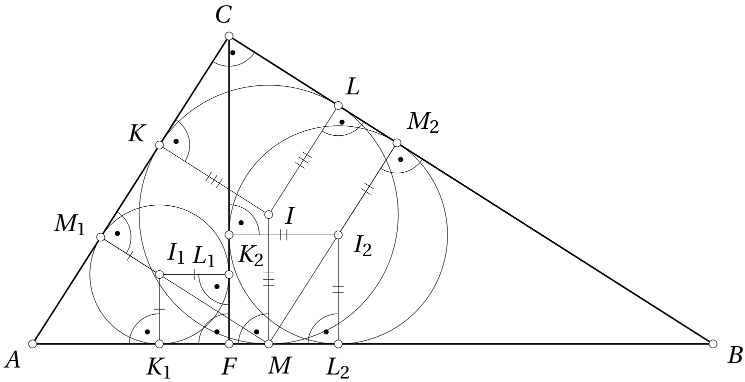


Fig. 37

Solution 1: Assume without the loss of generality that $AC \leqslant BC$. Let the radii of the incircles of ABC, ACF and BCF be r, r_1 and r_2 respectively. Let the tangency points of the incircle of ABC with the sides AC and BC be K and L respectively, the tangency points of the incircle of ACF with the sides AF and CF be K_1 and L_1 respectively and the tangency points of the incircle of BCF with the sides CF and BF be K_2 and L_2 respectively (Fig. 37).

We also denote $K_1M = l_1$ and $L_2M = l_2$. Notice that $KC = LC = r$ and similarly $K_1F = L_1F = r_1$ and $K_2F = L_2F = r_2$. So $l_1 + l_2 = K_1L_2 = r_1 + r_2$. But then the equality of tangents yields

$$\begin{aligned} l_1 - l_2 &= (AM - AK_1) - (BM - BL_2) \\ &= (AK - AM_1) - (BL - BM_2) \\ &= (CM_1 - CK) - (CM_2 - CL) \\ &= (CL_1 - r) - (CK_2 - r) \\ &= CL_1 - CK_2 \\ &= (CF - FL_1) - (CF - FK_2) \\ &= FK_2 - FL_1 \\ &= r_2 - r_1. \end{aligned}$$

The equations $l_1 + l_2 = r_1 + r_2$, $l_1 - l_2 = r_2 - r_1$ yield $l_1 = r_2$, $l_2 = r_1$. So I_1MK_1 and MI_2L_2 are equal right triangles with shorter sides of lengths r_1 and r_2.

The triangles ACF and CBF are similar to ABC due to having two pairs of equal angles. The similarity ratios are $\frac{AC}{AB}$ and $\frac{BC}{AB}$. So $\frac{r_1}{r} = \frac{AC}{AB}$ and $\frac{r_2}{r} = \frac{BC}{AB}$, which together yield $\frac{r_1}{AC} = \frac{r}{AB} = \frac{r_2}{BC}$. So the triangles I_1MK_1 and MI_2L_2 are also similar to ABC and furthermore $I_1M = I_2M = r = IM$. This means that M is the circumcenter of II_1I_2. We also have $\angle K_1MI_1 = \angle ABC$ and $\angle L_2MI_2 = \angle BAC$, which yield $MI_1 \parallel BC$ and $MI_2 \parallel AC$ and thus also $MI_1 \perp AC$ and $MI_2 \perp BC$. Therefore the extensions of MI_1 and MI_2 pass through M_1 and M_2 respectively, meaning that M is the intersection of M_1I_1 and M_2I_2.

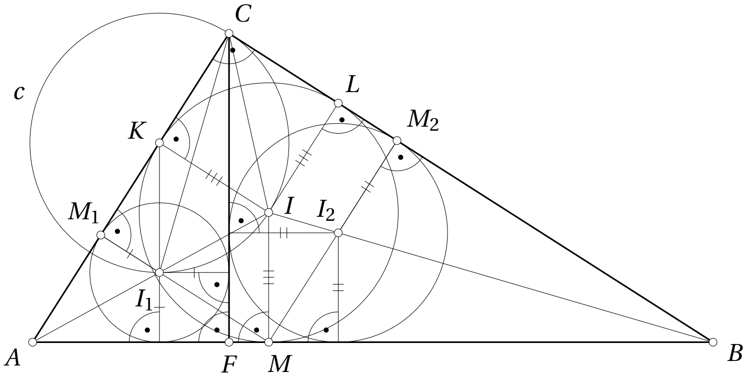


Fig. 38

Solution 2: Similarly to Solution 1 we define the points K and L and see that $CKIL$ is a square. Now consider the circle c with center K and passing through C and I (Fig. 38). The right angle CKI is thus the central angle subtending the arc CI.

Denote $\angle BAI = \angle CAI = \alpha$. Then $\angle AIK = \angle AIM = 90^\circ - \alpha$ and also $\angle BAI_1 = \angle CAI_1 = \alpha$, $\angle AI_1M_1 = 90^\circ - \alpha$. The triangle KIC is both right and isosceles, so $\angle KIC = 45^\circ$ and $\angle KCI = \angle ACI = 45^\circ$. Because of $\angle ACF = 90^\circ - \angle CAF = 90^\circ - 2\alpha$, we have $\angle ACI_1 = \frac{\angle ACF}{2} = 45^\circ - \alpha$.

Now, from the triangle CI_1I we see that

$$\begin{aligned} \angle CI_1I &= 180^\circ - \angle I_1CI - \angle I_1IC \\ &= 180^\circ - \angle ACI + \angle ACI_1 - \angle I_1IK - \angle KIC \\ &= 180^\circ - 45^\circ + (45^\circ - \alpha) - (90^\circ - \alpha) - 45^\circ = 45^\circ = \frac{\angle CKI}{2}. \end{aligned}$$

As $\angle CI_1I$ is half of the central angle subtending CI in c, I_1 must therefore lie on c. So $KI_1 = KI$. But as AIM and AIK are equal right triangles, the points M and K must be symmetric with respect to AI. As I_1 also lies on AI, this symmetry yields $MI_1 = MI$.

To show that M_1I_1 passes through M, we will show that $\angle M_1I_1M = 180^\circ$. The isosceles triangle IMI_1 yields $\angle II_1M = \angle I_1IM = 90^\circ - \alpha$. Since $\angle M_1I_1I = 180^\circ - \angle AI_1M_1 = 180^\circ - (90^\circ - \alpha) = 90^\circ + \alpha$, we indeed have

$$\angle M_1I_1M = \angle M_1I_1I + \angle II_1M = (90^\circ + \alpha) + (90^\circ - \alpha) = 180^\circ.$$

Analogously $MI = MI_2$ and $\angle M_2I_2M = 180^\circ$.

F27 *(Grade 12.)* We call a permutation $(\sigma_1, \sigma_2, \ldots, \sigma_n)$ of the numbers $1, 2, \ldots, n$ *alternating*, if $(-1)^i\sigma_i < (-1)^i\sigma_{i+1}$ holds for all $i = 1, \ldots, n-1$. For all positive integers n let α_n be the proportion of alternating permutations among all permutations of $1, 2, \ldots, n$. (For example $\alpha_1 = \frac{1}{1}$, $\alpha_2 = \frac{1}{2}$, $\alpha_3 = \frac{2}{6}$ etc.) Prove that there exist real numbers c_1, c_2 in the interval $(0;1)$ and a positive integer N, such that for all positive integers $n \geqslant N$ the inequality $(c_1)^n < \alpha_n < (c_2)^n$ holds.

Solution 1: We prove the existence of c_1, c_2 for $N = 2$.
We first show that c_1 exists. If $n = 2m$ for a positive integer m, consider permutations where the elements with an odd position are greater than m and the elements with an even position are at most m. Clearly all such permutations are alternating and there are $(m!)^2$ of them. As the total number of permutations is $n!$, we have $\frac{(m!)^2}{n!} \leqslant \alpha_n$. Now notice that

$$\frac{(m!)^2}{n!} = \frac{(m!)^2}{(1\cdot 3\cdot \ldots \cdot (n-1)) \cdot (2\cdot 4\cdot \ldots \cdot n)} > \frac{(m!)^2}{(2\cdot 4\cdot \ldots \cdot n)^2} = \frac{(m!)^2}{(2^m \cdot m!)^2} = \frac{1}{2^n}.$$

So $\left(\frac{1}{2}\right)^n < \alpha_n$.
If $n = 2m+1$, we set n to be the final element of the permutation and treat the other $2m$ elements analogously to the previous case. Therefore $\frac{(m!)^2}{n!} \leqslant \alpha_n$ and $\frac{(m!)^2}{n!} > \frac{1}{2^{n-1}\cdot n}$, like in the previous case. As $n < 2^n$, the final inequality yields $\frac{(m!)^2}{n!} > \frac{1}{2^n \cdot 2^n} = \left(\frac{1}{4}\right)^n$. In conclusion, $c_1 = \frac{1}{4}$ works.
Now we show that c_2 exists. We divide the elements of any alternating permutation into $\left\lfloor \frac{n}{2} \right\rfloor$ pairs consisting of consecutive elements (if n is odd, there will be one element left over). By choosing whether or not to swap the order of elements within each pair, we can generate $2^{\left\lfloor \frac{n}{2} \right\rfloor}$ distinct permutations from each alternating permutation. Furthermore, each permutation can only arise from at most one alternating permutation. Therefore $\alpha_n \leqslant \frac{1}{2^{\left\lfloor \frac{n}{2} \right\rfloor}} = \frac{1}{(\sqrt{2})^{2\cdot \left\lfloor \frac{n}{2} \right\rfloor}} \leqslant \frac{1}{(\sqrt{2})^{n-1}}$. By choosing $\varepsilon > 0$ such that $(1+\varepsilon)^2 < \sqrt{2}$, we have $(1+\varepsilon)^n < \left(\sqrt{2}\right)^{n-1}$. So $c_2 = \frac{1}{1+\varepsilon}$ works.
Solution 2: We prove the existence of c_1, c_2 for $N = 2$. For any positive integer k let s_k be the number of alternating permutations; then $s_k = \alpha_k \cdot k!$ and $s_0 = \alpha_0 = 1$.
The number n can only have an odd position. Denoting that by $2j+1$, we have C_{n-1}^{2j} ways of choosing the first $2j$ numbers. For each such choice, there are s_{2j} valid ways to arrange them. Similarly there are s_{n-2j-1} ways to fill the positions $2j+2, \ldots, n$ with the remaining numbers. As $2j+1 \leqslant n$, we have

$$s_n = \sum_{j=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} C_{n-1}^{2j} s_{2j} s_{n-1-2j},$$

where cancelling out $n!$ yields

$$\alpha_n = \frac{1}{n} \sum_{j=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \alpha_{2j} \alpha_{n-1-2j}.$$

We will show by strong induction that the inequality $(c_1)^n < \alpha_n$ holds for all $n \geqslant 1$, where e.g. $c_1 = \frac{1}{2}$ works. The base case $n = 1$ clearly holds. For

$n > 1$ we assume that the statement holds for all numbers less than n. Then

$$\alpha_n > \frac{1}{n}\sum_{j=0}^{\lfloor\frac{n-1}{2}\rfloor}(c_1)^{2j}(c_1)^{n-1-2j} = \frac{1}{n}\left\lfloor\frac{n+1}{2}\right\rfloor(c_1)^{n-1} \geqslant \frac{1}{2}(c_1)^{n-1} \geqslant (c_1)^n,$$

so the statement holds for all $n \geqslant 1$.
We now show by strong induction that for $n \geqslant 2$ the inequality $\alpha_n < (c_2)^n$ holds, where $c_2 = \frac{3}{4}$. For the base case $n = 2$ it holds. Now let $n > 2$ and assume that the statement holds for all numbers less than n (but greater than 1). If $n = 2k+1$, then (taking into account that $\alpha_0 = 1$)

$$\alpha_n = \alpha_{2k+1} \leqslant \frac{1}{2k+1}\sum_{j=0}^{k}(c_2)^{2j}(c_2)^{2k-2j} = \frac{k+1}{2k+1}(c_2)^{2k} < (c_2)^{2k+1} = (c_2)^n,$$

since $c_2 = \frac{3}{4} > \frac{2}{3} \geqslant \frac{k+1}{2k+1}$, if $k \geqslant 1$. If instead $n = 2k$, then (taking into account that $\alpha_0 = \alpha_1 = 1$)

$$\begin{aligned}\alpha_n = \alpha_{2k} &\leqslant \tfrac{1}{2k}\left(\left(\textstyle\sum_{j=0}^{k-2}(c_2)^{2j}(c_2)^{2k-1-2j}\right) + (c_2)^{2k-2}\right) = \\ &= \tfrac{k-1}{2k}(c_2)^{2k-1} + \tfrac{1}{2k}(c_2)^{2k-2} = \left(\tfrac{k-1}{2k} + \tfrac{1}{2k\cdot c_2}\right)(c_2)^{2k-1}.\end{aligned}$$

Since $c_2 = \frac{3}{4} \geqslant \frac{k+1}{2k}$, if $k \geqslant 2$, we have

$$\frac{k-1}{2k} + \frac{1}{2k\cdot c_2} \leqslant \frac{k-1}{2k} + \frac{1}{k+1} = \frac{k^2-1+2k}{2k(k+1)} < \frac{k^2+1+2k}{2k(k+1)} = \frac{k+1}{2k} \leqslant c_2,$$

which means that $\alpha_n < (c_2)^{2k} = (c_2)^n$. So the statement holds for all $n \geqslant 2$, as desired.

Selected Problems from the IMO Team Selection Contests

S1 Given a prime number p and integers x and y, find the remainder of the sum $x^0y^{p-1} + x^1y^{p-2} + \ldots + x^{p-2}y^1 + x^{p-1}y^0$ upon division by p.
Answer: 0 if $p \mid x - y$; 1 otherwise.
Solution 1: If $x \equiv y \pmod{p}$ then, for every $i = 0, 1, \ldots, p-1$,

$$x^iy^{p-1-i} \equiv x^{p-1} \pmod{p}.$$

As there are p summands,

$$x^0y^{p-1} + x^1y^{p-2} + \ldots + x^{p-2}y^1 + x^{p-1}y^0 \equiv p\cdot x^{p-1} \equiv 0 \pmod{p}.$$

Assume now that $x \not\equiv y \pmod{p}$. We know that

$$x^0y^{p-1} + x^1y^{p-2} + \ldots + x^{p-2}y^1 + x^{p-1}y^0 = \frac{x^p - y^p}{x - y}.$$

By Fermat's little theorem, $x^p \equiv x \pmod{p}$ and $y^p \equiv y \pmod{p}$, implying that $x^p - y^p \equiv x - y \pmod{p}$. By assumption, $x - y \not\equiv 0 \pmod{p}$,

implying that $\gcd(x-y,p)=1$. Hence $x^p - y^p \equiv x - y \pmod{p}$ implies
$$\frac{x^p - y^p}{x-y} \equiv 1 \pmod{p}.$$
All in all, the remainder of the sum under consideration upon division by p is 0 if $p \mid x-y$ and 1 otherwise.

Solution 2: We prove by induction on i that $C_{p-1}^{i} \equiv (-1)^i \pmod{p}$ for every $i = 0, 1, \ldots, p-1$. The induction base holds as $C_{p-1}^{0} = 1 = (-1)^0$. For the induction step, assume that the claim holds for some natural number i such that $i < p-1$. By Pascal's rule, $C_{p-1}^{i} + C_{p-1}^{i+1} = C_{p}^{i+1}$. But the latter is divisible by p whenever $0 \leqslant i < p-1$. Thus
$$C_{p-1}^{i+1} = C_{p}^{i+1} - C_{p-1}^{i} \equiv 0 - (-1)^i = (-1)^{i+1} \pmod{p}.$$
This completes the induction step.

The lemma just proved implies $x^i = (-1)^i(-x)^i \equiv C_{p-1}^{i}(-x)^i \pmod{p}$. Hence the sum under consideration is congruent modulo p to
$$C_{p-1}^{0}(-x)^0 y^{p-1} + C_{p-1}^{1}(-x)^1 y^{p-2} + \ldots + C_{p-1}^{p-2}(-x)^{p-2} y^1 + C_{p-1}^{p-1}(-x)^{p-1} y^0.$$
By binomial formula, the latter equals $(y-x)^{p-1}$. By Fermat's little theorem, $(y-x)^{p-1} \equiv 1 \pmod{p}$ whenever $p \nmid y-x$. If $p \mid y-x$ then obviously $(y-x)^{p-1} \equiv 0 \pmod{p}$.

S2 For any natural number n and positive integer k, we say that n is *k-good* if there exist non-negative integers $a_1, \ldots, a_k$ such that
$$n = a_1^2 + a_2^4 + a_3^8 + \ldots + a_k^{2^k}.$$
Is there a positive integer k for which every natural number is k-good?

Answer: No.

Solution: Let k be any positive integer. We show that there exists a natural number that is not k-good.

Choose an arbitrary integer $m > k$. Consider all expressions of the form $a_1^2 + a_2^4 + \ldots + a_k^{2^k}$ where $a_1, a_2, \ldots, a_k$ are non-negative integers and all summands are less than m^{2^k}. The last condition implies that $0 \leqslant a_i \leqslant m^{2^{k-i}} - 1$ for every $i = 1, 2, \ldots, k$. Thus for every $i = 1, 2, \ldots, k$, there are $m^{2^{k-i}}$ candidates for a_i. Consequently, the number of the expressions under consideration is $m^{2^{k-1}} \cdot m^{2^{k-2}} \cdot \ldots \cdot m$ which equals $m^{2^k - 1}$.

Whenever a number $n \leqslant m^{2^k}$ is k-good, its representation required by the definition of being k-good either is one just counted or contains one summand m^{2^k} while other summands being zeros. There are $m^{2^k-1} + k$ such representations. As $m > k \geqslant 1$, we have
$$m^{2^k-1} + k < m^{2^k-1} + m \leqslant m^{2^k-1} + (m-1) \cdot m^{2^k-1} = m^{2^k},$$
implying that there exist numbers less than m^{2^k} that cannot be represented in the required way.

S3 A convex quadrilateral $ABCD$ has $\angle BAC = \angle ADC$. Let M be the midpoint of the diagonal AC. The side AD contains a point E such that $ABME$ is a parallelogram. Let N be the midpoint of the line segment AE. Prove that the line AC touches the circumcircle of the triangle DMN at point M.

Solution 1: We have $AB \parallel EM$ as $ABME$ is a parallelogram. Hence

$$\angle AME = \angle BAM = \angle BAC = \angle ADC.$$

As the triangles AME and ADC have a common angle at the vertex A (Fig. 39), the triangles AME and ADC are similar.

As N is the midpoint of the side AE of the triangle AME and M is the midpoint of the side AC of the triangle ADC, the triangles AMN and ADM are similar, too. Hence $\angle AMN = \angle ADM = \angle NDM$, i.e., the inscribed angle subtending the arc MN of the circumcircle of the triangle DMN equals the angle between the chord MN and the line AM. Consequently, AM is tangent to the circumcircle of the triangle DMN.

Solution 2: We have $AB \parallel EM$ as $ABME$ is a parallelogram. Hence

$$\angle AME = \angle BAM = \angle BAC = \angle ADC = \angle EDC.$$

Thus $\angle EMC = 180^\circ - \angle AME = 180^\circ - \angle EDC$ (Fig. 40). Consequently, $EDCM$ is an inscribed quadrilateral, implying that $\angle ADM = \angle ACE$. As NM is a midline of the triangle AEC, we have $MN \parallel CE$, implying that $\angle ACE = \angle AMN$. Consequently, $\angle ADM = \angle AMN$. Hence AM is tangent to the circumcircle of the triangle DMN.

Solution 3: We have $AE \parallel BM$ as $ABME$ is a parallelogram. Let L be the intersection of the line BM with the side CD of the quadrilateral $ABCD$ (Fig. 41). Then $\angle BLC = \angle MLC = \angle ADC = \angle BAC$. Thus $ABCL$ is an inscribed quadrilateral, whence

$$MB \cdot ML = MA \cdot MC = AM^2.$$

On the other hand, ML is a midline of the triangle ACD since M is the midpoint of the side AC and $AD \parallel ML$. Hence $ML = \frac{1}{2}AD$, implying that

$$BM \cdot ML = AE \cdot \frac{1}{2}AD = \frac{1}{2}AE \cdot AD = AN \cdot AD.$$

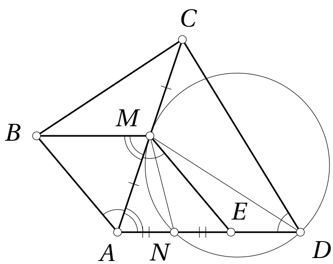


Fig. 39

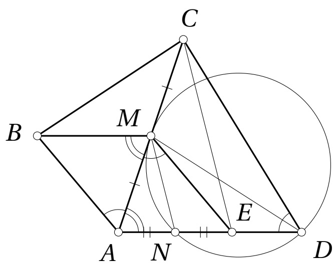


Fig. 40

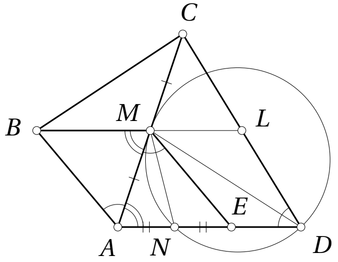


Fig. 41

Consequently, $AN \cdot AD = AM^2$, implying that AM is tangent to the circumcircle of the triangle DMN.

S4 We say that distinct positive integers n, m are *friends* if $|n-m|$ is a divisor of both n and m. Prove that, for any positive integer k, there exist k distinct positive integers such that any two of these integers are friends.

Solution 1: We prove the required claim by induction on k. If $k=1$, the single number 1 is suitable (actually any single positive integer does the job). If a tuple $(a_1,\dots,a_k)$ satisfies the conditions of the problem, whereby a_k is the largest number in the tuple, then also the tuple $(b_1,\dots,b_{k+1})$ satisfies the conditions of the problem, where

$$b_i = \begin{cases} a_k! + a_i & \text{if } 1 \leqslant i \leqslant k, \\ a_k! & \text{if } i = k+1. \end{cases}$$

Indeed, if $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant k$, then

$$b_i - b_j = (a_k! + a_i) - \left(a_k! + a_j\right) = a_i - a_j.$$

By the induction hypothesis, $a_i - a_j \mid a_i$ and $a_i - a_j \mid a_j$. As a_k is the largest number in $(a_1,\dots,a_k)$, we have $a_i \mid a_k!$ and $a_j \mid a_k!$. Thus $a_i \mid a_k! + a_i$ and $a_j \mid a_k! + a_j$. Consequently, $b_i - b_j \mid b_i$ and $b_i - b_j \mid b_j$. But if, for instance, $1 \leqslant i \leqslant k$, $j = k+1$, then

$$b_i - b_j = (a_k! + a_i) - a_k! = a_i.$$

As a_k is the largest number in $(a_1,\dots,a_k)$, we have $a_i \mid a_k!$. Thus also $a_i \mid a_k! + a_i$. Consequently, $b_i - b_j \mid b_i$ and $b_i - b_j \mid b_{k+1} = b_j$. This completes the induction step and the whole proof.

Solution 2: Define functions $f_0, f_1, f_2, \dots$ on positive integers by $f_0(l) = l$ and $f_{k+1}(l) = (f_k(l))!$, i.e., $f_1(l) = l!$, $f_2(l) = (l!)!$ etc.. Easy induction on k shows that $l < l'$ always implies $f_k(l) < f_k(l')$. Thus if k is positive and $l < l'$ then $f_k(l) \mid f_k(l')$. Clearly also $k < k'$ implies $f_k(l) \mid f_{k'}(l)$. We show that, for every positive integer k, the numbers in the list

$$\begin{aligned} &f_k(k), \\ &f_k(k) + f_{k-1}(k-1), \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots, \\ &f_k(k) + f_{k-1}(k-1) + \dots + f_1(1) \end{aligned}$$

are pairwise friends. This completes the solution since the numbers in the list are obviously distinct.

In order to prove the desired claim, we choose arbitrary positive integers i and j such that $i < j \leqslant k$ and find that

$$\begin{aligned}(f_k(k) + f_{k-1}(k-1) + \ldots + f_i(i)) - \big(f_k(k) + f_{k-1}(k-1) + \ldots + f_j(j)\big) \\ = f_{j-1}(j-1) + \ldots + f_i(i).\end{aligned}$$

As every summand in the list $f_{j-1}(j-1), \ldots, f_i(i)$ is a proper divisor of all preceding summands,

$$\begin{aligned} f_{j-1}(j-1) + \ldots + f_i(i) &\leqslant f_{j-1}(j-1) + \ldots + \tfrac{1}{2^{j-i+1}} f_{j-1}(j-1) \\ &= \left(\tfrac{1}{2^0} + \ldots + \tfrac{1}{2^{j-i+1}}\right) \cdot f_{j-1}(j-1) \\ &< 2 \cdot f_{j-1}(j-1) \leqslant f_{j-1}(j). \end{aligned}$$

For any integer l such that $j < l \leqslant k$, we have $f_{j-1}(j) < f_{l-1}(l)$. Thus $f_{j-1}(j-1) + \ldots + f_i(i) \mid (f_{l-1}(l))!$ for any integer l such that $j \leqslant l \leqslant k$. But $(f_{l-1}(l))! = f_l(l)$. Consequently, the number $f_{j-1}(j) + \ldots + f_i(i)$ divides all numbers $f_j(j), \ldots, f_{k-1}(k-1), f_k(k)$ and hence it divides also the number $f_k(k) + f_{k-1}(k-1) + \ldots + f_j(j)$. Thus $f_k(k) + f_{k-1}(k-1) + \ldots + f_i(i)$ and $f_k(k) + f_{k-1}(k-1) + \ldots + f_j(j)$ are friends.

Solution 3: We prove the claim by induction on k. If $k = 1$, the single number 1 is suitable (actually any single positive integer does the job). Whenever any tuple $(a_1, \ldots, a_k)$ satisfies the conditions of the problem, also the tuple $(b_1, \ldots, b_{k+1})$ satisfies the conditions of the problem, where $b_i = a_1 \ldots a_{i-1} (a_i + 1) a_{i+1} \ldots a_k$ for every $i = 1, \ldots, k$ and $b_{k+1} = a_1 \ldots a_k$. Indeed, if $1 \leqslant i < j \leqslant k$ then

$$\begin{aligned} b_i - b_j &= a_1 \ldots a_{i-1} a_{i+1} \ldots a_{j-1} a_{j+1} \ldots a_k \left((a_i + 1) a_j - a_i \left(a_j + 1\right)\right) = \\ &= a_1 \ldots a_{i-1} a_{i+1} \ldots a_{j-1} a_{j+1} \ldots a_k \left(a_j - a_i\right). \end{aligned}$$

By the induction hypothesis, $a_j - a_i \mid a_i$ and $a_j - a_i \mid a_j$, implying that $a_j - a_i \mid a_i \left(a_j + 1\right)$ and $a_j - a_i \mid (a_i + 1) a_j$. Hence $b_i - b_j \mid b_i$ and $b_i - b_j \mid b_j$. For every $i = 1, \ldots, k$, we also have

$$b_i - b_{k+1} = a_1 \ldots a_{i-1} a_{i+1} \ldots a_k \left((a_i + 1) - a_i\right) = a_1 \ldots a_{i-1} a_{i+1} \ldots a_k$$

and the latter is a common divisor of b_i and b_{k+1}. Thus every two members of the tuple $(b_1, \ldots, b_{k+1})$ are friends. They are also distinct because they are obtained from $a_1 \ldots a_k$ by multiplying by distinct factors $\frac{a_1+1}{a_1}, \ldots, \frac{a_k+1}{a_k}, 1$.

Solution 4: We prove the claim by induction on k. If $k = 1$, the single number 1 is suitable (actually any single positive integer does the job). Assume that the tuple $(a_1, \ldots, a_k)$ satisfies the conditions of the problem. Let v be a common multiple of all differences $a_i - a_j$, where $i \neq j$, such that $v > a_i$ for all $i = 1, \ldots, k$. Let w be a common multiple of all differences $v - a_i$, such that $w > a_i$ for all $i = 1, \ldots, k$. Define $b_i = a_i + vw - v$ for every $i = 1, \ldots, k$ and $b_{k+1} = vw = v + vw - v$. We show that the tuple $(b_1, \ldots, b_{k+1})$ satisfies the conditions of the problem. Indeed, if $1 \leqslant i < j \leqslant k$ then $b_i - b_j = a_i - a_j$ and $a_i - a_j$ divides both b_i and b_j since it divides both a_i and a_j, as well as v. For every $i = 1, \ldots, k$ also $b_i - b_{k+1} = a_i - v$, but the latter divides w. Consequently, $b_i - b_{k+1}$ divides b_{k+1}, implying that $b_i - b_{k+1}$ also divides b_i. Thus every two members of the tuple $(b_1, \ldots, b_{k+1})$ are friends. They are also distinct since they are obtained from distinct numbers $a_1, \ldots, a_k, v$ by adding the same constant $vw - v$.

Solution 5: Note that n and m are friends if and only if $|n - m| = \gcd(n, m)$. Indeed, if $|n - m| = \gcd(n, m)$ then $|n - m|$ is a common divisor of n and m, meaning that n and m are friends. Conversely, if n and m are friends then $|n - m|$ is a common divisor of n and m, implying that $|n - m|$ is a divisor of $\gcd(n, m)$. But as $\gcd(n, m)$ divides $|n - m|$, this is possible only if $|n - m| = \gcd(n, m)$.

We prove the claim by induction on k. If $k = 1$, the single number 1 is suitable (actually any single positive integer does the job). Assume that a tuple $(a_1, \ldots, a_k)$ such that $a_1 < \ldots < a_k$ satisfies the conditions of the problem. Define $b_i = 2^{a_k} - 2^{a_k - a_i} = (2^{a_i} - 1) \cdot 2^{a_k - a_i}$ for every $i = 1, \ldots, k$ and $b_{k+1} = 2^{a_k}$. We show that the tuple $(b_1, \ldots, b_{k+1})$ satisfies the conditions of the problem. If $1 \leqslant i < j \leqslant k$ then the induction hypothesis together with the well-known fact that $\gcd\left(x^i - 1, x^j - 1\right) = x^{\gcd(i,j)} - 1$ imply

$$\begin{aligned}
\gcd(b_i, b_j) &= \gcd\left((2^{a_i} - 1) \cdot 2^{a_k - a_i}, (2^{a_j} - 1) \cdot 2^{a_k - a_j}\right) = \\
&= 2^{a_k - a_j} \cdot \gcd\left((2^{a_i} - 1) \cdot 2^{a_j - a_i}, 2^{a_j} - 1\right) = \\
&= 2^{a_k - a_j} \cdot \gcd\left(2^{a_i} - 1, 2^{a_j} - 1\right) = \\
&= 2^{a_k - a_j} \cdot \left(2^{\gcd(a_i, a_j)} - 1\right) = \\
&= 2^{a_k - a_j} \cdot \left(2^{a_j - a_i} - 1\right) = 2^{a_k - a_i} - 2^{a_k - a_j} = b_j - b_i.
\end{aligned}$$

For every $i = 1, \ldots, k$ we also have

$$\gcd\left(b_i, b_{k+1}\right) = \gcd\left((2^{a_i} - 1) \cdot 2^{a_k - a_i}, 2^{a_k}\right) = 2^{a_k - a_i} = b_{k+1} - b_i.$$

Thus every two members of the tuple $(b_1, \ldots, b_{k+1})$ are friends. The numbers $b_1, \ldots, b_{k+1}$ are also pairwise distinct since 2 occurs with distinct exponents $a_k - a_1, \ldots, a_k - a_{k-1}, 0, a_k$ in their canonical representation.